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# Integrable Quantum Field Theories Perturbed by $T\bar{T}$

Olalla Castro-Alvaredo


**Athens XmasWorkshop in Theoretical Physics**

**17-19 December 2025**

This talk will be a quick summary of a program of research spanning the past 3 years. Find out more in the papers below!

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1. O. C.-A., Stefano Negro and Fabio Sailis, Completing the Bootstrap Program for $T\bar{T}$-Deformed Massive Integrable Quantum Field Theories, J. Phys. **A57** 265401 (2024); 2305.17068
2. O. C.-A., Stefano Negro and Fabio Sailis, Form Factors and Correlation Functions of $T\bar{T}$-Deformed Integrable Quantum Field Theories, JHEP **09** 2023 048; 2306.01640
3. O. C.-A., Stefano Negro and Fabio Sailis, Entanglement Entropy from Form Factors in $T\bar{T}$-Deformed Integrable Quantum Field Theories, JHEP **11** 2023 129; 2306.11064
4. O. C.-A., Stefano Negro and István M. Szécsényi, On the Representation of Minimal Form Factors in Integrable Quantum Field Theory, Nucl. Phys. **B1000** (2024) 116459; 2311.16955
5. Riccardo Travaglino, Michele Mazzoni and O. C.-A., Generalised Hydrodynamics of $T\bar{T}$-Deformed Integrable Quantum Field Theories, JHEP 2024 (8), 1-33; 2405.06564
6. O. C.-A., Stefano Negro and Fabio Sailis, Boundary Quantum Field Theories Perturbed by $T\bar{T}$: Towards a Form Factor Program, Nucl. Phys. **B1017** (2025) 116924. 2501.11647
7. Fabio Sailis, O. C.-A. and Stefano Negro, Complete Minimal Form Factors for Irrelevant Deformations of Integrable Quantum Field Theory, Nucl. Phys. **B1019** (2025) 117131; 2506.20517

Stefano Negro (York)

Michele Mazzoni (Bologna)

Fabio Sailis (City)

István M. Szécsényi (Stockholm)

Riccardo Travaglino (SISSA)

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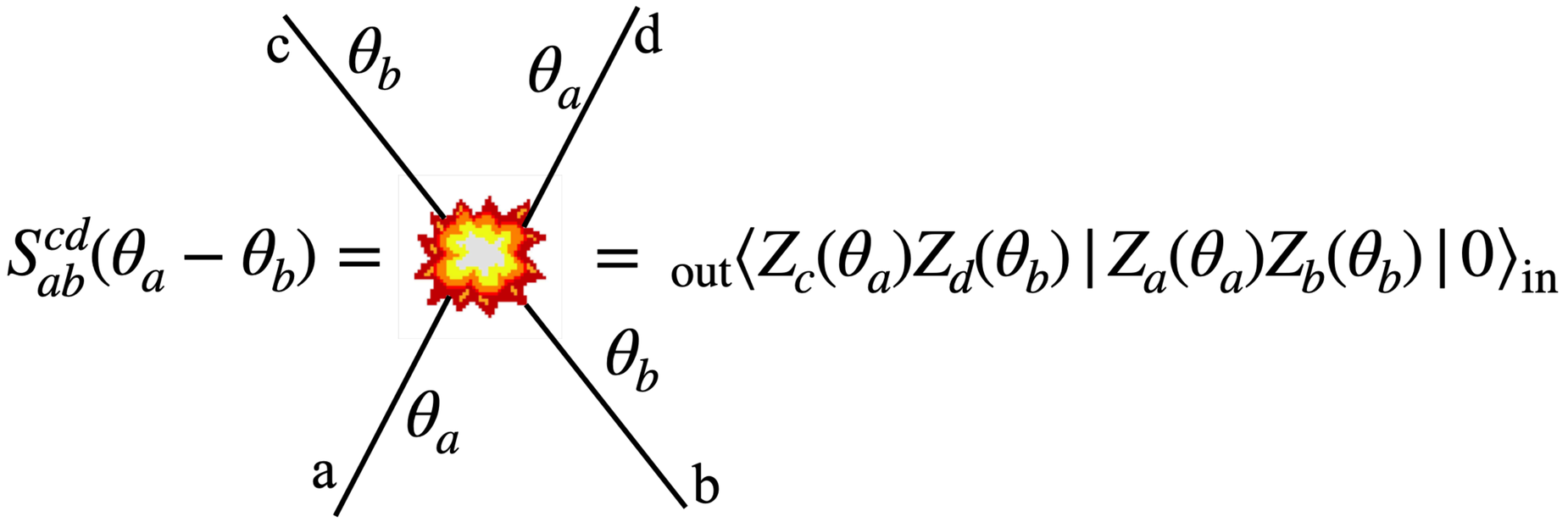

