

Carrollian conformal dynamics

a flat-holographic story

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Based on [2505.00077] with A. Fiorucci, M. Petropoulos and M. Vilatte (to appear in *PRL*)



An invitation: conservation laws in physics

Many systems obey *relativistic conservation laws*. [Classical fields, fluid mechanics, Einstein's equations...]

Lagrangian $\mathcal{L}[\phi] \rightarrow$ canonical stress-tensor:

$$t^\mu{}_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \partial_\nu \phi^\alpha - \delta^\mu{}_\nu \mathcal{L}.$$

Conserved and symmetric:

$$\partial_\mu t^\mu{}_\nu \approx 0, \quad t^{\mu\nu} = t^{(\mu\nu)}.$$

Translation and Lorentz invariance (up to BR).

Systems with conformal invariance:

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Scale and special conformal invariance (up to BR).

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Scale and special conformal invariance (up to BR).

Link and geometric origin of improvement terms [Iosifidis, Karydas, Petkou, Siampos '25].

Curved metric $g \rightarrow$ minimal coupling $S[\phi; g]$

$$T^{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}.$$

Covariantly conserved (already symmetric):

$$\nabla_\mu T^{\mu\nu} \approx 0.$$

Diffeomorphism invariance.

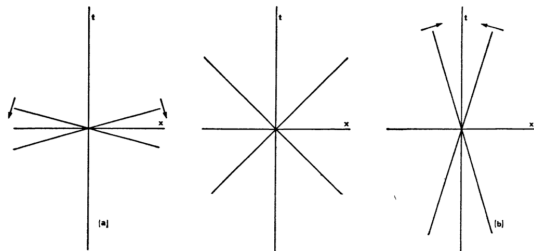
For conformally-coupled theories:

$$g_{\mu\nu} T^{\mu\nu} \approx 0.$$

Weyl invariance (CFTs with no Weyl anomaly).

Non-Lorentzian *relativities* and possible kinematics

Non-Lorentzian relativities: *Galilean* ($c \rightarrow \infty$) [Bargmann '54] or *Carrollian* ($c \rightarrow 0$) [Lévy-Leblond '65].



Credit: [Lévy-Leblond '65].

Treat c as a parameter \rightarrow explore different regimes of Lorentzian theories.

[particles, gauge theories, (conformal) field theories, gravity, higher-spins, strings...]

[Bergshoeff et al.], [Duval, Gibbons, Horvathy et al.], [Figueroa-O'Farrill et al.], [Hartong et al.], [Henneaux et al.], [Obers et al.], ...

Role of non-Lorentzian “conservation” (better called *evolution*) laws increasingly studied.

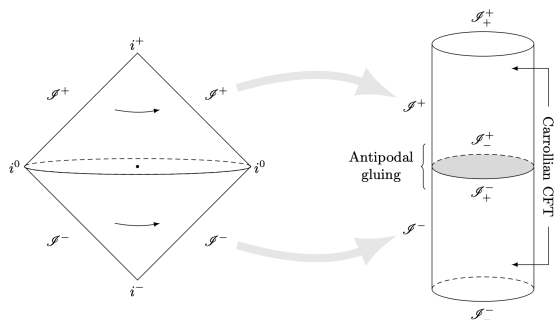
[Duval, Künzle '76], [Bagchi, Gopakumar '09], [Bekaert, Morand '14], [Duval et al. '14], [de Boer et al. '18], ...

Carrollian physics: motivation from flat holography

Null (light-like) hypersurfaces described by **Carrollian** physics [Duval et al. '14].

Application to *flat holography*: null infinity (\mathcal{I}) is a Carrollian + conformal hypersurface.

[Penrose '63], [Ashtekar et al.], [Duval et al.], [Barnich et al.], [Petrooulos et al.], ...



Credit: [Donnay, Fiorucci, Herfray, Ruzziconi '22].

This talk: *Carrollian conformal evolution equations* with application (BMS evolution equations).

