

# A 'Reverse Biot-Savart' method for calculating the coil shapes of iron free magnets

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ISIS Neutron and Muon Source



## Background

An FFA is a candidate accelerator for the ISIS-II project, a proposed megawatt scale pulsed spallation neutron source in the UK.

R&D towards ISIS-II includes a low energy, high intensity FFA demonstrator using the Front End Test Stand (FETS), a linac test facility at RAL, as an injector. This is the FETS-FFA.

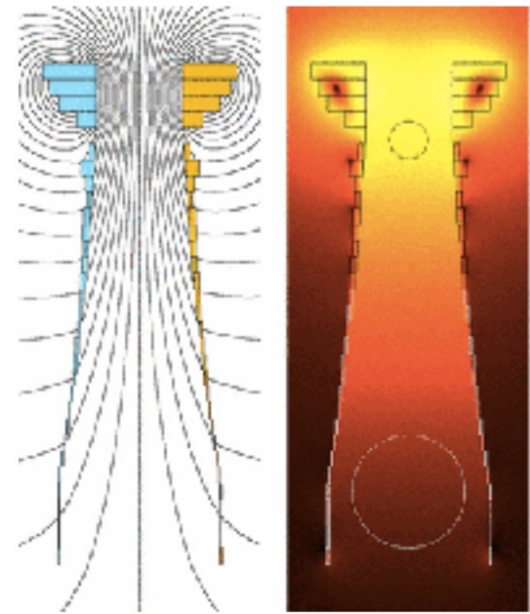
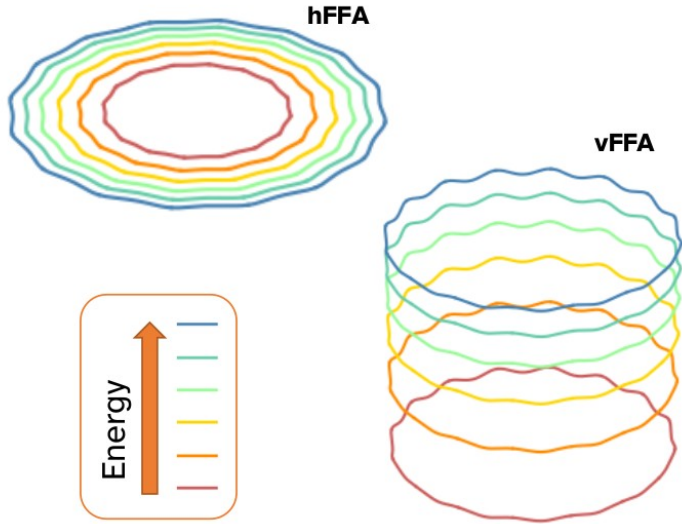
A vertical excursion FFA (vFFA) was seriously considered before being abandoned in favour of a more traditional horizontal FFA due to the technological challenges involved in the low energy demonstrator.

A great deal was learnt about the dynamics in a vFFA and several new techniques were developed including the 'Reverse Biot-Savart' method outlined here.



## Vertical FFA

A vertical FFA is a scaling machine in which a change of momentum leads to a vertical displacement of the beam.



A vertical mid-plane field of the form  $B_y = B_0 e^{m(y-y_0)}$  produces the required vertical motion and a skew quadrupole focussing term.

