Critical Dynamics of Superfluids

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Plan of the talk

1. Introduction
2. Dynamics of critical superfluids
3. Holographic superfluids
4. Conclusion

Introduction

Hydrodynamics

Hydrodynamics describes finite temperature, many-body systems, out of thermodynamic equilibrium.

It is an effective theory, valid at long wavelengths

Main idea: At sufficiently **large time and length scales**, all microscopic degrees of freedom have equilibrated, leaving **conserved charges** to dominate the effective description.

- This provides a **universal framework**, irrespective of the details of the microscopic theory
- It is a **classical theory**, thermal fluctuations are ignored (but can be included with Keldysh-Schwinger formalism [Liu and Glorioso, 2018])

Hydro variables and Equations of Motion

Assuming Poincaré symmetry and a global $U(1)$ invariance, the corresponding Noether currents are $T^{\mu\nu}$, J^{μ}.

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0 \rightarrow$ eoms of hydro.

What are the effective variables?

- Thermodynamics: temperature T, chemical potential μ, fluid velocity u^μ.
- Hydrodynamics: Promote thermodynamic variables to slowly varying functions of spacetime. Concretely, e.g. $l_m\, \partial_\mu T \ll 1$.

Express conserved currents in terms of hydrodynamic variables, in a derivative expansion.

$$T^{\mu\nu} = \underbrace{T^{\mu\nu}_{eq}}_{\mathcal{O}(\partial^0)} + \underbrace{T^{\mu\nu}_1}_{\mathcal{O}(\partial)} + \underbrace{T^{\mu\nu}_2}_{\mathcal{O}(\partial^2)} + \cdots\,, \quad J^\mu = \underbrace{J^\mu_{eq}}_{\mathcal{O}(\partial^0)} + \underbrace{J^\mu_1}_{\mathcal{O}(\partial)} + \underbrace{J^\mu_2}_{\mathcal{O}(\partial^2)} + \cdots$$

Normal fluid hydrodynamics

For **normal fluids**, up to first order in derivatives (background $g_{\mu\nu}$, A_μ):

$$T^{\mu\nu}_{eq} = \epsilon\, u^\mu u^\nu + p\, P^{\mu\nu}, \quad J^\mu_{eq} = \varrho\, u^\mu$$

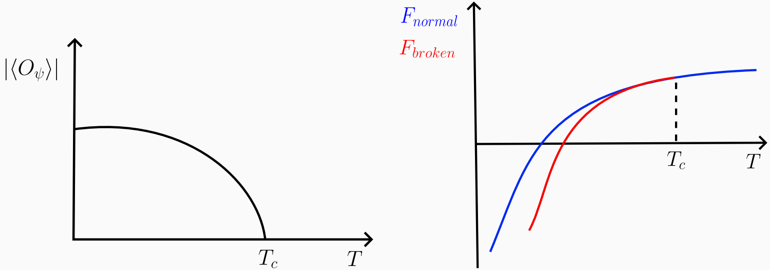
$$T^{\mu\nu}_1 = -\eta\, \sigma^{\mu\nu} - \zeta P^{\mu\nu} \nabla_\rho u^\rho\,, \quad J^\mu_1 = -T\sigma P^{\mu\nu} \left(\nabla_\nu \left(\frac{\mu}{T} \right) - \frac{F_{\nu\rho}\, u^\rho}{T} \right)$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\,, \sigma^{\mu\nu} = P^{\mu\rho} P^{\nu\sigma} \left(2\, \nabla_{(\rho} u_{\sigma)} - \tfrac{2}{d-1} g_{\rho\sigma} \nabla_\tau u^\tau \right)$$

- η, ζ, σ are examples of *transport coefficients*. Arbitrary func. of T, $\mu \to$ fixed by UV theory.
- We have two constraints:
    1. Local form of **second law of thermodynamics**: $\nabla_\mu s^\mu \geq 0$
    2. **Onsager reciprocity** relations, i.e. microscopic time reversal.

Superfluids

Superfluid: Phase of matter with spontaneous breaking of a global $U(1)$. Order parameter: $\langle O_\psi \rangle = |\langle O_\psi \rangle| e^{iq_e\,\theta} \neq 0$ for $T < T_c$.