Plan

- 1 Lorentzian case
- 2 Carrollian case
- 3 Application: *gravitational evolution* equations
- 4 Outlook

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Lorentzian conformal geometry: frame and local transformations

Smooth manifold \mathcal{M} of dimension $(d + 1)$

Local frame and co-frame:

$$\{\mathbf{e}_a\}_{a=0,1,\dots,d} \in \Gamma(T\mathcal{M}),$$

dual to

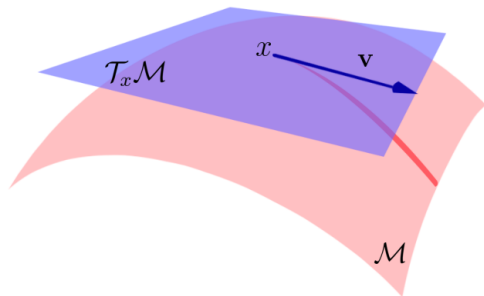
$$\{\boldsymbol{\theta}^a\}_{a=0,1,\dots,d} \in \Gamma(T^*\mathcal{M}).$$

Curved metric $\mathbf{g} = \eta_{ab}\boldsymbol{\theta}^a \otimes \boldsymbol{\theta}^b$.

Flat tangent space metric η_{ab} .

Transformations of the local (co-)frame

- Local Lorentz: $\delta_\lambda \boldsymbol{\theta}^a = \lambda^a_b \boldsymbol{\theta}^b$; ($\lambda^{ab} = \lambda^{[ab]}$)
- Local Weyl: $\delta_B \boldsymbol{\theta}^a = -B \boldsymbol{\theta}^a$; ($\delta_B \mathbf{g} = -2B \mathbf{g}$)
equivalence class of metrics: $\mathbf{g} \sim \mathcal{B}^{-2} \mathbf{g}$
- Diffeomorphisms: $\delta_\xi \boldsymbol{\theta}^a = \mathcal{L}_\xi \boldsymbol{\theta}^a$.
acts only on the base-space indices



Credit: [Battiloro et al. '23].

Lorentzian conformal geometry: connection, non-metricity, torsion

Spin connection ω^a_b

- Local Lorentz: $\delta_\lambda \omega^a_b = \mathbf{D} \lambda_b^a$; ($\mathbf{D} := d + \omega^a_b$)
- Local Weyl: $\delta_B \omega^a_b = \delta^a_b dB$. ($\mathcal{D} := \mathbf{D} + \frac{w}{d+1} \omega^a_a$)

Assume Weyl-metricity: $\omega^{\langle ab \rangle} := \omega^{(ab)} - \frac{1}{d+1} \eta^{ab} \omega^c_c \stackrel{!}{=} 0$.

Generalisation of LC theorem: $\mathbf{K}^a := \mathbf{D} \theta^a \stackrel{!}{=} 0 \implies \omega^a_b$ fixed uniquely.

Subtle point: Weyl-connection ω^a_a extra but **pure-gauge** (special conformal transformations).

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Spoiler: *not* the true in Galilean/Carrollian world! [(Weyl-)metric & torsionless $\not\Rightarrow$ unique.]

Lorentzian on-shell variation and evolution equations

First-order *metric-affine* variation (on-shell for ϕ^α) [Iosifidis '20]

$$\delta S[\phi; \theta, \omega] \approx \int_{\mathcal{M}} \mu \left(\delta \theta^a [\mathbf{T}_a] + \delta \omega^a{}_b [\mathbf{\Omega}_a{}^b] \right),$$

$\mu = \frac{1}{(d+1)!} \theta^{a_0} \wedge \dots \wedge \theta^{a_d} \varepsilon_{a_0 \dots a_d}$ volume form. \mathbf{T}_a : momenta, $\mathbf{\Omega}_a{}^b$: hypermomenta.

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- Local Lorentz & Weyl transformations $\delta_\lambda S = 0 = \delta_B S \implies$ constraints:

$$T_{[ab]} = -\mathcal{D} \cdot \Omega_{[ba]}, \quad T^a{}_a = -\mathcal{D} \cdot \Omega_a{}^a.$$

- Diffeomorphisms $\delta_\xi S = 0 \implies$ *evolution equations* [and charge]:

$$\mathcal{D}[\mathbf{T}_a] = R^b{}_c[\mathbf{e}_a, \mathbf{\Omega}_b{}^c].$$

For $\mathbf{K}^a \neq 0$ and $\mathbf{\Omega}_a{}^b \neq 0$: cannot be recast as conservation (part of connection is independent).

[Useful e.g. for fermionic fields, spin fluids, (cosmological) hyperfluids, teleparallel gravity...]

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An incursion into the Galilean world

Galilean case describes “non-relativistic” hydrodynamics: studied long ago (Landau-Lifschitz!).