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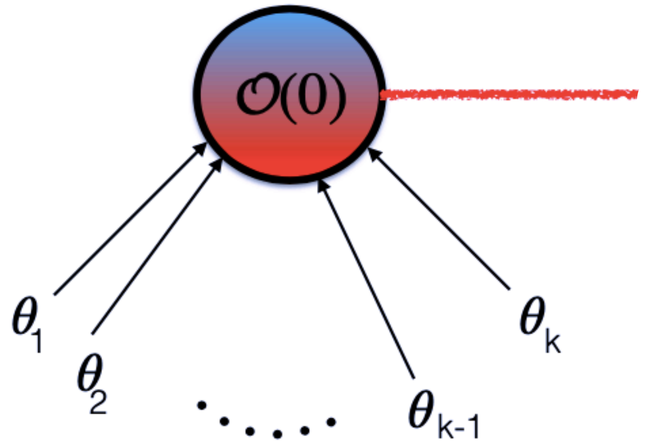


$$= \langle 0 | \mathcal{O}(0) | Z_{a_1}(\theta_1), \ldots, Z_{a_k}(\theta_k) \rangle$$

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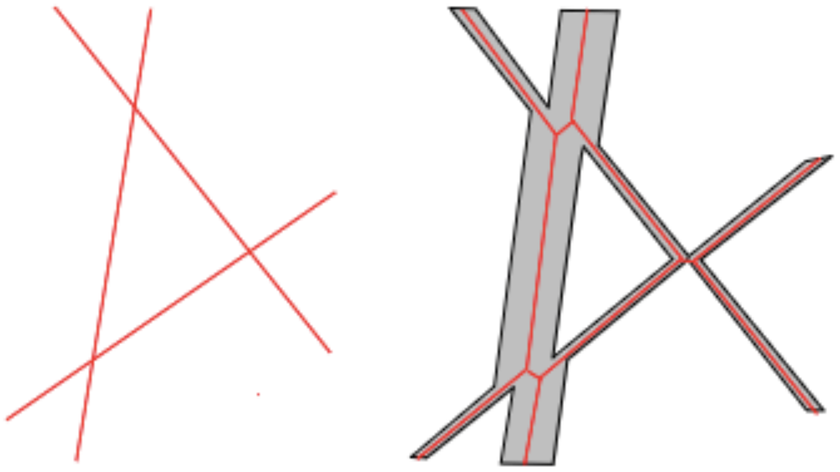

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$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n; \boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta_i - \theta_{i+1}) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n; \boldsymbol{\alpha})$$

$$F_n^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2 \ldots, \theta_n; \boldsymbol{\alpha}) = F_n^{\mathcal{O}}(\theta_2, \ldots, \theta_n, \theta_1; \boldsymbol{\alpha})$$

$$\lim_{\bar{\theta} \to \theta} (\bar{\theta} - \theta) F_{n+2}^{\mathcal{O}}(\bar{\theta} + i\pi, \theta, \theta_1, \ldots, \theta_n; \boldsymbol{\alpha}) = i \left( 1 - \prod_{j=1}^{n} S_{\boldsymbol{\alpha}}(\theta - \theta_j) \right) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha})$$

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- In our earlier works, we arbitrarily chose $\boldsymbol{\beta} = \mathbf{0}$ and studied higher particle form factors and correlation functions in this setting.
- Today, I present a systematic construction which will fix the $\boldsymbol{\beta}$ in a more natural way.