Assuming $\partial_\mu \ll T - T_c$:

- We have one **extra hydro dof**: The superfluid velocity $\sim \partial_\mu \theta$ (massless Goldstone boson due to the SSB of $U(1)$)

Superfluid Hydrodynamics

- At zeroth order in derivatives ($m_\mu = \nabla_\mu \theta + A_\mu$, $n^\mu = P^{\mu\nu} m_\nu$):

$$T^{\mu\nu}_{eq} = \epsilon\, u^\mu u^\nu + p\, P^{\mu\nu} - 2\mu\, \chi_{JJ}\, n^{(\mu} u^{\nu)} + \chi_{JJ}\, n^\mu n^\nu$$

$$J^\mu_{eq} = \varrho\, u^\mu - \underbrace{\chi_{JJ}\, n^\mu}_{Supercurrent}$$

$$\mu = u^\mu m_\mu \quad \text{“Josephson relation”=time evolution of } \theta$$

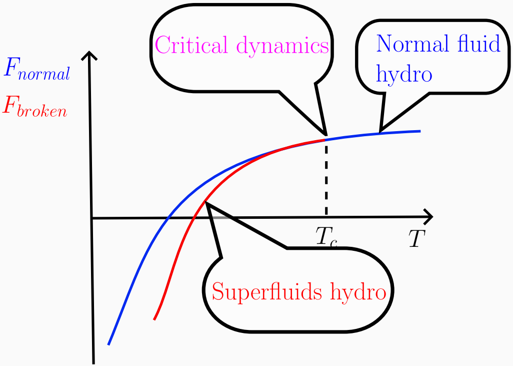
- At first order in derivatives [Herzog et al., 2011], [Donos and Kailidis, 2022]:

$$T^{\mu\nu}_1 = -\eta\, \sigma^{\mu\nu} - \zeta_1 P^{\mu\nu} \nabla_\rho u^\rho + \zeta_2 P^{\mu\nu} \nabla_\rho \left(\chi_{JJ} n^\rho\right)$$

$$J^\mu_1 = -T\sigma P^{\mu\nu} \left(\nabla_\nu \left(\frac{\mu}{T}\right) - \frac{F_{\nu\rho}\, u^\rho}{T} \right)$$

$$\mu = u^\mu m_\mu + \zeta_2 \nabla_\mu u^\mu - \zeta_3 \nabla_\mu \left(\chi_{JJ}\, n^\mu\right)$$

Hydrodynamics and its breakdown



Close to T_c the corresponding bulk viscosities diverge ($\zeta_i \sim \frac{1}{T-T_c} \to \infty$) and Hydrodynamics **breaks down**!

Apart from the phase of O_ψ, its **amplitude** $|\langle O_\psi \rangle|$ becomes a **nearly massless** dof$\to$ include it in the EFT

Dynamics of critical superfluids

Setup

Statement of the problem: Study the nearly critical dynamics, in the presence of small, slowly varying external sources $g_{\mu\nu}$, A_μ, s_ψ.

Generating functional $W[g_{\mu\nu}, A_\mu, s_\psi] = \int d\mu_f \, e^{-F}$

$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}$, $J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta A_\mu}$, $\psi = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta s_\psi^*}$

- Diffeo's and $U(1)$ gauge invariance lead to Ward id.:

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu + \frac{1}{2} \left(\psi^* \, D^\nu s_\psi + \psi \, D^\nu s_\psi^* \right) \, ,$$
$$\nabla_\mu J^\mu = \frac{q_e}{2i} \left(\psi^* \, s_\psi - s_\psi^* \, \psi \right) \, .$$

- We need a further EOM for ψ.

Effective theory-Equilibrium

In **equilibrium**: Timelike Killing vector K, such that Lie derivative $\mathcal{L}_K$ acts trivially on the external sources (e.g. $\mathcal{L}_K A_\mu = \nabla_\mu \lambda_K$)

- Thermod. variables:
$$T = \frac{T_0}{\sqrt{-K^2}}, \quad u^\mu = \frac{K^\mu}{\sqrt{-K^2}}, \quad \mu = \frac{K^\mu A_\mu - \lambda_K}{\sqrt{-K^2}}$$
- For the order parameter:
$\mathcal{L}_K |\psi| = 0, \quad \mathcal{L}_K \, \theta = -\lambda_K \Rightarrow u^\mu \hat{D}_\mu \psi = 0.$

($\hat{D}_\mu = \nabla_\mu + iq_e A_\mu + iq_e \mu \, u_\mu$)