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Revisited in [Duval, Künzle '76] using Newton-Cartan geometry (similar to Lorentz with torsion).

But by (4.4) and (4.9) this can be written as

$$\nabla_{\beta} T^{\beta}{}_{\alpha} = \int \gamma_{\alpha\beta}^{\vee} \dot{v}^{\beta} \quad (4.12)$$

provided one defines

$$T^{\beta}{}_{\alpha} := \tilde{T}^{\beta}{}_{\alpha} + \int v^{\beta} v_{\alpha} - \frac{1}{2} \rho v^2 \psi_{\alpha} v^{\beta} \quad (4.13)$$

Equation (4.12) is the closest analogue to the relativistic conservation law (2.2) one can find in the Newtonian theory, since it is not possible to absorb the right hand side in the divergence. The relativistic stress energy tensor $T^{\alpha\beta}$ therefore decouples in the Newtonian theory in a tensor $T^{\beta}{}_{\alpha}$ describing stresses and energy flow, and the matter

No LC theorem (part of connection independent) \implies hypermomenta \implies *non-conservation*.

Carrollian (conformal) symmetries & structure

Carrollian relativity: $c \rightarrow 0$ (effective speed of light, thought of as $\Delta x \gg c\Delta t$).

Carroll group in $(d + 1)$ dimensions (rotation $R^i_j \in SO(d)$, boost $\lambda_i \in \mathbb{R}^d$, translations $a^i \in \mathbb{R}^d$, $a^0 \in \mathbb{R}$)

$$\begin{cases} x^i \rightarrow x'^i = R^i_j x^j + a^i, \\ x^0 \rightarrow x'^0 = x^0 + a^0 - \lambda_i R^i_j x^j. \end{cases}$$

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Conformal extension [Duval, Gibbons, Horvathy '14]

Conformal extension of Carroll in $(d + 1)$ dims \simeq Poincaré in $(d + 2)$ dims.

Infinite-dimensional extension \rightarrow BMS_{d+2} super-translations.

Flat-space analogue of conformal symmetries in $(d + 1)$ dims \simeq AdS isometries $(d + 2)$ dims.

[Carrollian conformal symmetry is the relevant group for flat holography].

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From AdS/CFT perspective [Bekaert, Campoleoni, Ciambelli, Donnay, Fiorucci, Herfray, Nguyen, Petropoulos, Ruzzi...]]

Flat limit (bulk) \longleftrightarrow Carrollian limit (boundary).

Carrollian (conformal) geometry & evolution equations

Carrollian structure $(\mathcal{M}, \mathbf{g}, \mathbf{n})$: degenerate metric \mathbf{g} and vector \mathbf{n} in its kernel [Henneaux '79]:

$$\mathbf{g}(\mathbf{n}, \cdot) = 0.$$

Conformal class: $(\mathbf{g}, \mathbf{n}) \sim (\mathcal{B}^{-2}\mathbf{g}, \mathcal{B}\mathbf{n})$ [\mathbf{g} has weight -2 and \mathbf{n} has weight $+1$].

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Can a connection be Weyl- (\mathbf{g}, \mathbf{n}) -compatible and torsion-free? [Ciambelli et al. '18]

- 1 Geometric shear vanishes $(\mathcal{L}_{\mathbf{n}}\mathbf{g})^{\text{TF}} \stackrel{!}{=} 0$;
- 2 Temporal Weyl-connection fixed;
- 3 No LC thm: *parts of connection are free*.

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Co-frame $\{\mathbf{n}, \mathbf{e}_i\}_{i=1, \dots, d}$ and Weyl-compatible, torsion-free connection $\{\omega^0_0, \omega^0_i, \omega^{[ij]}\}$:

- Natural to consider on-shell metric-affine variation *with hypermomenta* [Fiorucci et al. '25];
- Evolution equations $\mathcal{D}[\mathbf{T}_0] = \mathbf{R}^a_b[\mathbf{n}, \Omega_a^b]$, $\mathcal{D}[\mathbf{T}_i] = \mathbf{R}^a_b[\mathbf{e}_i, \Omega_a^b]$ not conservation laws.

