Progress in the study of scalar CFTs

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Introduction

Based on the following work:

Perturbative

- Henriksson, Herzog, Kousvos, Roosmale Nepveu 2507.12518 [hep-ph]
- Henriksson, Kousvos, Roosmale Nepveu 2511.16740 [hep-th]

Non-perturbative

- Kousvos, Stergiou 2507.05338 [hep-th]

Outline

- Introduction: Wilson-Fisher CFTs, the Conformal Bootstrap and Motivation
- Perturbative results
- Non-perturbative results
- Conclusion and Outlook

Wilson-Fisher Theories

We will discuss **CFTs that can be obtained as controlled fixed points of (multi) scalar Lagrangians**. The simplest such Lagrangian is

$$\mathcal{L} = \mathcal{L}_{kin} + \frac{\lambda}{4!}\phi^4$$

One can compute the beta function of this theory in $d = 4 - \varepsilon$

$$\beta(\lambda) = -\varepsilon\lambda + 3\lambda^2 + O(\lambda^3)$$

as Wilson and Fisher noticed, this has **an interacting fixed point** when $\varepsilon \neq 0$

$$\beta(\lambda^\star) = 0 \to \lambda^\star = \frac{\varepsilon}{3} + O(\varepsilon^2)$$

fixed points are **scale invariant** and, more often than not, **conformal invariant**

Wilson-Fisher Theories

CFT are described by two sets of data: **scaling dimensions** and **OPE coefficients**. At the fixed point, engineering dimensions are generically corrected by anomalous dimensions

$$\Delta_\phi = \frac{d-2}{2} + \frac{\varepsilon^2}{108} + O(\varepsilon^3)$$

$$\Delta_{\phi^2} = d - 2 + \frac{\varepsilon}{3} + O(\varepsilon^2)$$

$$\Delta_{\phi^4} = 2(d-2) + 2\varepsilon + O(\varepsilon^2)$$

with notable exceptions such as $T^{\mu\nu}$.

Three dimensional physics can then be computed by **resumming the series** in ε and plugging in $\varepsilon = 1$. One simple choice is Padé$_{m,n}$ approximants: $(a_0 + a_1\varepsilon + \ldots + a_m\varepsilon^m)/(1 + b_1\varepsilon + \ldots + b_n\varepsilon^n)$

Wilson-Fisher Theories

We can discuss more generally CFTs that arise as fixed points of the following generic Lagrangian

$$\mathcal{L} = \mathcal{L}_{kin} + \frac{\lambda_{ijkl}}{4!}\phi_i\phi_j\phi_k\phi_l$$

in $d = 4 - \varepsilon$. **Any scalar field theory with a controlled fixed point in $d = 4 - \varepsilon$ can be brought into this form.**

For example, one can recover the $O(n)$ vector models if we set

$$\lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl}$$

and plugging $n = 1$ into this, one recovers the previous example (Ising CFT).

Why

These CFTs are interesting because they:

- provide a **laboratory for studying non-integrable theories** where we can compute (perturbatively and non-perturbatively)
- they describe systems undergoing phase transitions in the **real world**

The above quoted scaling dimensions Δ_ϕ and Δ_{ϕ^2} can be used to determine critical exponents describing real-world phase transitions in $3D$.

Critical Exponents

Consider a magnet cooled down from a high temperature phase, to some critical temperature T_c where the spins spontaneously align. We have

- $\beta = \frac{\Delta_\phi}{d-\Delta_{\phi^2}}$, magnetisation exponent $|M| \sim (T - T_c)^\beta$
- $\nu = \frac{1}{d-\Delta_{\phi^2}}$, correlation length exponent $\xi \sim (T - T_c)^{-\nu}$

The bootstrap

In recent years, an **axiomatic** non-perturbative method called the **conformal bootstrap** [Rattazzi + Rychkov + Tonni + Vichi, 2008] **emphasizing the whole spectrum** was revived, which can be used to study the theory directly in $d = 3$ (or in fact any value of d).