Effective theory-Equilibrium

$F = \int d^d x \sqrt{-g}\, f_{tot}$ "Ginzburg-Landau potential"

$$f_{tot} = \underbrace{\frac{w_0(\mu,T)}{2}|D^{\perp}_{\mu}\psi|^2}_{\text{gradient term}} + \underbrace{f(\mu,T,|\psi|^2)}_{\text{Landau f.e.}} - \underbrace{\frac{1}{2}\left(s^*_{\psi}\,\psi + s_{\psi}\,\psi^*\right)}_{\text{source term}}$$

$$J^{\mu}_{eq} = \varrho\, u^{\mu} + q_e w_0\,\mathrm{Im}(\psi D^{\perp\mu}\psi^*)\,,$$

$$T^{\mu\nu}_{eq} = \epsilon\, u^{\mu}u^{\nu} + p\, P^{\mu\nu} + 2\, w_0\, q_e\, \mu\, u^{(\mu}\,\mathrm{Im}\left(\psi D^{\perp\nu)}\psi^*\right) + w_0 D^{\perp(\mu}\psi D^{\perp\nu)}\psi^*\,,$$

In the saddle point approx., $\mathcal{F}^*_{\psi} \equiv \frac{1}{\sqrt{-g}}\frac{\delta F}{\delta\psi} = 0$

Effective theory-Dissipation

In the presence of **dissipation**, we expect corrections

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + T^{\mu\nu}_{diss}, \quad J^{\mu} = J^{\mu}_{eq} + J^{\mu}_{diss}, \quad u^{\mu}\hat{D}_{\mu}\psi = E_{diss}\,.$$

To construct the perturbative expansion we have **two** small parameters available:

1. $\lambda \leftrightarrow$ number of spacetime derivatives
2. $\varepsilon \leftrightarrow$ distance from critical point (gap of ampl. mode$\sim \varepsilon^2$):
   $T(\varepsilon) = T_c(\mu) + \mathcal{O}(\varepsilon^2)\,,\, \mu(\varepsilon) = \mu + \mathcal{O}(\varepsilon^2)\,,\, \psi \sim \varepsilon$

Effective theory-Dissipation

The constitutive relations read

$$T^{\mu\nu}_{diss} = -\,\eta\,\sigma^{\mu\nu} - Z_1\,P^{\mu\nu}\nabla_\rho u^\rho - 2\,\mathrm{Re}\left[Z_3\,\mathcal{F}_\psi\,\psi^*\right]P^{\mu\nu} + \mathcal{O}(\varepsilon^6)$$

$$J^\mu_{diss} = -\,T\sigma P^{\mu\nu}\left(\nabla_\nu\left(\frac{\mu}{T}\right) - \frac{F_{\nu\rho}\,u^\rho}{T}\right) + \mathcal{O}(\varepsilon^6)\,,$$

$$E_{diss} = -\,2\,\overline{\Gamma}_0\,\mathcal{F}_\psi - Z_n\,\psi^2\,\mathcal{F}^*_\psi + Z_2\,\psi\,\nabla_\mu u^\mu - Z_\pi\,u^\mu\,\hat{D}_\mu\mathcal{F}_\psi + \mathcal{O}(\varepsilon^7)$$

Note: The last term in E_{diss} was missed by [Khalatnikov and Lebedev, 1978].

Relative scaling: Since $u^\mu\hat{D}_\mu\psi = \lambda\varepsilon \sim \mathcal{F}_\psi = \varepsilon^3 \rightarrow$ Choose $\lambda \sim \varepsilon^2$.

Phenom. constraints

Constraints on transport coefficients:

1. Onsager reciprocity fixes: $Z_3 = Z_2\,, \quad \mathrm{Im}(Z_n) = 0\,.$
2. Imposing postive entropy production:

$$\eta \geq 0\,, \sigma \geq 0,\; Z_1 \geq 0\,,$$
$$2\,\mathrm{Re}\,(\Gamma_0) + \mathrm{Re}(Z_n)|\psi|^2 \geq 0\,,$$
$$2\,\mathrm{Re}\,(\Gamma_0) - \mathrm{Re}(Z_n)|\psi|^2 \geq 0\,,$$
$$2|\psi|^2\,\mathrm{Im}(Z_2)^2 \leq \left(2\,\mathrm{Re}(\Gamma_0) - \mathrm{Re}(Z_n)|\psi|^2\right) Z_1\,.$$

Limits of the effective theory

- To write the const. rel. of the nearly critical theory we took $\lambda \sim \varepsilon^2$.
- Consistency check: Consider two extreme limits of the EFT:
  1. $\lambda \ll \varepsilon^2$: **amplitude** $|\psi|$ **decouples**.$\rightarrow$ "integrate it out" $\rightarrow$ we find **superfluid hydrodynamics** with transport coefficients fixed in terms of the the coefficients of the nearly critical theory, e.g.

$$\zeta_1 = \frac{(s\,\nu_{\rho s} + \varrho\,\nu_{\rho\varrho} + \rho_v\,\mathrm{Re}Z_2)^2}{\mathrm{Re}\Gamma_0 + \frac{\rho_v^2}{2}\,\mathrm{Re}Z_n} + Z_1$$

  2. $\lambda \gg \varepsilon^2$: **whole order parameter** ψ **decouples** at leading order. We find **normal fluid hydrodynamics** with transport coefficients fixed in terms of the the coefficients of the n.c. theory, e.g. $\zeta = Z_1$.

