

Amplitudes, Zigzags, and Curves

Marcus Spradlin, Brown University



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Introduction

In recent years we've seen remarkable progress on the problem of unlocking the hidden mathematical structure of quantum field theories, both for its own sake (**beauty**) and for the desire to develop new methods of practical importance for comparison of theory to experiment (**truth**).

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(1) some recent developments on tree-level amplitudes for gluons, where, perhaps surprisingly, the final story is not yet written.

This part is based on [2512.09704](#) [Paranjape, MS, Volovich, Weng] which applies some recent formalism in [2506.11895](#) [Pokraka, MS, Volovich, Weng] to the older formula [hep-th/0503198](#) [Britto, Feng, Roiban, MS, Volovich] that expresses split-helicity gluon amplitudes using [zigzag diagrams](#).

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(2) showing how to extend the “**curve integral formalism**” of **Arkani-Hamed, Frost, Plamondon, Salvatori, Thomas** for scalar particles to introduce fermions with Yukawa interactions.

This part is based on **2406.04411 [De, Pokraka, Skowronek, MS, Volovich]**.

An Apology

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Sometimes, very occasionally, new formalism can help us discover new physics (example: Lagrangian mechanics);

more sometimes it can just be a useful (sometimes, extremely useful) for better understanding “known” physics.

Tree-Level Gluon Amplitudes

Let's think about the question: what is the computational complexity of computing the **tree-level scattering amplitude** of $n = n_+ + n_-$ gluons with helicities ± 1 .

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Real Motivation: Is there something about the quantum field theory of spin-one particles that we fundamentally do not understand?

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In fact, from physics we know where all of the possible poles are, so if we define a rescaled amplitude

$$\tilde{A}_{n_+,n_-} = \left(\prod_{i < j} \langle ij \rangle [ij] \right) A_{n_+,n_-}$$

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it is guaranteed to be a **polynomial**. So if we want to capture the attention of a mathematician or computer scientist, we can declare that all of our kinematic variables are **integers**; then the amplitude is also an **integer**, and the problem of computing this amplitude is the problem of evaluating a certain function

$$\tilde{A}_{n_+,n_-} : \mathbb{Z}^{4n} \mapsto \mathbb{Z}$$

Tree-Level Gluon Amplitudes

The BCFW recursion expresses the answer as a sum of

$$\frac{1}{n-3} \binom{n-3}{k-1} \binom{n-3}{k-2} \quad k = \min(n_+, n_-)$$

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Reminder: a cluster algebra is generated by sets (called clusters) of variables (called cluster variables), with the variables in the sets related to each other in a specific way (called mutation); and two variables are called **compatible** if there is a cluster containing both.
[Fomin, Zelevinsky; Scott]

At tree-level, **cluster adjacency** means that the denominator in each individual term is a **product of compatible cluster variables**.

Mathematicians Even-Zohar, Lakrec, Parisi, Sherman-Bennett, Tessler, Williams have recently proven this to be correct.

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However, in our paper [arXiv:1906.10682](#) we worked in momentum twistor variables, which naturally expose the Grassmannian structure.

Motivated by recent work exploring the connection between the cluster algebra associated to the partial flag variety $\text{Fl}(2, n-2; n)$ that describes n -particle spinor helicity variables, we wanted to revisit this question for amplitudes expressed in terms of the more familiar spinor helicity variables.

We focused to start with on [split-helicity](#) which are simpler, and can be written in closed form.

Zigzags for Split Helicity

In hep-th/0503198 [Britto, Feng, Roiban, MS, Volovich] we showed that

$$A(1^-, \dots, q^-, (q+1)^+, \dots, n^+) = \sum_{k=0}^{\min(q-3, n-q-2)} \sum_{A_k, B_{k+1}} \frac{N_1 N_2 N_3}{D_1 D_2 D_3}$$

$$N_1 = \langle 1 | P_{2,b_1} P_{b_1+1,a_1} P_{a_1+1,b_2} \cdots P_{b_{k+1}+1,q-1} | q \rangle^3,$$