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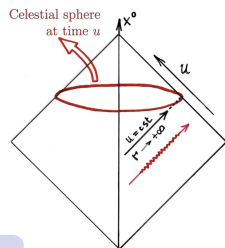
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Asymptotically flat space-times: Bondi gauge

Asymptotically Minkowski₄ space-time [Bondi, van der Burg, Metzner '62], [Sachs '62]

$$ds^2 = -\frac{R}{2} du^2 - 2dudr + r^2 \gamma_{AB} dx^A dx^B + r C_{AB} dx^A dx^B + \dots$$

with $\gamma_{AB}(x)$ metric on S^2 [curvature R] and $\gamma^{AB} C_{AB} = 0$.



Metric at \mathcal{I}

Boundary metric $0 \times du^2 + \gamma_{AB} dx^A dx^B$ is **Carrollian**.

Conformal compactification \Rightarrow conformal class.

- Asymptotic **shear** C_{AB} [news $N_{AB} := \partial_u C_{AB}$];
- *Bondi mass aspect*: $(\dots + \frac{2}{r} M + \dots) du^2$;
- *Angular momentum aspect*: $(\dots - \frac{2}{3r} N_A + \dots) du dx^A$.

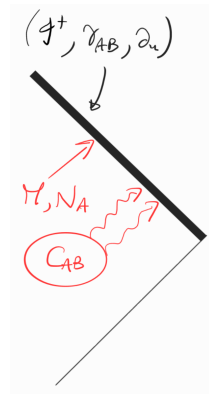
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Asymptotically 4d flat space-times (Bondi gauge)

- Carrollian + conformal boundary $(\mathcal{I}, \gamma_{AB}, \partial_u)$;
- Einstein's equations (evolution equations) [Tamburino, Winicour '66]

$$\partial_u M = \frac{1}{4} D_A D_B N^{AB} - \frac{1}{8} N_{AB} N^{AB} + \frac{1}{8} D_A D^A R,$$

$$\begin{aligned} \partial_u N_A &= \partial_A M - \frac{1}{4} N^{BC} D_A C_{BC} + \frac{1}{16} \partial_A (C_{BC} N^{BC}) \\ &\quad + \frac{1}{2} D^B (D_{[A} D^C C_{B]C} + C^C_{[A} N_{B]C}) + \frac{1}{4} C_{AB} \partial^B R. \end{aligned}$$



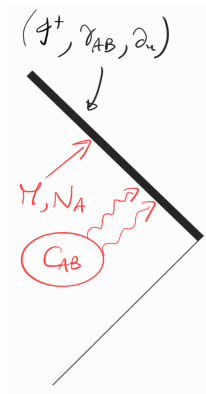
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Einstein's equations in AIAdS₄ \iff conservation of *holographic* stress-tensor. [Balasubramanian, Kraus '99]

Flat limit \longrightarrow *evolution equations* (not conservation). [Compère, Fiorucci, Ruzziconi '19], [Campoleoni et al. '23]

Can be recast as Carrollian conformal evolution equations at \mathcal{I} ?

Carrollian conformal dynamics at \mathcal{I} : setup

\mathcal{I} is a Carrollian conformal manifold with specific connection [Ciambelli et al. '18], [Campoleoni et al. '23].

Kinematic arena

- 1 Einstein's equations \implies Weyl-compatible, torsion-free connection at \mathcal{I} ;
- 2 **Ambiguity** in connection \longleftrightarrow asymptotic **shear** C_{AB} ; [Geroch '77], [Ashtekar '81]

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Flux of radiation at \mathcal{I} (AS potential) \longleftrightarrow **hypermomenta** $\neq 0$ for C_{AB} [Ashtekar, Streubel '81]

The need for hypermomentum

$$\delta S_{\text{EH}} \ni \vartheta_{\text{AS}} = \frac{1}{32\pi G} \int_{\mathcal{I}} du d^2x \sqrt{\gamma} \delta C_{AB} N^{AB} .$$

Variation of connection \times **hypermomentum**.

Carrollian conformal dynamics at \mathcal{I} : dictionary

Bondi frame ($d = 2$): $\delta_{ij}\theta^i \otimes \theta^j = \gamma_{AB}dx^A dx^B$, $\tau = du$.

Ambiguity in connection: $\omega^0_{\langle i}[\mathbf{e}_j] := -\frac{1}{2}C_{ij}$ (asymptotic shear).