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- However, we also know that the S-matrix of the sinh-Gordon model is also a CDD factor. It is a solution to the unitary and crossing relations with no poles on the physical strip (for $B \in [0,2]$).

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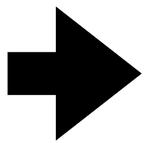

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$$-i\frac{d}{d\theta}\log S(\theta) = \frac{4\cos\frac{\pi b}{2}\cosh\theta}{\cos(\pi b) + \cosh(2\theta)}$$

B=b-1

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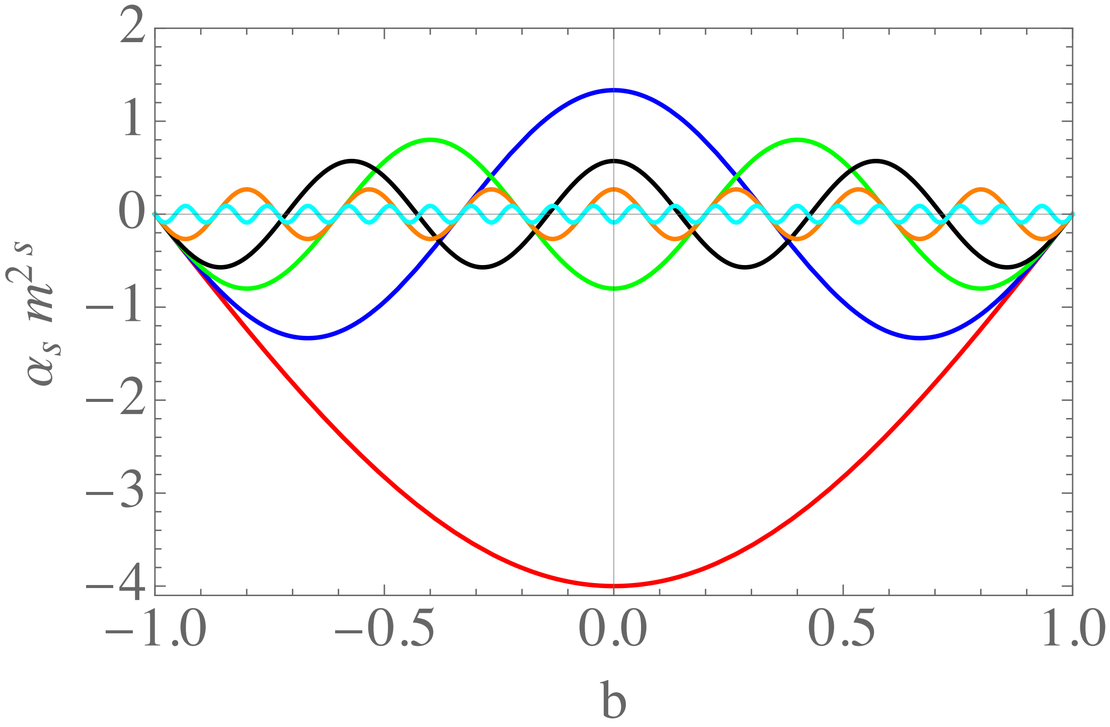