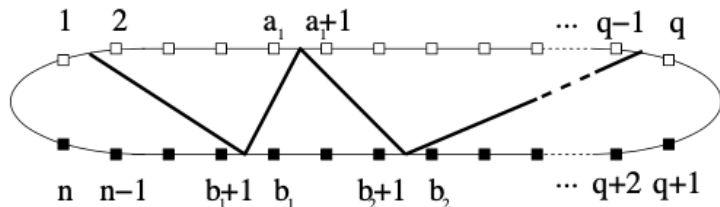
$$N_2 = \langle b_1+1 \ b_1 \rangle \langle b_2+1 \ b_2 \rangle \cdots \langle b_{k+1}+1 \ b_{k+1} \rangle,$$

$$N_3 = [a_1 \ a_1+1] \cdots [a_k \ a_k+1],$$

$$D_1 = P_{2,b_1}^2 P_{b_1+1,a_1}^2 P_{a_1+1,b_2}^2 \cdots P_{b_{k+1}+1,q-1}^2,$$

$$D_2 = F_{q,1} \bar{F}_{2,q-1},$$

$$D_3 = [2 | P_{2,b_1} | b_1+1 \rangle \langle b_1 | P_{b_1+1,a_1} | a_1 \rangle [a_1+1 | P_{a_1+1,b_2} | b_2+1 \rangle \cdots \langle b_{k+1} | P_{b_{k+1}+1,q-1} | q-1 \rangle]$$



Zigzags for Split Helicity

One big difference between the $\text{Gr}(4, n)$ and $\text{Fl}(2, n - 2; n)$ cluster algebras for describing, respectively, momentum twistor and spinor helicity variables: the latter breaks the natural cyclic structure of tree-level gluon amplitudes (cyclic permutations of the gluons).

Therefore we have to check, in principle, arbitrary permutations.

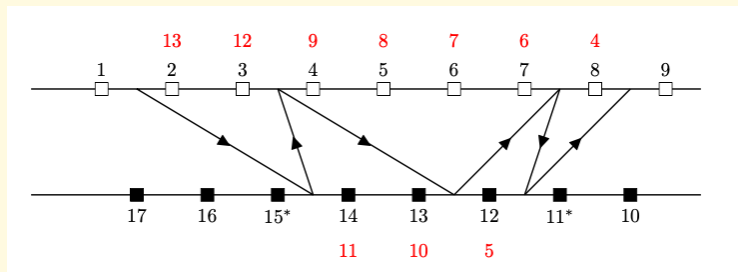
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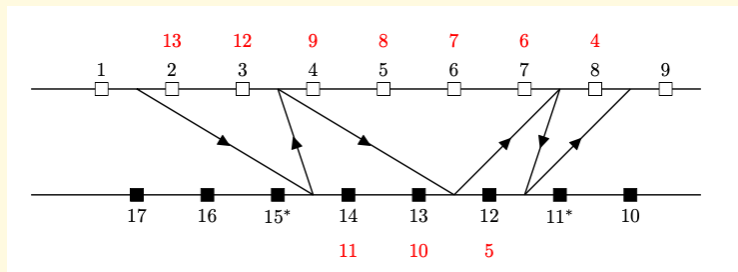
Based on “numerical experimentation”, we conjecture that cluster adjacency holds for each term (i.e., for each zigzag diagram) for a very specific set of permutations that can be read off from the diagram:

Zigzags for Split Helicity



The red numbers are fixed and the remaining numbers (1,2,3 in the left on the right and 14, 15, 16, 17 in the block on the left)

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TODO: (1) prove it, (2) check other cluster structures for describing spinor helicity variables, including one by [Chicherin](#), [Henn](#), and [Papathanasiou](#), (3) extend to other helicity amplitudes (the full “momentum amplituhedron”), (4) understand why it happens and what it means!

A Long Motivational Introduction

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What does it mean to “evaluate a loop integral”?

In practice, it often means to express it as a linear combination of “known” special functions, like

$$\text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln(1-t)$$

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Here I’m putting on an unusually practical hat – imagining I’m someone who wants to get digits of precision in order to compare to some data.

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A practically minded person might be happy to consider a loop integral to be evaluated if there is a fast algorithm for either (1) writing down a rapidly convergent series expansion, around some useful points or at least (2) processing it down into an integral formula with **as few integrations remaining as possible** – then one can **define** the things that appear as a new class of “special functions” and consider the job done.

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In D spacetime dimensions, L -loop Feynman integrals are expressible as

$$\int d^D \ell_1 d^D \ell_2 \cdots d^D \ell_L [\text{rational function of } p_1, \dots, p_n \text{ and } \ell_1, \dots] \\ = \pi^{DL-d} \int d^d \vec{x} [\text{rational function of } p_1, \dots, p_n \text{ and } \vec{x}]$$

where p_1, \dots, p_n are the momenta of the particles and $d \leq DL/2$ but there are two big things currently lacking.

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That's a big problem because one can have a long expression, resulting from a sum over many Feynman diagrams – maybe gigabytes long – that integrates to something very simple, or even to zero!