Interlude

Interlude

So far, we have presented a general EFT, with minimal assumptions about the microscopics.

- Can we **reproduce** this effective theory using some tractable **microscopic** theory? What about the **extra term**?
- What are the **transport coefficients** as functions of background quantities?

As our microscopic theory we choose holography.

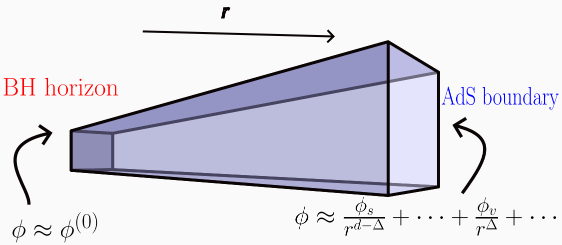
Holographic superfluids

Holographic duality

Holography connects a d-dimensional field theory (without gravity) with a $d + 1$-dimensional gravitational theory.

Precise mapping between bulk fields and field theory operators

- $\langle T_{\mu\nu} \rangle \leftrightarrow g_{\mu\nu}\,, \quad \langle J_\mu \rangle \leftrightarrow A_\mu, \quad \langle O_\phi \rangle \leftrightarrow \phi$



- $\phi_s \leftrightarrow$ source, $\phi_v \leftrightarrow \langle O_\phi \rangle$
- We can introduce temperature in our field theory by adding a black hole in the bulk spacetime.

Holographic superfluids

General remark: Duality between gauge symmetries in the bulk and global symmetries in the boundary (for us $U(1)$)

Superfluids can be described **holographically** ([Gubser, 2008], [Hartnoll, Herzog, and Horowitz, 2008]) with an Einstein-Maxwell-charged scalar action

$$S_{bulk} = \int d^4x \sqrt{-g} \left(R - V - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}|D_\mu \psi|^2 - \frac{1}{4}\tau \left(F_{\mu\nu}\right)^2 \right) .$$

$\psi \leftrightarrow \langle O_\psi \rangle =$ order parameter

- **Thermal state=aAdS Black hole solution**, labelled by (T, μ, ϕ_s).
- For $T \leq T_c$ we have two solutions:
    1. ψ=0 (normal phase)
    2. $\psi \neq 0$ (superfluid phase)

Hydro perturbation

Goal: Study spacetime dependent linear fluctuations $\delta_{Hydro}\mathcal{F} = e^{-i\omega(t+S(r))+ik_i x^i}\delta_H f(r)$ around a nearly critical background, in the scaling regime $\omega \sim |\vec{k}| \sim \varepsilon^2$. Expand the hydro perturbation in ε:

$$\delta_H f(r) = \delta f_{Thermo}(r)\,\varepsilon^2 + \delta f_{subl}(r)\,\varepsilon^4 + \mathcal{O}(\varepsilon^6)\,.$$

The nontrivial problem is how to find the subleading part $\delta f_{subl}(r)$...

Symplectic current

Our basic tool is the **symplectic current** of [Crnkovic and Witten, 1986], constructed out of ANY two solutions of the linearised eom of the Lagrangian density $\mathcal{L}(\phi^I, \partial_\mu \phi^I)$:

$$P^\mu_{\delta_1,\delta_2} = \delta_1 \phi^I \, \delta_2 \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I \right)} \right) - \delta_2 \phi^I \, \delta_1 \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^I \right)} \right)$$

Properties of P^μ:

1. $\partial_\mu P^\mu_{\delta_1,\delta_2} = 0$
2. As $r \to \infty : P^r \to \delta\phi^I_{1(s)} \, \delta\phi^I_{2(v)} - \delta\phi^I_{2(s)} \, \delta\phi^I_{1(v)}$

Integrating along the radial variable we connect the boundary to the black hole horizon

We can read the vevs of bulk fields without the need to solve the linearised eoms in the bulk [Donos, Kailidis, and Pantelidou, 2022]

To study hydro: Pick $\delta\phi^I_1$ hydro mode, $\delta\phi^I_2$ static

Analytic Results

Method: Apply symplectic current $\to$ calculate $\delta f_{subl}(r)$ $\to$ find $\delta\langle T_{\mu\nu}\rangle$ and $\delta\langle J^{\mu}\rangle$ and the eom of $\delta\psi$, in an ε expansion