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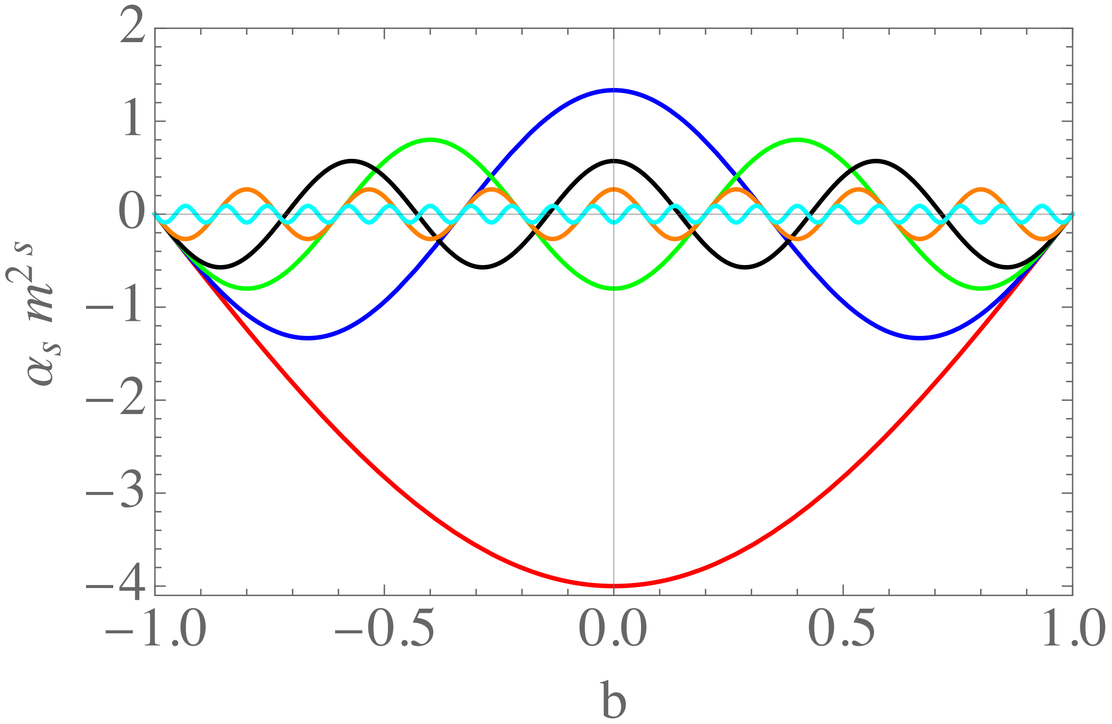


- The sinh-Gordon model can be seen as the Ising model perturbed by an infinite number of irrelevant perturbations for carefully chosen couplings [LeClair'21; Ahn & LeClair'22]

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- This is equivalent to replacing in the integral $e^{-mt}\sinh t \mapsto e^{-t(m+1)}$ and again can be generalised to any spin s.

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  - The formula interpolates smoothly between the regimes of finite and infinite number of perturbations.