The idea is to impose **self consistency** and **spectrum assumptions**, and see what spectra can satisfy these constraints. This provides non-perturbative bounds that constrain observables (scaling dimensions, OPE coefficients). These bounds sometimes **effectively determine** these observables, with **rigorous error bars**.

The Conformal Bootstrap

Due to the extra symmetry CFTs enjoy, two and three point functions of operators are completely fixed

$$\langle O(x_1)O(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}$$

The Conformal Bootstrap

Four point functions are not completely fixed, but can be studied using the Operator Product Expansion (OPE)

$$O(x)O(y) \sim \sum_{O'} c_{OO}^{O'} P(x-y, \partial_y) O'(y)$$

which leads to

$$\begin{aligned} &\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle \\ = \sum_{O'} (c_{OO}^{O'})^2 &\frac{1}{|x_{12}|^{2\Delta_O}|x_{34}|^{2\Delta_O}} g_{\Delta_{O'},l}(u,v) \end{aligned}$$

where u, v are conformally invariant combinations of the coordinates x_i.

Consistency Constraints

OPE associativity is the statement that however we choose to take the OPE, the 4-point function is the same.

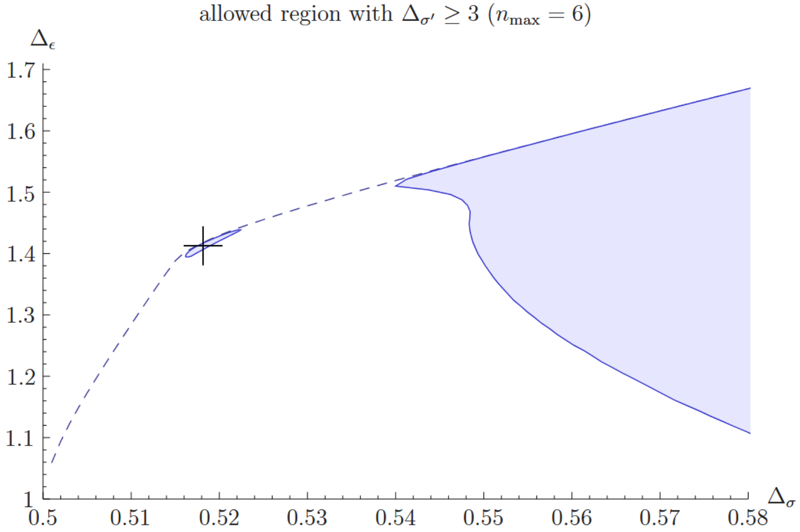
$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \sum_{O'} (c_{OO}^{O'})^2 \frac{1}{|x_{12}|^{2\Delta_O}|x_{34}|^{2\Delta_O}} g_{\Delta_{O'},l}(u,v)$$

This results in:

$$\sum_{O'} (c_{OO}^{O'})^2 \frac{1}{|x_{14}|^{2\Delta_O}|x_{23}|^{2\Delta_O}} g_{\Delta_{O'},l}(v,u) = \sum_{O'} (c_{OO}^{O'})^2 \frac{1}{|x_{12}|^{2\Delta_O}|x_{34}|^{2\Delta_O}} g_{\Delta_{O'},l}(u,v)$$

The Ising Island

From [Kos+Poland+Simmons-Duffin 2014], $\Delta_\epsilon = \Delta_{\phi^2}$ and $\Delta_\sigma = \Delta_\phi$



The Ising Island

Over the years there have been many of efforts to make this island shrink. The current record is [Chang et al - 2025]

- $\Delta_\phi = 0.518148806(24)$
- $\Delta_{\phi^2} = 1.41262528(29)$

Interlude

We saw two methods for computing in a CFT

- **Perturbation Theory**: Can compute systematically, non-rigorous error bars
- **Conformal Bootstrap**: Can compute rigorously, precisely, but not systematic, some observables harder than others.