Dictionary (responses to variations)

- Hypermomenta: *geometry* (boundary Riemann)

$$\Omega_0^i := (N^{ij} + \frac{1}{2}\delta^{ij}R) \mathbf{e}_j, \quad \Omega_{[ij]} := D_{[i}C_{j]}^k \mathbf{e}_k.$$

- Momenta: *charge aspects* (with geometric decorations)

$$T^0_0 \equiv 4M, \quad T^0_i \equiv 2N_i + \frac{1}{16}D_i(C_{jk}C^{jk}), \quad T_{\langle ij \rangle} := -\frac{1}{4}RC_{ij}.$$

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With this identification:

Bondi evolution equations \equiv Carrollian conformal evolution equations.

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Summary & future directions

Take-home message

- General evolution equations on non-Lorentzian manifolds ;
- Clarified status of BMS evolution equations ;
- **Geometric** understanding of radiation and charges from boundary viewpoint .

Open questions

- Dictionary from holographic renormalisation [Hartong et al. '25] ;
- Comparison with relaxed Bondi gauge [Geiller, Zwickel '22] ;
- **Microscopic** theory coupling to connection reproducing this dictionary ;
- Implications for scattering amplitudes & Carrollian holography .

Thank you!



Nikos Engonopoulos, *Trois philosophes*, 1937.

Plan

5 Supplementary material

Reminder: Lorentzian stress-tensor from metric variation

Theory for relativistic fields ϕ^α in $(d + 1)$ dimensions, action $S[\phi] = \int d^{d+1} \mathcal{L}[\phi]$

$$S[\phi] \xrightarrow{\text{min. coupling}} S[\phi; g], \quad \text{such that } S[\phi; \eta] = S[\phi].$$

General variation:

$$\delta S[\phi; g] = \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left(\delta\phi^\alpha \mathcal{E}_\alpha + \frac{1}{2} \delta g_{\mu\nu} T^{\mu\nu} \right).$$

- On-shell for the fields ($\mathcal{E}_\alpha \approx 0$), $T^{\mu\nu}$ is covariantly conserved: $\nabla_\mu T^{\mu\nu} \approx 0$.
Can be deduced from evaluating $\delta S[\phi; g]$ on a general diffeomorphism ξ (boundary term δQ_ξ is the charge):

$$\delta_\xi S[\phi; g] \approx \frac{1}{2} \int d^{d+1}x \sqrt{-g} (\mathcal{L}_\xi g)_{\mu\nu} T^{\mu\nu} = \delta_\xi Q - \int d^{d+1}x \sqrt{-g} \xi_\nu \nabla_\mu T^{\mu\nu},$$

- In case theory is scale/conformal invariant, conformal coupling implies $g_{\mu\nu} T^{\mu\nu} \approx 0$.
Conformal coupling: $S[\phi; g]$ is also invariant under $\delta_B g_{\mu\nu} = -2B g_{\mu\nu}$

$$\delta_B S[\phi; g] \approx - \int d^{d+1}x \sqrt{-g} B g_{\mu\nu} T^{\mu\nu}.$$

In the following, consider **on-shell variations** (“integrate out ϕ^α ”): $\delta S_{\text{on-shell}}[g] \approx \delta S[\phi; g]$.

Asymptotically locally AdS₄: FG gauge and Einstein's equations

Consider pure 4d Einstein gravity with $\Lambda = -\frac{3}{\ell^2}$.

AIAdS₄ space-times in the Fefferman-Graham gauge

$$ds^2 = \frac{\ell^2}{\rho^2} \left(d\rho^2 + g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \right),$$

with $g_{\mu\nu}(\rho, x) = g_{\mu\nu}^{(0)}(x) + \rho g_{\mu\nu}^{(1)}(x) + \rho^2 g_{\mu\nu}^{(2)}(x) + \rho^3 g_{\mu\nu}^{(3)}(x) + \mathcal{O}(\rho^4)$.

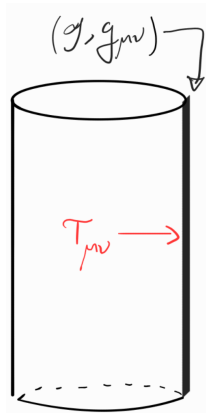
- $g_{\mu\nu}^{(0)}(x)$ is the boundary metric.
- $g_{\mu\nu}^{(1)} = 0$.
- $g_{\mu\nu}^{(2)}$ is fixed in terms of $g_{\mu\nu}^{(0)}$ (Schouten tensor).
- $g_{\mu\nu}^{(3)}$ is conserved and traceless.
- all higher orders are fixed in terms of $g_{\mu\nu}^{(0)}$ and $g_{\mu\nu}^{(3)}$.