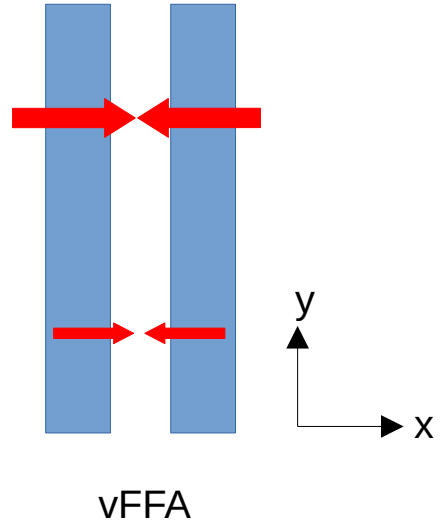
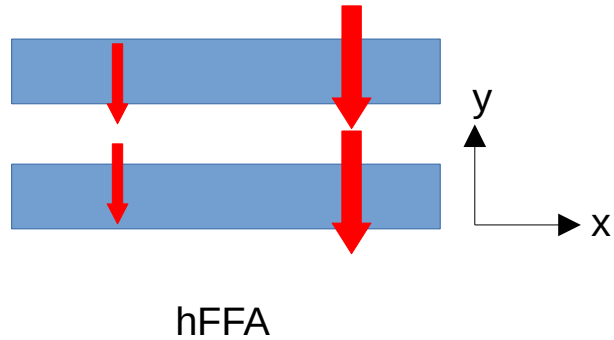
For the FETS-vFFA the necessary field strength dictated a superconducting, iron-free magnet.



## vFFA Fields

The vFFA magnet has a scaling vertical field and zero horizontal field on the mid-plane.

A dipole magnet where the horizontal fields either side of the gap cancel.

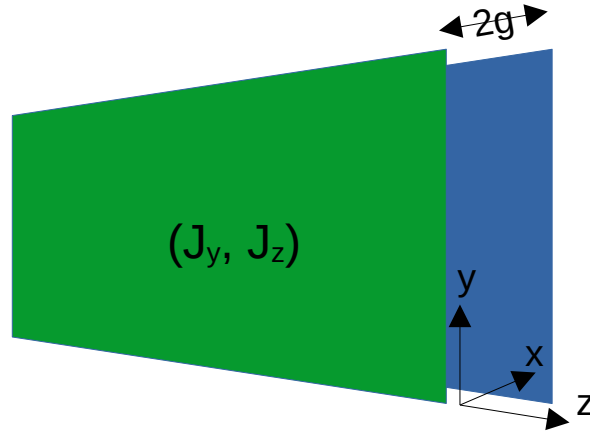


Given the desired mid-plane field shape is there a direct method to calculate the required shape of the coils?



## Mid-plane field of complex exponential current sheets

Consider two infinite parallel current sheets in  $(y, z)$  placed at  $x = \pm g$



On the mid-plane the vertical field component  $B_y$  is only proportional to the longitudinal current component  $J_z$ .

$$J_z|_{x>0} = -J_z|_{x<0}$$

If the longitudinal current density has the form

$$J_z = e^{j\omega_y y} e^{j\omega_z z}$$

The mid-plane vertical field component at a point  $(0, p_y, p_z)$  can be found from Biot-Savart:

$$B_y = \frac{\mu_0}{2\pi} \iint_{-\infty}^{\infty} e^{j\omega_y y} e^{j\omega_z z} \frac{g}{(g^2 + (p_y - y)^2 + (p_z - z)^2)^{3/2}} dy dz$$

There are no standard form solutions to this integral.

However the integral looks like an inverse Fourier transform of the form

$$B_y = k \mathcal{F}^{-1} \left( \frac{a}{(a^2 + \omega^2)^{3/2}} \right)$$

But shifted from  $\omega=0$ .



A 2D radial function  $f(r)$  has a 2D radial FT

$$f(r) \rightarrow \mathcal{F}(\omega_r)$$

where

$$\omega_r^2 = \omega_y^2 + \omega_z^2$$

The 2D FT can be reduced to a 1D FT through the Fourier slice theorem. The resulting 1D FT is the sum of a series of Hankel transforms of the original function

$$\mathcal{H}_\nu\{f(r), \omega\} = \int_0^\infty f(r) J_\nu(\omega r) r dr$$

For a purely radial function with no azimuthal dependence, the FT is just the zeroth order Hankel transform.

Unlike the Biot-Savart integral the Hankel transform does have a standard result.

For

$$f(r) = e^{-ar}$$

The zeroth order Hankel transform is

$$\mathcal{H}_0(\omega_r) = \frac{ka}{(a^2 + \omega_r^2)^{3/2}}$$

So the Biot-Savart integral should have a solution that includes a term like  $e^{-ar}$ .