Solution: **compute in both!**

Results: Perturbative

Before our work the state of the art was the six-loop results of [Bednyakov+Pikelner 2021] for

$$\mathcal{L} = \mathcal{L}_{kin} + t_a\phi^a + \frac{m^2_{ab}}{2}\phi^a\phi^b + \frac{h_{abc}}{3!}\phi^a\phi^b\phi^c + \frac{\lambda_{abcd}}{4!}\phi^a\phi^b\phi^c\phi^d$$

which allows one to compute the scaling dimensions of **any** ϕ, ϕ^2, ϕ^3 and ϕ^4 type operator, in **any** "$\lambda\phi^4$" type theory to sixth order in ε.

In our work we extended this to

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{kin} + \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d + \mathcal{L}^{(\Delta<4)} + \mathcal{L}^{(\ell=0)} + u^\mu \mathcal{L}_\mu^{(\ell=1)} \\ &+ v^{(\mu\nu)}\mathcal{L}_{(\mu\nu)}^{(\ell=2)} + w^{[\mu\nu]}\mathcal{L}_{[\mu\nu]}^{(\ell=\{1,1\})}\end{aligned}$$

where

$$\mathcal{L}^{(\Delta<4)} = t_a\phi^a + \frac{m_{ab}^2}{2}\,\phi^a\phi^b + \frac{h_{abc}}{3!}\,\phi^a\phi^b\phi^c\,,$$

and

Results: Perturbative

[JH+FH+**SRK**+JRN, 2025] and [JH+**SRK**+JRN, 2025]

$$\mathcal{L}^{(\ell=0)} = \frac{c^{(5,0)}_{abcde}}{5!}\,\phi^a\phi^b\phi^c\phi^d\phi^e + \frac{c^{(6,0)}_{abcdef}}{6!}\,\phi^a\phi^b\phi^c\phi^d\phi^e\phi^f - \frac{c^{(6,0)}_{abcd}}{4}\,\phi^a\phi^b\partial_\mu\phi^c\partial^\mu\phi^d\,,$$

$$\mathcal{L}^{(\ell=1)}_\mu = c^{(3,1)}_{ab}\,\phi^a\partial_\mu\phi^b + \frac{c^{(4,1)}_{abc}}{2}\phi^a\phi^b\partial_\mu\phi^c + \frac{c^{(5,1)}_{abcd}}{3!}\phi^a\phi^b\phi^c\partial_\mu\phi^d + c^{(5,1)}_{ab}\left(3\,\partial_\mu\phi^a\partial^2\phi^b - \phi^a\partial_\mu\partial^2\phi^b\right) + \frac{c^{(6,1)}_{abcde}}{4!}\phi^a\phi^b\phi^c\phi^d\partial_\mu\phi^e\,,$$

$$\mathcal{L}^{(\ell=2)}_{(\mu\nu)} = c^{(4,2)}_{ab}\left(\phi^a\partial_\mu\partial_\nu\phi^b - 2\,\partial_\mu\phi^a\partial_\nu\phi^b\right) + 2\,c^{(5,2)}_{abc}\left(\phi^a\phi^b\partial_\mu\partial_\nu\phi^c - 2\,\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\right) + c^{(6,2)}_{abcd}\left(\phi^a\phi^b\phi^c\partial_\mu\partial_\nu\phi^d - 2\,\phi^a\phi^b\partial_\mu\phi^c\partial_\nu\phi^d\right) + c^{(6,2)}_{ab}\partial_\mu\partial_\nu\phi^a\partial^2\phi^b\,,$$