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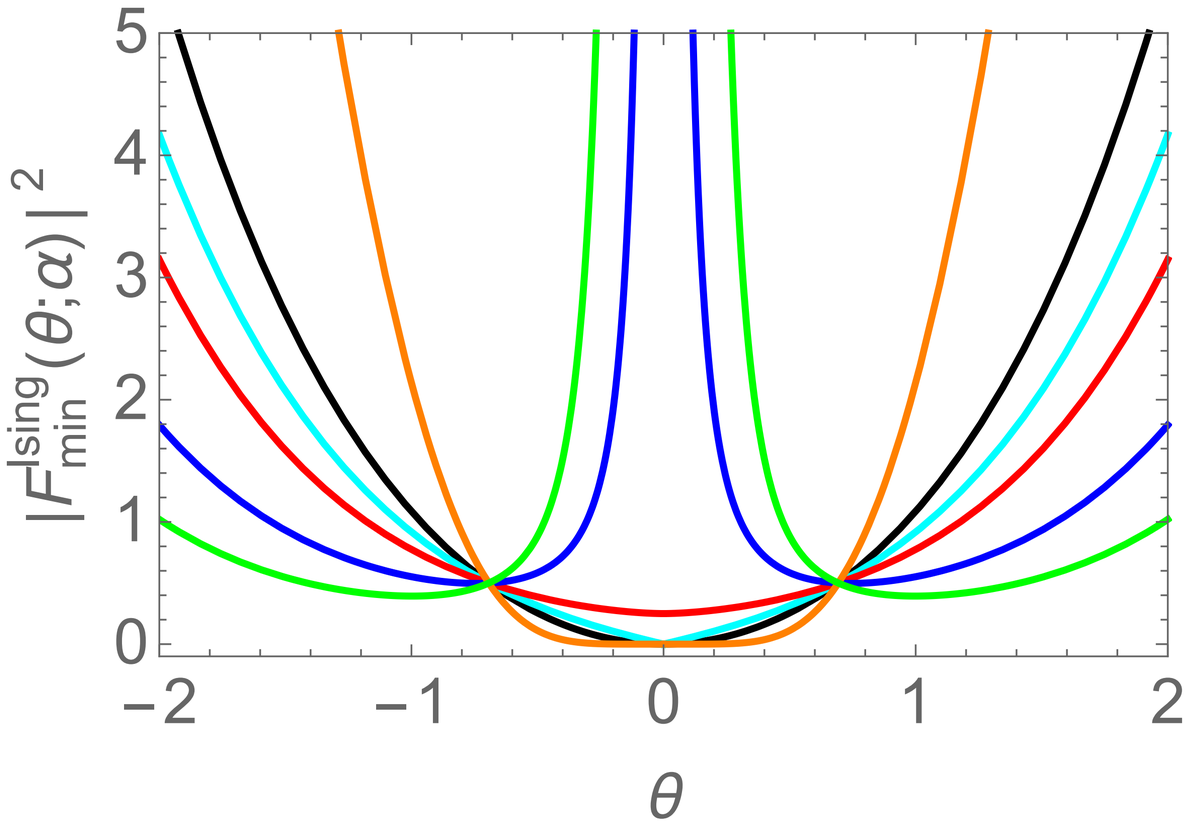

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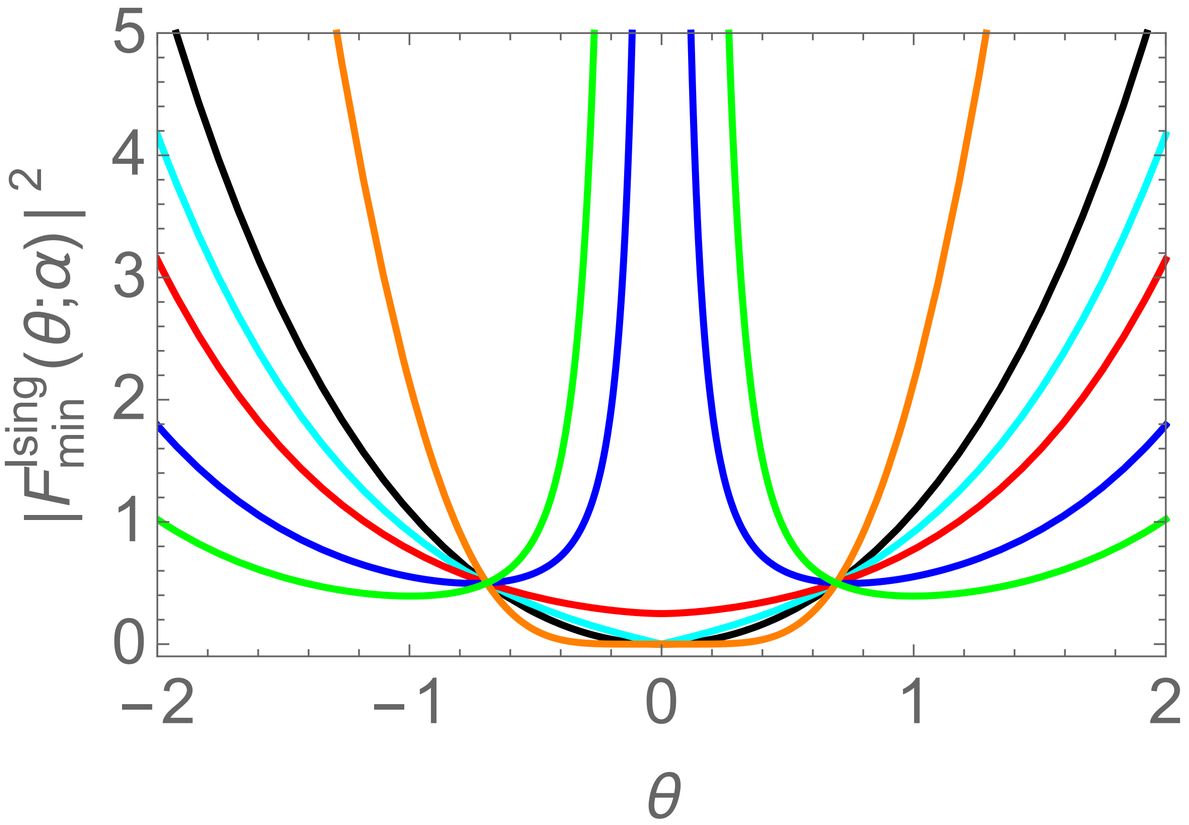