Let

$$g(\omega_r) = \frac{ka}{(a^2 + \omega_r^2)^{3/2}}$$

We want  $f(r)$  such that

$$\mathcal{F}\{f(\sqrt{y^2 + z^2})\} = g(\sqrt{\omega_y^2 + \omega_z^2})$$

This is equivalent to

$$\mathcal{H}_0\{f(r)\} = g(\omega_r)$$

If

$$f(r) = e^{-ar}$$

then

$$\int_0^{\infty} e^{-ar} J_0(\omega_r r) r dr = g(\omega_r)$$



$$\int_0^{\infty} e^{-ar} J_0(\omega_r r) r dr = g(\omega_r)$$

The LHS has the form of a Laplace transform

$$\mathcal{L}\{J_0(\omega_r r) r\}(a) = g(\omega_r)$$

The Laplace transform of  $J_0$  is

$$\mathcal{L}\{J_0(\omega_r r)\}(a) = \frac{1}{\sqrt{a^2 + \omega_r^2}}$$

Multiplication by  $r$  in the transform is equivalent to differentiation so

$$\mathcal{L}\{J_0(\omega_r r) r\}(a) = \frac{a}{(a^2 + \omega_r^2)^{3/2}} = g(\omega_r)$$



The Biot-Savart integral has the form of an inverse FT of a term like

$$\frac{k a}{(a^2 + \omega_r^2)^{3/2}}$$

except the integral is of a radial function centred on  $(p_y, p_z)$  not  $(0, 0)$ .  
Rewrite the integral explicitly as an IFT

$$B_y = \frac{\mu_0}{2\pi} \iint_{-\infty}^{\infty} e^{jy\omega_y} e^{jz\omega_z} \frac{g}{(g^2 + (\Omega_y - \omega_y)^2 + (\Omega_z - \omega_z)^2)^{3/2}} d\omega_y d\omega_z$$

This is the IFT of a radial function of frequency centred at  $\Omega_y, \Omega_z$ .

The shift of centre is convolution with a 2D delta function at  $\Omega_y, \Omega_z$ .

ie

$$B_y = \frac{\mu_0}{2\pi} \iint_{-\infty}^{\infty} e^{jy\omega_y} e^{jz\omega_z} \frac{g}{(g^2 + \omega_y^2 + \omega_z^2)^{3/2}} * \delta(\Omega_y) * \delta(\Omega_z) d\omega_y d\omega_z$$

$$B_y = k e^{jy\Omega_y} e^{jz\Omega_z} e^{-gr}$$

$$r = \sqrt{y^2 + z^2}$$

Undo the symbol substitution

$$B_y = k e^{j\omega_y p_y} e^{j\omega_z p_z} e^{-g\omega_r}$$

$$\omega_r = \sqrt{\omega_y^2 + \omega_z^2}$$

## Recap:

For two infinite parallel current sheets in  $(y, z)$  placed at  $x = \pm g$

If the longitudinal current density has the form

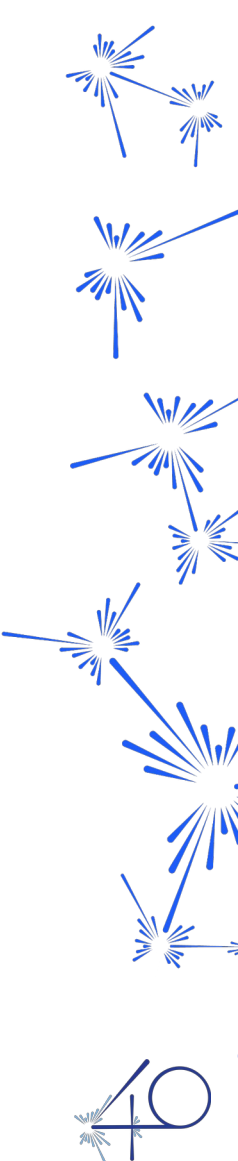
$$J_z = k_j e^{j\omega_y y} e^{j\omega_z z}$$

The mid-plane vertical field component has the form

$$B_y = k_B e^{j\omega_y y} e^{j\omega_z z} e^{-g\sqrt{\omega_y^2 + \omega_z^2}}$$

The mid-plane field is a filtered version of the current density.