Results: Perturbative

[JH+FH+**SRK**+JRN, 2025] and [JH+**SRK**+JRN, 2025]

and, lastly

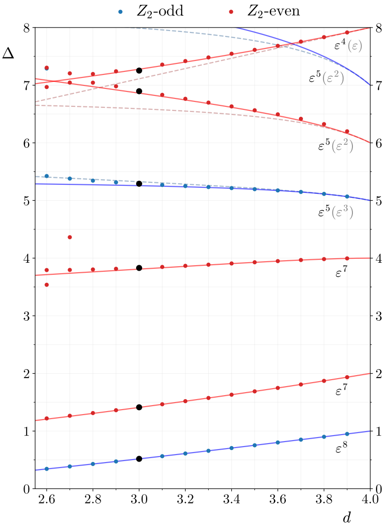
$$\mathcal{L}_{[\mu\nu]}^{(\ell=\{1,1\})} = -c_{abc}^{(5,\{1,1\})}\,\phi^a\partial_\mu\phi^b\partial_\nu\phi^c - c_{abcd}^{(6,\{1,1\})}\,\phi^a\phi^b\partial_\mu\phi^c\partial_\nu\phi^d$$

The anomalous dimensions of all these operators have been computed to 5-loop order, with the exception of $c_{abcde}^{(6,1)}$ and $c_{abcdef}^{(6,0)}$ which are given at 4-loops. This presents the current state of the art for generic scalar field theory.

Our general results are released in **Mathematica** and **FORM** format. However, we also extracted and compared data for specific models!

Let us, for example, **compare to our earlier non-perturbative work** [Henriksson + Kousvos + Reehorst, 2022] the new results for Ising

Comparison to the Ising CFT



Black Dots = [Simmons-Duffin, 2016], ε^7= [Schnetz, 2016, 2022], ε^8= [Schnetz, 2022]

Specific Theories

To summarise: In addition to the results for general theories, we performed an exhaustive spectrum extraction (all operators upto dimension $\Delta = 6$ and Lorentz rank 2 at **five loops**) for specific theories, in particular:

- Ising CFT
- $O(n)$ CFTs
- Hypercubic $S_n \ltimes (Z_2)^n$ CFTs (≥ 100 operators!)

$$\mathcal{L}_{hypercubic} = \mathcal{L}_{kin} + \lambda_1 \delta_{ij}\delta_{kl}\phi_i\phi_j\phi_k\phi_l + \lambda_2 \delta_{ijkl}\phi_i\phi_j\phi_k\phi_l$$

for $n = 3$

$$\mathcal{L}_{cubic} = \mathcal{L}_{kin} + \lambda_1({\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2)^2 + \lambda_2({\phi_1}^4 + {\phi_2}^4 + {\phi_3}^4)$$

While we already had a non-perturbative handle on the Ising CFT, **our perturbative results allowed us to make significant non-perturbative progress in the Cubic CFT.**

The Cubic Theory

[Kousvos + Stergiou, 2025]

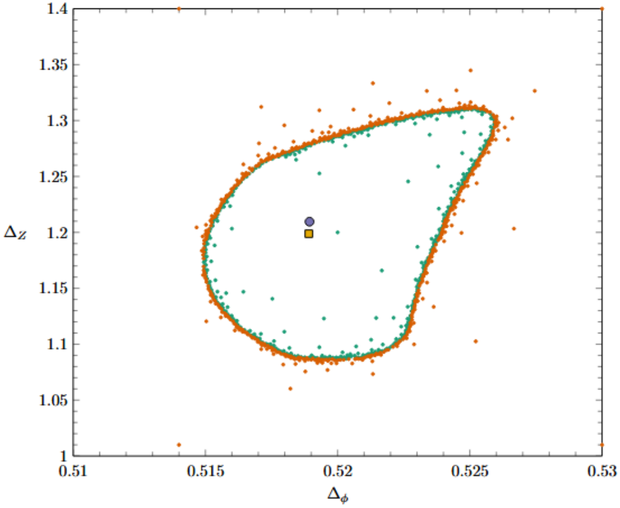
The Cubic theory describes **phase transitions in three-dimensional ferromagnets**. However, despite its physical importance no bootstrap island had been produced for it. With the results discussed above, we finally obtained a bootstrap island.

