LoopFest XI

Multi-parton NLO calculations.

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11 May Pittsburgh 2012

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Motivation

- Multi-jet final states play an important role for the experiments at the LHC.
- Jet observables can be easily modelled at leading order (LO).
- To improve the accuracy we include higher order corrections in perturbative theory.
- Next-to-leading order (NLO) corrections contain two parts: real corrections and the virtual corrections.
- The virtual corrections involve a one-loop integral.
- The past years have seen a significant progress in calculating virtual corrections with many external legs.
- This was achieved mainly thru perfection of the traditional Feynman graph approach or algorithms based on unitary methods.

Introduction

In this talk we present...

- ... an algorithm for the numerical calculation of one-loop amplitudes.
- ... the subtraction method of the virtual sector.
- ... the local subtraction terms for the infrared singularities of an one loop amplitude.
- ... the guiding principles for constructing local subtraction terms for the ultraviolet singularities of an one-loop amplitude.
- ...a method for contour deformation.
- ... results for the process $e^+e^- \longrightarrow n$ jets.

The subtraction method

- The contributions of an infrared observable at next-to-leading order (NLO) with n final state particles can be written as

$$\langle O \rangle^{NLO} = \int\limits_{n+1} O_{n+1} d\sigma^R + \int\limits_{n} O_n d\sigma^V.$$

  - $d\sigma^R$: real emission contribution.
  - $d\sigma^V$: virtual contribution.

- Usually one introduces subtraction terms to perform the phase space integrations by Monte Carlo methods.
- We extend this subtraction method to the virtual sector.
- The renormalised one-loop amplitude is related to the bare amplitude by

$$\mathcal{A}^{(1)} = \mathcal{A}^{(1)}_{bare} + \mathcal{A}^{(1)}_{CT},$$

  where $\mathcal{A}^{(1)}_{CT}$ denotes the ultraviolet counterterm from renormalisation.
- The bare amplitude involves the loop integration

$$\mathcal{A}^{(1)}_{bare} = \int \frac{d^D k}{(2\pi)^D} \mathcal{G}^{(1)}_{bare}.$$

The subtraction method II

- We can write the NLO contribution as a sum of three finite pieces.

$$\langle O\rangle^{NLO} = \langle O\rangle^{NLO}_{real} + \langle O\rangle^{NLO}_{virtual} + \langle O\rangle^{NLO}_{insertion}$$

- For the *real* part we have

$$\langle O\rangle^{NLO}_{real} = \int\limits_{n+1} \left(O_{n+1} d\sigma^R - O_n d\sigma^A \right).$$

- For the *virtual* part we have

$$\langle O\rangle^{NLO}_{virtual} = 2\int d\phi_n \,\mathrm{Re} \int \frac{d^4k}{(2\pi)^4} \left[\mathcal{A}^{(0)^*} \left(\mathcal{G}^{(1)}_{bare} - \mathcal{G}^{(1)}_{IR} - \mathcal{G}^{(1)}_{UV} \right) \right] O_n,$$

where $\mathcal{G}^{(1)}_{IR}$ and $\mathcal{G}^{(1)}_{UV}$ are the local subtraction terms for the IR and UV divergences of the *bare* one-loop amplitude.

- For the *insertion* part we have

$$\langle O\rangle^{NLO}_{insertion} = \int\limits_{n} O_n \left(\mathbf{I} + \mathbf{L} \right) \otimes d\sigma^B,$$

- The notation $\otimes$ includes colour correlation due to soft gluons.
- The sum of the insertion operators $\mathbf{I}$ and $\mathbf{L}$ is finite.

$$\mathbf{L} \otimes d\sigma^B = 2\,\mathrm{Re} \left[\mathcal{A}^{(0)^*} \left(\mathcal{A}^{(1)}_{CT} + \mathcal{A}^{(1)}_{IR} + \mathcal{A}^{(1)}_{UV} \right) \right] d\phi_n, \quad \mathbf{I} \otimes d\sigma^B = \int\limits_1 d\sigma^A.$$