This is intuitively correct and the analysis gives the exact form of the filtering.



This leads to a possible method to directly determine the required current density for a given mid-plane field.

If the required field can be written as

$$B_y = f(y)g(z) = \sum_i a_i e^{j\omega_{yi}y} \sum_k b_k e^{j\omega_{zk}z}$$

The necessary current density is

$$J_z = \sum_i \left( a_i e^{j\omega_{yi}y} \sum_k b_k e^{j\omega_{zk}z} e^{g\sqrt{\omega_{yi}^2 + \omega_{zk}^2}} \right)$$

Which can be calculated from a 2D FFT of  $B_y$

$J_y$  can be calculated from  $\nabla \cdot J = 0$

$$J_y = \int \frac{\partial J_z}{\partial z} dy$$

Because

$$J_z \Rightarrow B_x, B_y$$

And

$$J_y \Rightarrow B_x, B_z$$

$J$  can therefore be made divergence free without effecting  $B_y$



Large  $dB/dy,z$  implies high frequency components in  $B$

For a practical magnet the factor  $e^{g\sqrt{\omega_{yi}^2 + \omega_{zj}^2}}$  means that rapid changes in  $B$  outside of the good field region can lead to large, oscillatory  $J$  which cannot practically be realised with a finite number of discrete coils.

This method relies on specifying the field everywhere not just the good field region. The shape and magnitude of the field outside the good field region needs to be carefully considered.

To produce closed coils a negative field region for the coil return path must be included.

From  $\nabla \times B = 0$  it follows that

$$B_z = \int_{-\infty}^y \frac{\partial B_y}{\partial z} dy$$

If

$$B_y = f(y)g(z) \Big|_{y < y_0}$$
$$B_y = B_0 e^{m(y-y_0)} g(z) \Big|_{y \geq y_0}$$