A Long Motivational Introduction

One way in which problem (2) could, hypothetically, be solved, is: consider the infinite set of different ways of writing a given integral

$$I = \int d^d \vec{x} \text{ [rational function of } p_1, \dots, p_n \text{ and } \vec{x}]$$

that differ from each other by combinations of arbitrary changes of variables and/or integration by parts.

If there were an algorithm for picking, from among this infinite dimensional equivalence class, a certain preferred representative – let's suggestively call it a **canonical form** – then we could easily compare whether two things are equal, or if some long combination of objects sums to zero.

A Long Motivational Introduction

Aside: A **period** is a number that is the volume of some region in \mathbb{R}^n carved out by polynomial inequalities with coefficients in \mathbb{Q} .

Example:

$$\pi = \int_{\mathbb{R}^2: x^2+y^2 < 1} dx dy$$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset$ algebraic numbers \subset periods \subset transcendental numbers

Indeed, a giant open conjecture is that if two periods are equal to each other, then there is an “explanation” for it: i.e., the two defining integrals can be mapped into each other by some combination of changes of variables and integration by parts (that involve only algebraic numbers). (**Kontsevich, Zagier**) Problem (2) mentioned above is a reflection of this...

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These are the so-called **multiple polylogarithm functions**, and the tool for unlocking their structure is called the **symbol**. (Goncharov, Spradlin, Vergu, Volovich).

Curve Integrals

The **curve integral formalism** is applicable to any **colored** theory; that means all particles are in the adjoint representation of some group (let's say $U(N)$), but they can be massless or massive, and the spacetime dimension is arbitrary (so we can use dim reg if we need to regulate IR or UV divergences).

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It provides a combinatorial algorithm for **writing down** an expression of the form

$$\int d^{n+3(L-1)}\vec{t} \text{ [rational function of } p_1, \dots, p_n \text{ and } \vec{t}]$$

that represents the **sum of all L -loop Feynman diagrams** to the n -particle amplitude. (One can restrict to planar, or any order in the $1/N$ expansion.)

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The complexity of computing the integrand is $\mathcal{O}(n^2)$ rather than $\mathcal{O}(4^n)$ as it would be for summing tri-valent Feynman graphs.

Curve Integrals

The complexity is $\mathcal{O}(n^2)$ because the key ingredients that enter are not individual Feynman diagrams, but curves that can be drawn on a surface. For example, consider all non-planar Feynman diagrams that contribute to the three-point amplitude shown here:



Individual Feynman diagrams correspond to triangulations of this surface, but the curve integral (representing the sum over all Feynman diagrams) can be written down by computing certain quantities associated to **compatible curves** on the surface; the curve from 1 to 3 is shown in purple.

Exampleritute

The 2-point 1-loop bubble diagram:

$$A(p^2) = \int_{t_1+t_2 \leq 0} dt_1 dt_2 \exp\left(\frac{p^2}{t_1 + t_2} (\max(0, t_2) - t_2 - \max(0, t_1))^2 + m^2(2 \max(0, t_2) + t_1) - \frac{D}{2} \log(t_1 + t_2)\right)$$

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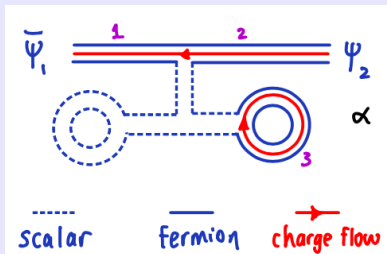
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(2) however, a big drawback of the fact that D is just a parameter is that none of the very special technology (spinor helicity, momentum twistors) specially tuned to $D = 4$ can help out here.

(3) The non-analytic integrand may frighten you, but there are recently-developed techniques (tropical sampling; M. Borinsky) for numerically evaluating integrals precisely of this type extremely fast (see G. Salvatori's talk at Amplitudes 2025).

Colored Yukawa Theory

In order to take this formalism one step closer to the real world, in [arXiv:2406.04411](https://arxiv.org/abs/2406.04411) we should how to incorporate (adjoint) fermions with Yukawa interactions: we gave an explicit formula for the curve integral (sum of all Feynman diagrams, at any loop order and any order in $1/N$) involving certain **determinants**.



Each curve can be assigned to be either bosonic or fermionic; for an L loop amplitude we must sum over the 2^L possible assignments (bosonic or fermionic) for each puncture.

Summary and Conclusion

Dramatic progress has been made in recent decades, but many fundamental questions about the structure of quantum field theory remain unsolved.

Seeking new formalisms may help to shed light on some of these questions and perhaps some may even ultimately be useful!