Results:

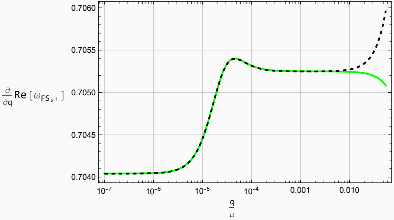
1. We find complete **agreement** with the predictions of the **effective theory**. Important: $Z_\pi \neq 0$ in holography.
2. All **transport coefficients** of the EFT are **fixed** in terms of background quantities. E.g. $Z_1 = \frac{s}{4\pi}\left(\varrho\,\partial_\varrho\phi^{(0)} + s\,\partial_s\phi^{(0)}\right)^2 + \mathcal{O}(\varepsilon^2)$
   - They are fixed perturbatively in an ε expansion and they all remain finite close to the transition(as $\varepsilon \to 0$).
   - They explicitly obey the phenom. constraints

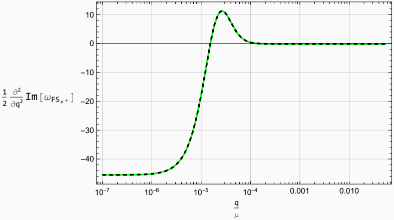
Numerical checks

Quasinormal modes$\leftrightarrow$ Poles of retarded Green's func's.

Solve the bulk eoms numerically (exactly) and compare with analytic predictions.

Ex.: For **first sound** mode, very close to T_c





Interpolating behaviour:

- For $q \ll \varepsilon^2 : \omega_{FS}(q) \approx c_{sf}\, q - iD_{sf}\, q^2$ (sound mode of superfluids)
- For $q \gg \varepsilon^2 : \omega_{FS}(q) \approx c_{nf}\, q - iD_{nf}\, q^2$(sound mode of normal fluid)

Conclusion

Summary

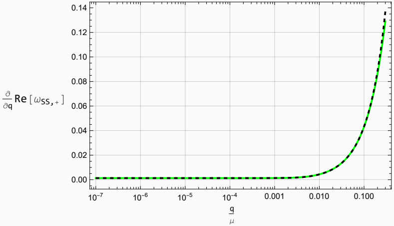
- We have constructed an EFT for relativistic superfluids close to T_c
- We have identified an extra coefficient, compared to the treatment by Khalatnikov and Lebedev.
- We have also studied holographic superfluids analytically and found the same EFT
  Performed numerical checks concerning the dominant quasinormal modes of the system.
- (Not discussed here) We arrived at the same theory also using the Keldysh-Schwinger framework: dynamical KMS, entropy current, fluctuations...

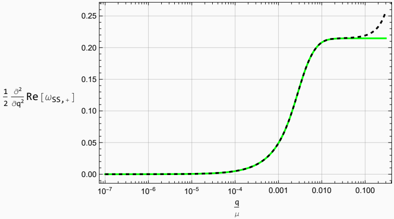
Outlook

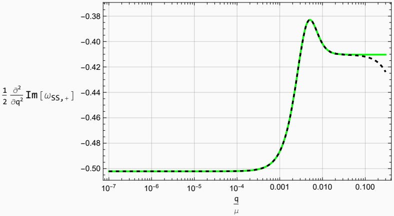
- Use the symplectic current in holography to extract the effective description for other systems. Most notable example: Describe *critical dynamics* in systems with **spontaneously broken translations**.
- Analyse the statistical fluctuations in the KS framework

Thank you for your attention!

Numerics: Second sound mode

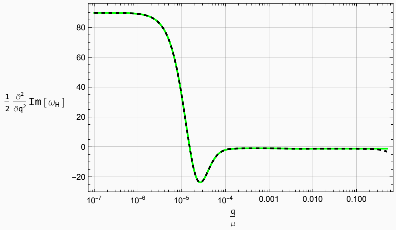






- For $q \ll \varepsilon^2 \; \omega_{SS}(q) \rightarrow$ Goldstone mode of superfluids
- For $q \gg \varepsilon^2 \; \omega_{SS}(q) \rightarrow$ mode related to fluctuations of ψ in the normal phase

Numerics: Higgs/Amplitude mode



Interpolating behaviour:

- For $q \ll \varepsilon^2\ \omega_H(q) \to$ Amplitude/Higgs mode
- For $q \gg \varepsilon^2\ \omega_H(q) \to$ charge diffusion of normal phase

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