Then

$$B_z(y_0) = g'(z) \int_{-\infty}^{y_0} f(y) dy$$

And

$$B_z(y_1) = B_z(y_0) + g'(z) B_0 \int_{y_0}^{y_1} e^{m(y-y_0)} dy$$

To achieve scaling

$$\frac{B_z(y_1)}{B_z(y_0)} = \frac{B_y(y_1)}{B_y(y_0)}$$

Requires

$$\int_{-\infty}^{y_0} f(y) dy = \frac{B_0}{m}$$

For all coils to be closed

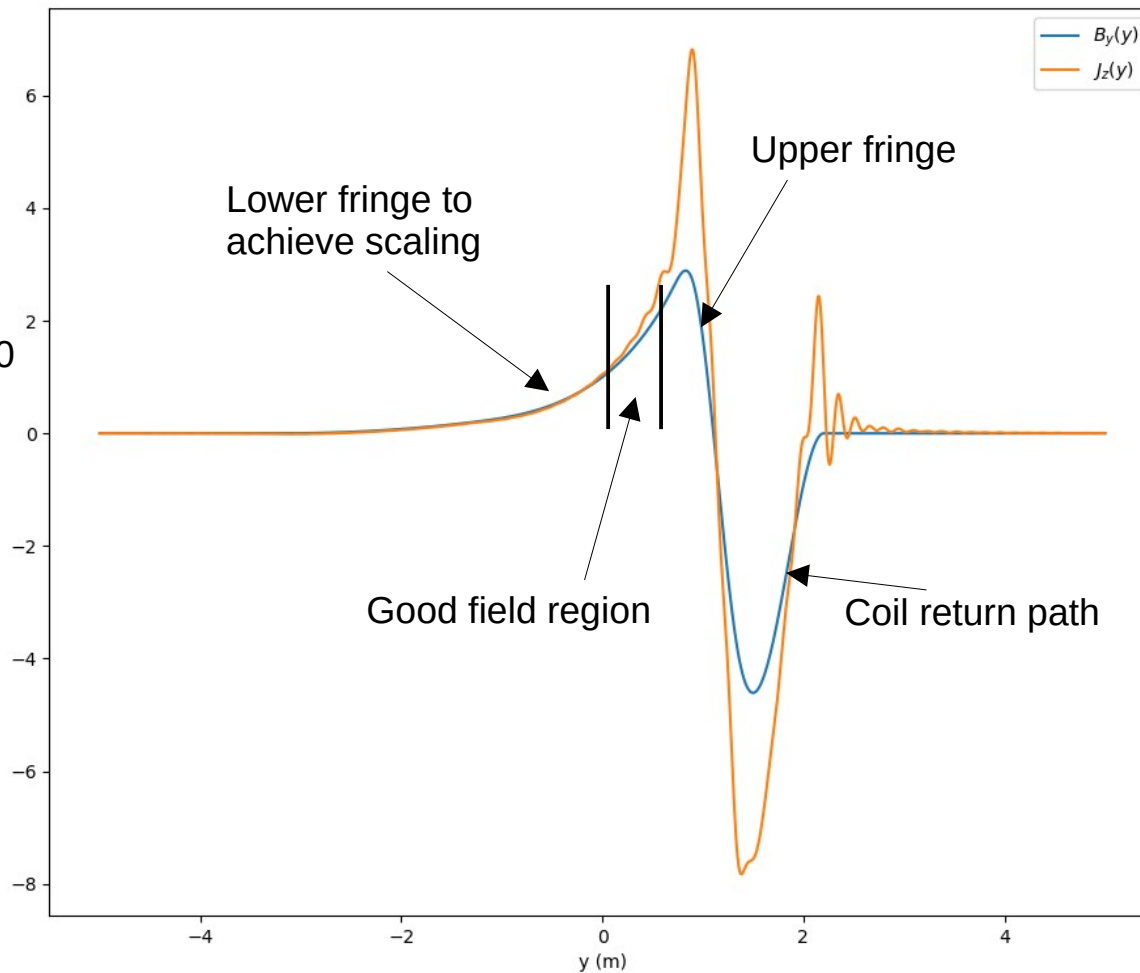
$$\int_{y_{min}}^{y_{max}} J_z dy = 0$$

$$\int_{z_{min}}^{z_{max}} J_y dz = 0$$