Colour decomposition

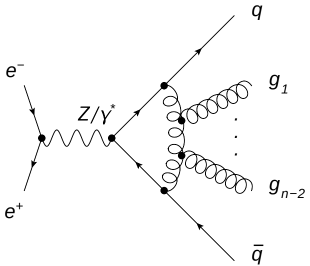
- Amplitudes in QCD may be decomposed into group-theoretical factors (carrying the colour structures) multiplied by kinematic factors called partial amplitudes.
- At one-loop level partial amplitudes can be further decomposed into primitive amplitudes.

$$\mathcal{A}^{(1)} = \sum_j C_j A_j^{(1)}$$

The colour structures are denoted by C_j, while the primitive amplitudes are denoted by $A_j^{(1)}$.

  - Primitive amplitudes are gauge invariant.
  - Primitive amplitudes have a fixed cyclic ordering of the external legs and a definite routing of the of the external fermion lines.
  - This ensures that the type of each loop propagator is uniquely defined, being either a quark or a gluon/ghost propagator.
- Reconstructing the full amplitude out of primitive amplitudes is a purely combinatorial problem.
- Therefore we will focus in the remaining talk on the calculation of primitive amplitudes.

- In the leading colour approximation of the process $e^+e^- \longrightarrow n - jets$, only one primitive amplitude occur.

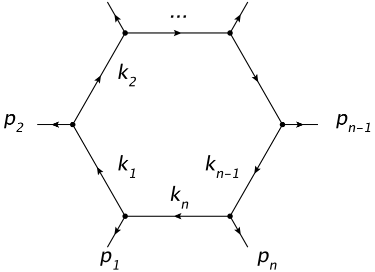


- Remark: A primitive amplitude can be constructed via Berends-Giele type recurrence relations.

Kinematics

- In a bare primitive amplitude with n external legs, $A^{(1)}_{bare}$, only n different propagators occur in the loop integral.
- We define the kinematics as follows:

$$
\begin{aligned}
k_j &= k - q_j, \\
q_j &= \sum_{l=1}^{j} p_l .
\end{aligned}
$$



- We define the bare one-loop integrand $G^{(1)}_{bare}$ via:

$$
A^{(1)}_{bare} = \int \frac{d^D k}{(2\pi)^D} G^{(1)}_{bare}, \qquad G^{(1)}_{bare} = P(k) \prod_{j=1}^{n} \frac{1}{k_j^2 - m_j^2 + i\delta}
$$

The infrared subtraction terms

- For massless QCD the soft and collinear subtraction terms are given by

$$
\begin{aligned}
\mathcal{G}^{(1)}_{soft} &= 4\imath \sum_{j\in I_g} \frac{p_j \cdot p_{j+1}}{k_{j-1}^2 k_j^2 k_{j+1}^2} \mathcal{A}^{(0)}_j \\
\mathcal{G}^{(1)}_{coll} &= -2\imath \sum_{j\in I_g} \left[\frac{S_j g_{UV}\left(k_{j-1}^2, k_j^2\right)}{k_{j-1}^2 k_j^2} + \frac{S_{j+1} g_{UV}\left(k_j^2, k_{j+1}^2\right)}{k_j^2 k_{j+1}^2} \right] \mathcal{A}^{(0)}_j
\end{aligned}
$$

- $j \in I_g$ denotes all gluon propagators in the loop.
- S_j are symmetry factors:

$$
S_j = \begin{cases} 1 & \text{quark} \\ 1/2 & \text{gluon} \end{cases}
$$

- g_{UV} ensures the UV finiteness of the collinear subtraction term.

$$
\lim_{k_{j-1}||k_j} g_{UV}\left(k_{j-1}^2, k_j^2\right) = 1, \qquad \lim_{k\to\infty} g_{UV}\left(k_{j-1}^2, k_j^2\right) = \mathcal{O}\left(\frac{1}{|k|}\right),
$$

- The IR subtraction terms are formulated at amplitude level and can be easily integrated analytically over the loop momentum.