$T_{\mu\nu} := \frac{3\ell^{-1}}{16\pi G} g_{\mu\nu}^{(3)}$ is the stress-tensor of the dual CFT.

Asymptotically locally AdS₄: well-posed variational principle

Conformal boundary \mathcal{I} from compactification [Penrose '65]

$$ds^2 = \Omega^2 \widetilde{ds}^2, \quad \Omega|_{\mathcal{I}} = 0, \quad d\Omega|_{\mathcal{I}} \neq 0.$$



Holographic renormalisation in AdS₄ [de Haro et al. '00]

Well-posed EH variational principle \rightarrow reduces to boundary term.

Data at \mathcal{I} are the metric $g_{\mu\nu}^{(0)}$ and holographic stress-tensor $T_{\mu\nu}$

$$\delta S_{\text{ren}} = \frac{1}{2} \int_{\mathcal{I}} d^3x \sqrt{-g} \delta g_{\mu\nu}^{(0)} T^{\mu\nu}.$$

Dirichlet b.c. make the action stationary (see also [Compère, Marolf '07]).

Covariant conservation of stress-tensor ($\nabla_{\mu}^{(0)} T^{\mu\nu} = 0 = g_{\mu\nu}^{(0)} T^{\mu\nu}$)
[Balasubramanian, Kraus '99] equivalent to the bulk Einstein's equations.

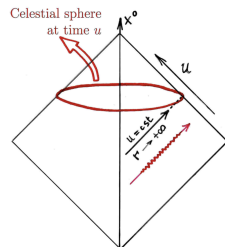
Asymptotically flat space-times: Bondi gauge

Asymptotically Minkowski₄ space-time [Bondi, van der Burg, Metzner '62], [Sachs '62]

$$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + g_{AB} (dx^A - U^A du) (dx^B - U^B du),$$

with $g_{AB} = r^2 \gamma_{AB} + r C_{AB} + \mathcal{O}(1)$ and $\partial_r \det(r^{-2} g_{AB}) = 0$.

[Boundary conditions: $\frac{V}{r} = \mathcal{O}(1)$, $U^A = \mathcal{O}(\frac{1}{r^2})$, $\beta = \mathcal{O}(\frac{1}{r^2})$.]



Metric at \mathcal{I}

Boundary metric $0 \times du^2 + \gamma_{AB} dx^A dx^B$ is **Carrollian**.

Conformal compactification \Rightarrow conformal class.

- Boundary metric γ_{AB} , *transverse* connection is LC (cov. derivative D_A , curvature R);
- Asymptotic **shear** C_{AB} (traceless) [News $N_{AB} := \partial_u C_{AB}$];
- *Bondi mass aspect*: $\frac{V}{r} = -\frac{R}{2} + \frac{2}{r} M + \mathcal{O}(r^{-2})$;
- *Angular momentum aspect*: $U^A = -\frac{1}{2r^2} D_B C^{AB} - \frac{2}{3r^3} (N^A - \frac{1}{2} C^{AB} D^C C_{BC}) + \mathcal{O}(r^{-4})$.

Attempts at defining a flat holographic stress-tensor

- Holographic stress-tensor for asymptotically flat space-times.

[Chandrasekaran et al. '21] [Donnay, Fiorucci, Herfray, Ruzziconi '22] [Freidel, Riello '24] [Ruzziconi, Saha '24] [Bhambure, Krishna '24] [Ciambelli '25] [Hartong et al. '25], ...

- Here, we define a variational principle *intrinsic to the boundary* (see also [Hartong et al. '25]).

Employ methods & tools of metric-affine gravity → **hypermomentum**

Proceeding this way, we obtain

- 1 the BMS evolution equations ;
- 2 the presymplectic potential of GR & asymptotic charges ;
- 3 a geometric setup to treat objects in a covariant fashion.