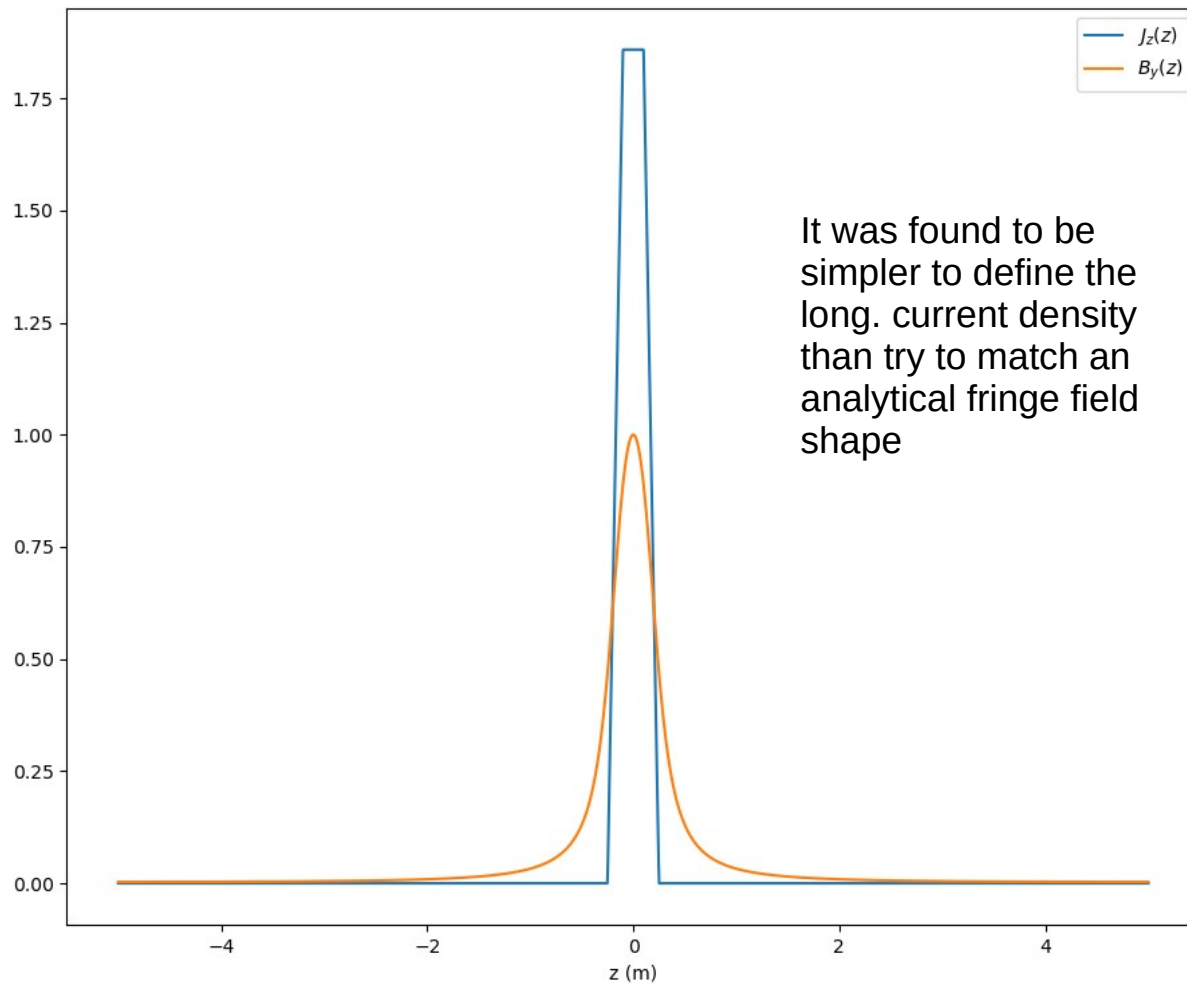


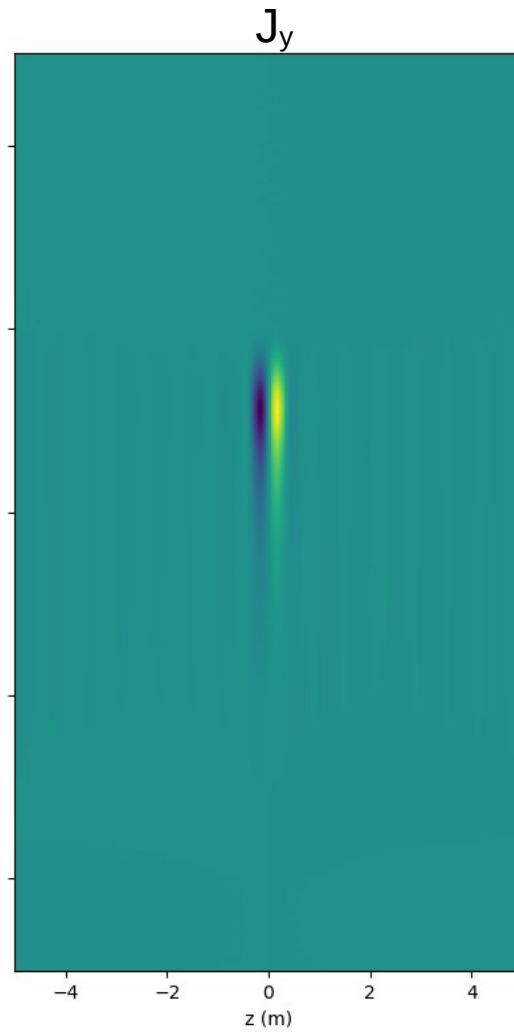
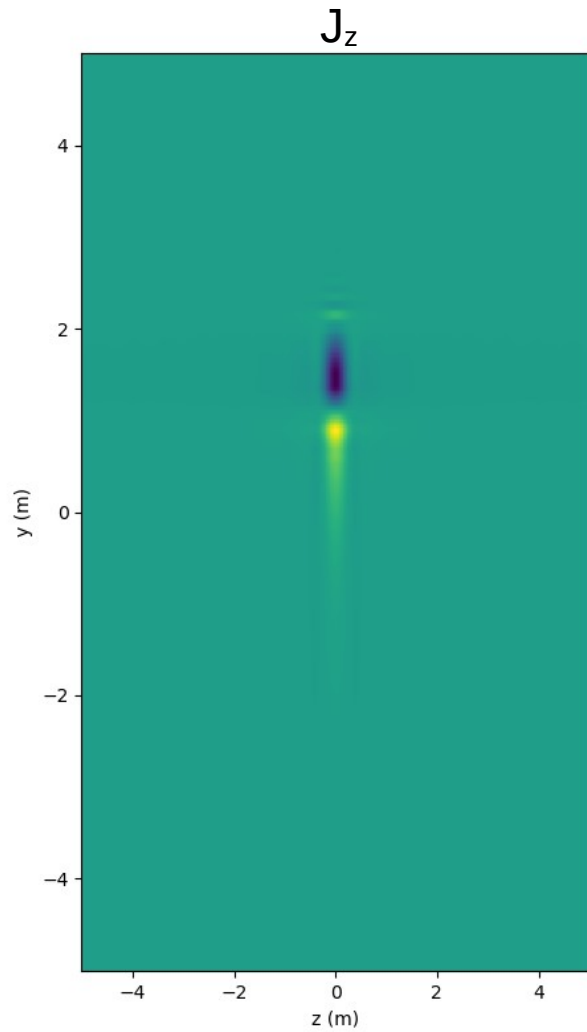
Example for  $m = 1.3$ , good field region = 0.6 m and  $g = \pm 15\text{cm}$