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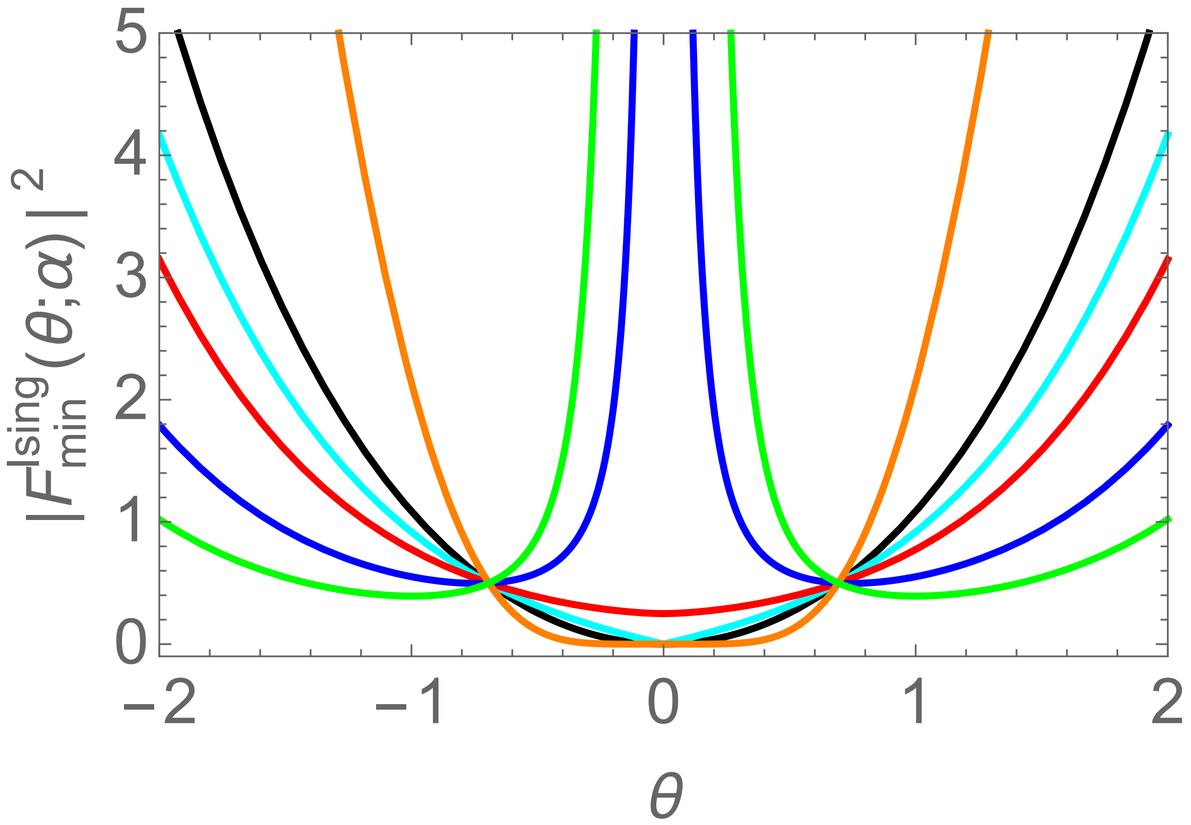