Carrollian conformal geometry: metric, frame and local transformations

Carrollian structure (\mathbf{g}, \mathbf{n}) : degenerate metric \mathbf{g} and preferred vector in its kernel \mathbf{n} :

$$\mathbf{g}(\mathbf{n}, \cdot) = 0.$$

Conformal class of structures: $(\mathbf{g}, \mathbf{n}) \sim (\mathcal{B}^{-2}\mathbf{g}, \mathcal{B}\mathbf{n})$ (\mathbf{g} has weight -2 and \mathbf{n} has weight 1).

In Cartan's frame:

$$\{\theta^a\}_{a=0,1,\dots,d} = \{\tau \equiv \theta^0, \theta^i\}_{i=1,\dots,d},$$

dual to

$$\{\mathbf{e}_a\}_{a=0,1,\dots,d} = \{\mathbf{n} \equiv \mathbf{e}_0, \mathbf{e}_i\}_{i=1,\dots,d}.$$

Carrollian structure: $(\mathbf{g} = \delta_{ij}\theta^i \otimes \theta^j, \mathbf{n})$.

Tangent space metric is now δ_{ij} .

Local transformations of the frame

- Local rotation: $\delta_r \theta^i = r^i_j \theta^j$;
with $r^{ij} = r^{[ij]} \in \mathfrak{so}(d)$.
- Local boosts: $\delta_\lambda \tau = -\lambda_i \theta^i$;
freedom in choosing τ dual to \mathbf{n} .
- Local Weyl: $\delta_B \theta^a = -B \theta^a$;
(equivalently $\delta_B \mathbf{e}_a = +B \mathbf{e}_a$).
- Diffeomorphisms: $\delta_\xi \theta^a = \mathcal{L}_\xi \theta^a$.

Carrollian conformal geometry: connections, metricity, torsion

Carrollian connection

Introduce $\omega^a{}_b$ the spin-connection one-form

- Local rotations: $\delta_r \omega^i{}_j = \mathbf{D} r_j^i$; $\mathbf{D} := d + \omega^i{}_j$
- Local boosts: $\delta_\lambda \omega^0{}_i = \mathbf{D} \lambda_i$;
- Local Weyl: $\delta_B \omega^a{}_b = \delta^a{}_b dB$;
- Weyl-non-metricities: $\omega^{\langle ij \rangle}$, $\omega^i{}_0$, $\omega^0{}_0 - \frac{1}{d} \omega^i{}_i$;
- Torsion two-forms: $\mathbf{K}^a := d\theta^a + \omega^a{}_b \wedge \theta^b$.

We can always require the connection to be Weyl-metric-compatible:

$$\omega^{\langle ij \rangle} \stackrel{!}{=} 0, \quad \omega^i{}_0 \stackrel{!}{=} 0, \quad \omega^0{}_0 \stackrel{!}{=} \frac{1}{d} \omega^i{}_i.$$

Imposing $\mathbf{K}^a \stackrel{!}{=} 0$, crucial differences wrt Lorentzian setting [Duval et al. '14]:

- 1 Geometric shear of the metric has to vanish $(\mathcal{L}_n \theta_{\langle i} \rangle)[e_j] = 0$;
- 2 Temporal Weyl connection $\omega^0{}_0[n]$ does not necessarily vanish;
- 3 LC theorem does not apply! Connection always contain an arbitrary part $\omega^0{}_{(i}[e_j])$.

Again, some components ($\omega^0{}_i[e^i]$ and $\omega^0{}_0[e_i]$) are pure-gauge (under special conformal transformations).

Carrollian conformal variational principle: evolution equations

Naturally led to on-shell variation *with hypermomenta* [Fiorucci et al. '25]

$$\delta S[\theta, \omega] \approx \int_{\mathcal{M}} \mu \left(\delta \theta^a [\mathbf{T}_a] + \delta \omega^a{}_b [\Omega_a{}^b] \right).$$

- Local rotation, Carroll boost & Weyl \implies constraints:

$$T^i{}_0 = -D \cdot \Omega_0{}^i, \quad T_{[ij]} = -D \cdot \Omega_{[ij]}, \quad T^a{}_a = -D \cdot \Omega_a{}^a.$$

- Diffeomorphisms \implies evolution equations:

$$D \cdot T_0 = R^a{}_b[n, \Omega_a{}^b], \quad D \cdot T_i = R^a{}_b[e_i, \Omega_a{}^b].$$

Can *not* be recast as conservation laws in general [Duval, Künzle '76].

Because connection contains independent part, conservation if $\theta^{(i}[\Omega_0^{j)}] = 0$.