Vertical profile at  $x=0, z=0$



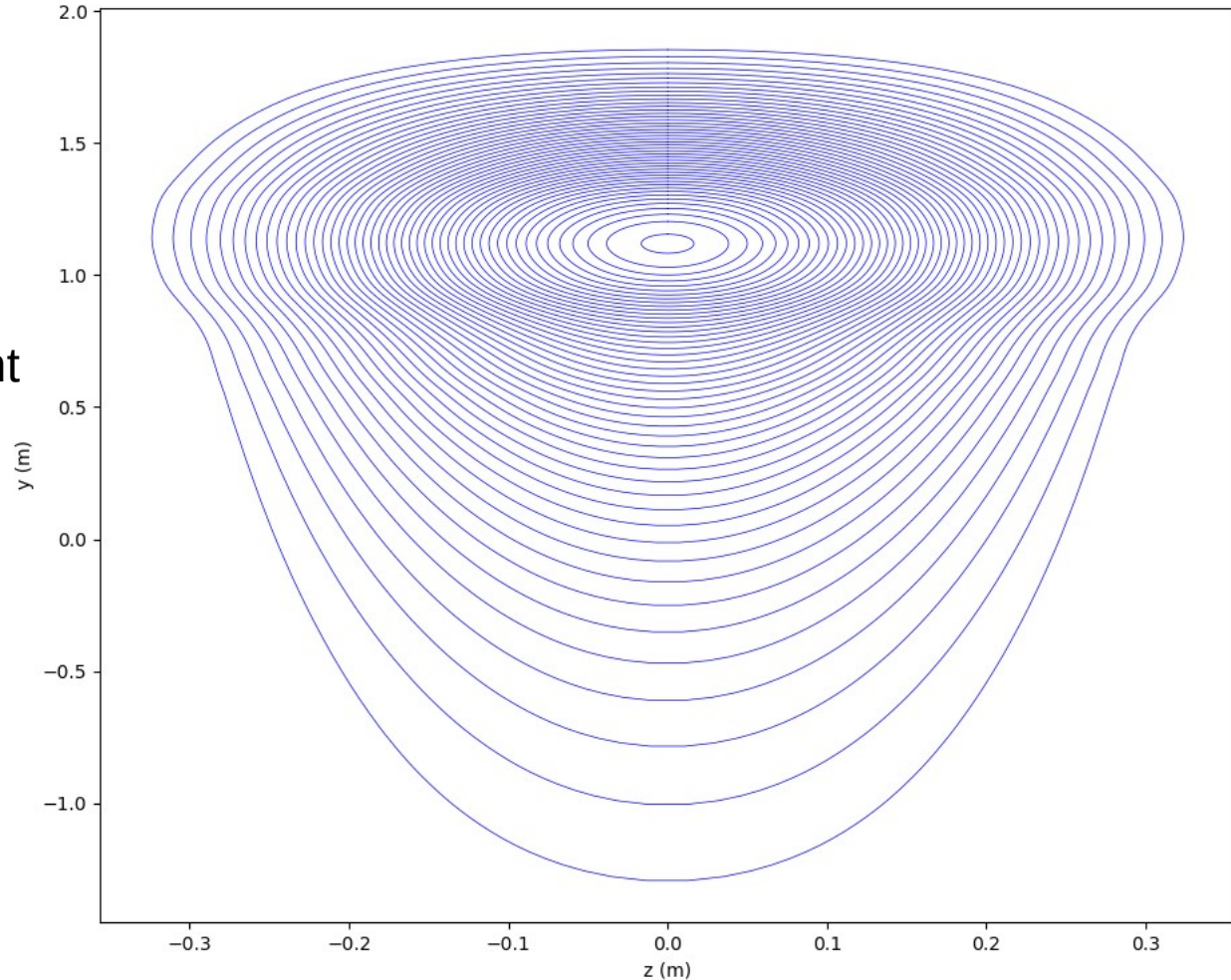
# Longitudinal fringe field



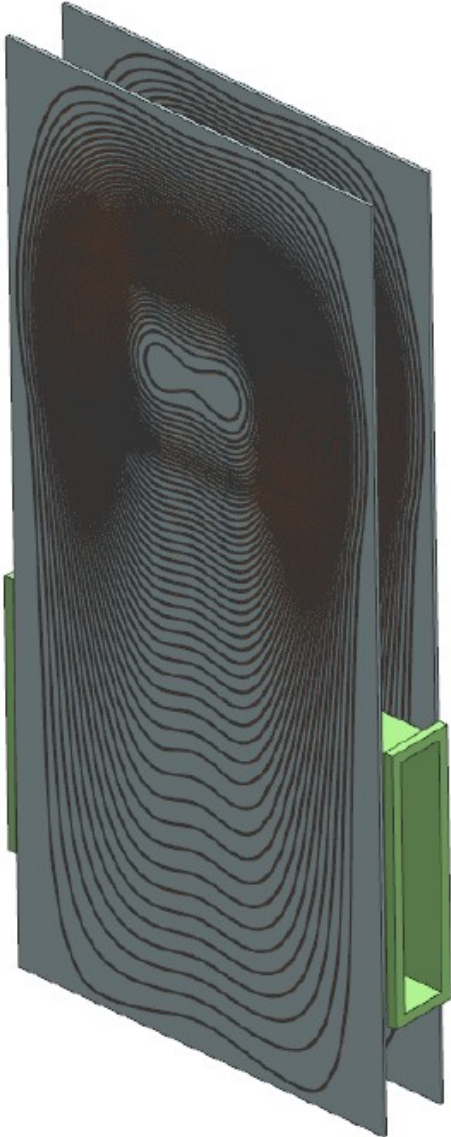


To generate coils from the quasi-continuous current density,  $J_z$  is integrated from  $y_{min}$  at  $z=0$ .

Wires are placed at the centre of each equi-current interval of the integral and follow streamlines of  $J$ .



Prototype magnet design produced by this method.



## Summary

For an iron-free magnet the required current density is directly related to the mid-plane field in the frequency domain.

A method for evaluating the current density for a vFFA magnet has been presented.

The current density can be approximated to the required accuracy by a (possibly large) number of discrete coils.

With certain simplifying assumptions a similar method can be applied to other magnet configurations.

Although derived in the absence of iron, the method is likely to give a good estimate of coils on the face of a flat pole eg trim coils.