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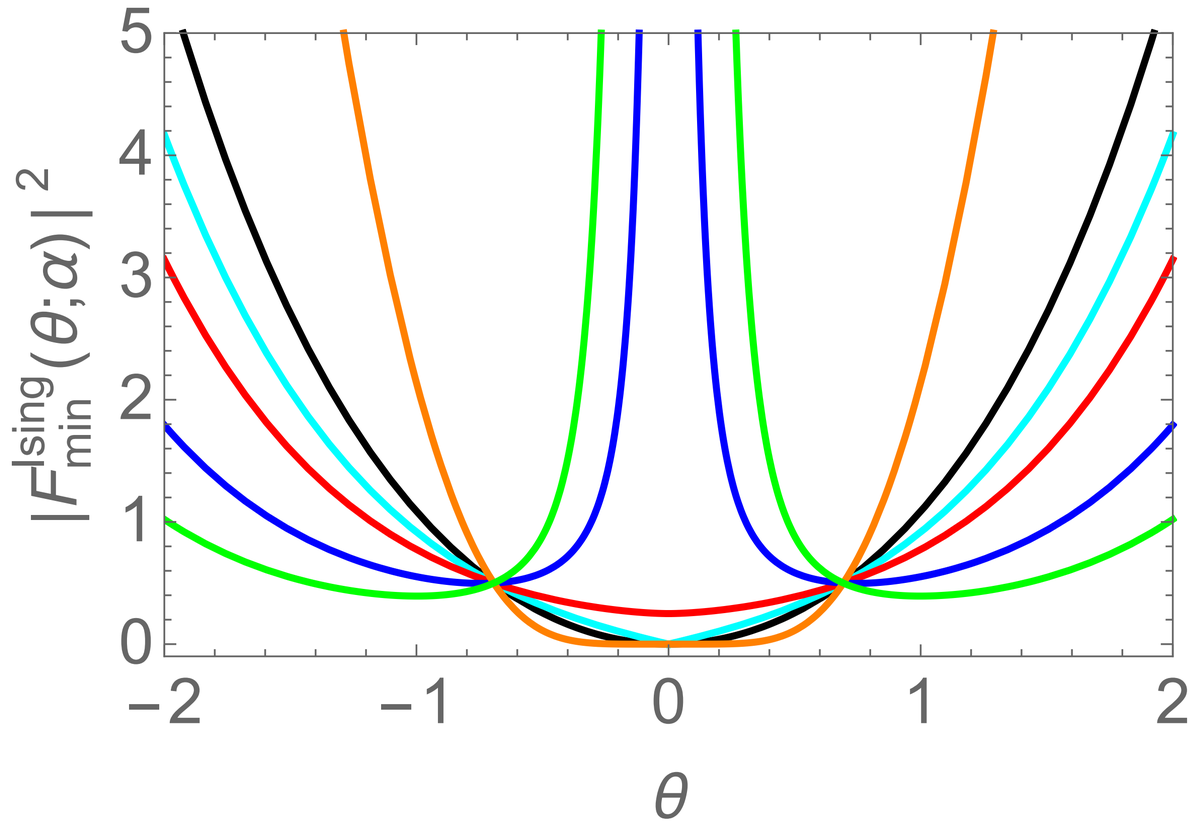


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- Should we think of new FF equations or better solution techniques?

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- The construction of higher-particle FFs is not fully understood yet, and might involve rethinking the whole program!


Merry
Christmas