In particular, we constrained the following three operators in an island

- ϕ_i
- $Z_{ij} \sim \phi_i \phi_j$, $i \neq j$
- $X_{ij} \sim \phi_i \phi_j - \frac{\delta_{ij}}{n} \phi_k \phi_k$, $i = j$

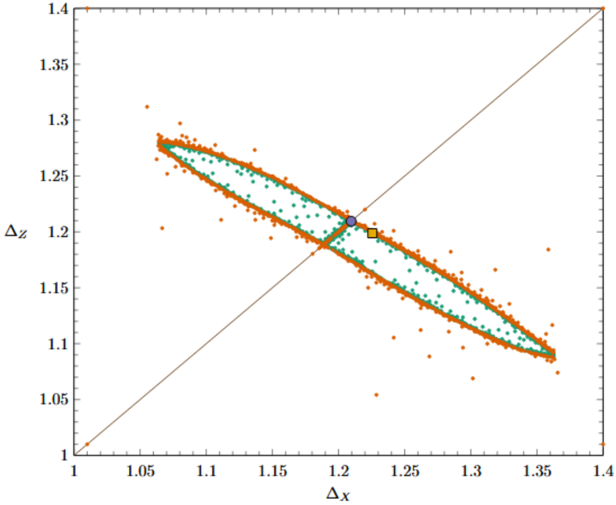
These are the three lightest operators in the theory.

The Cubic Theory, [Kousvos + Stergiou, 2025]



$$\Delta_S \geq 1.5,\ \Delta_{\phi'} \geq 1.5,\ \Delta_{Z_3} \geq 1.5,\ \Delta_{XV} \geq 1.5,\ \Delta_{Z'} \geq 2.8,\ \Delta_{T'_{\mu\nu}} \geq 4.0$$

The Cubic Theory, [Kousvos + Stergiou, 2025]



$\Delta_S \geq 1.5$, $\Delta_{X'} \geq 2.5$, $\Delta_{Z'} \geq 2.5$, $\Delta_{\phi'} \geq 1.5$, $\Delta_B \geq 4.0$, $\Delta_{T'_{\mu\nu}} \geq 4.0$, $\Delta_\phi = 0.51893$

Conclusion and outlook

- We (significantly) extended the state of the art for generic scalar field theories
- We extracted state of the art spectra for physically important theories (at 5 loops)
- We used these results to obtain an island for the Cubic theory, a long standing open problem in the conformal bootstrap

For the future:

- Make the Cubic island shrink to get precision data (exponents)
- Use the $d = 3 - \varepsilon$ expansion ($\lambda\phi^6$ theory) to understand/explore non-integrable CFTs in $d = 2$
- Provide the first island for non-integrable theories in $d = 2$ (first steps carried out in [Kousvos + Piazza + Vichi, 2024], $S_3 \ltimes (S_3)^3$ theories)
- Extract more data using microscopic methods (Monte Carlo, Fuzzy Sphere Regularisation, Quantum Spin Chains)

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Consistency constraints

Probing the correlator $\langle OOOO \rangle$ for self consistency, we obtain the equation:

$$0 = \sum_{O'} (c_{OO}^{O'})^2 F_{\Delta_{O'}, l_{O'}}^{\Delta_O}$$

this provides a constraint that all CFT data in a consistent CFT must satisfy.

The conformal bootstrap systematically probes this equation to see which values of scaling dimensions and OPE coefficients violate it. These are then excluded from parameter space.

$$0 = \sum_{O'} (c_{OO}^{O'})^2 a \left(F_{\Delta_{O'}, l_{O'}}^{\Delta_O} \right)$$

If a functional a exists such that each term above is positive, the inserted set of Δ_O, $\Delta_{O'}$ and $c_{OO}^{O'}$ lead to a contradiction ($0 =$positive) and are thus excluded from parameter space.