The ultraviolet subtraction terms I

- Only propagator and vertex corrections are UV divergent.
- We expand the relevant propagators around a new UV propagator:

$$\begin{aligned}\frac{1}{(k-p)^2} &= \frac{1}{\bar{k}^2-\mu_{UV}^2}+\frac{2\bar{k}\cdot(p-Q)}{\left(\bar{k}^2-\mu_{UV}^2\right)^2}-\frac{(p-Q)^2+\mu_{UV}^2}{\left(\bar{k}^2-\mu_{UV}^2\right)^2}\\ &\quad+\frac{\left(2\bar{k}\cdot(p-Q)\right)^2}{\left(\bar{k}^2-\mu_{UV}^2\right)^3}+\mathcal{O}\left(\frac{1}{|\bar{k}|^5}\right)\end{aligned}$$

  where $\bar{k}=k-Q$.
- Applying this expansion to a vertex or propagator correction, we get:

$$F_n(k) = \frac{N(k)}{(\bar{k}^2-\mu_{UV}^2)^n}\left(1+\sum_{j=1}^{l}\frac{X_j(\bar{k})}{(\bar{k}^2-\mu_{UV}^2)^j}\right)+\mathcal{O}\left(\frac{1}{|\bar{k}|^5}\right)$$

  where $X_j(\bar{k})$ is a polynomial of degree j in $\bar{k}$.

The ultraviolet subtraction terms II

- Integration yields

$$\int \frac{d^D k}{(2\pi)^D} F_n(k) \quad = \quad C\left(\frac{1}{\epsilon} - \ln\frac{\mu_{UV}^2}{\mu^2}\right) + R + \mathcal{O}(\epsilon).$$

- Add finite pieces to get rid of R.

$$S_{UV}(k) \quad = \quad \frac{N(k)}{(\bar{k}^2 - \mu_{UV}^2)^n}\left(1 + \sum_{j=1}^{l} \frac{X_j(\bar{k})}{(\bar{k}^2 - \mu_{UV}^2)^n}\right) - \frac{-2\mu_{UV}^2 R}{(\bar{k}^2 - \mu_{UV}^2)^3} + \mathcal{O}\left(\frac{1}{|\bar{k}|^5}\right)$$

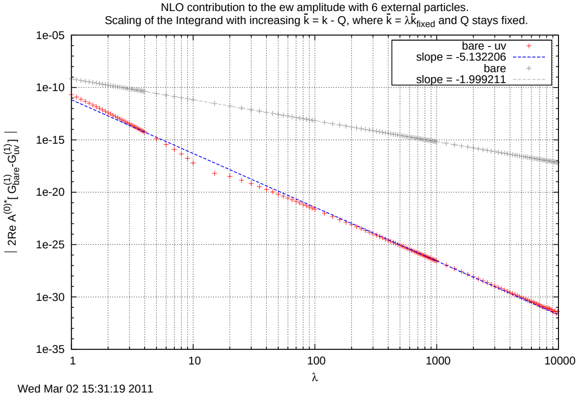
- The integrated subtraction term is:

$$\int \frac{d^D k}{(2\pi)^D} S_{UV}(k) \quad = \quad C\left(\frac{1}{\epsilon} - \ln\frac{\mu_{UV}^2}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

- The unintegrated total UV subtraction term $G_{UV}^{(1)}$ can be constructed efficiently via Berends-Giele type recurrence relations.
- The complete integrated subtraction term is proportional to a Born amplitude.

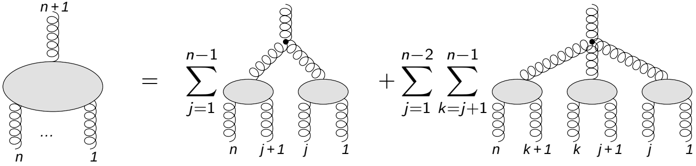
Consistency check of the UV subtraction

- The plot shows $|2\,\mathrm{Re}(A^{(0)}G^{(1)}_{bare})|$ and $|2\,\mathrm{Re}(A^{(0)}(G^{(1)}_{bare} - G^{(1)}_{UV}))|$ over the UV scaling parameter λ for the process $e^+e^- \longrightarrow 4jets$.
- The *bare* Amplitude decrease like $1/k^2$ and is therefore quadratic divergent.
- The $(bare - UV)$ Amplitude decrease like $1/k^5$ and is therefore UV-safe.



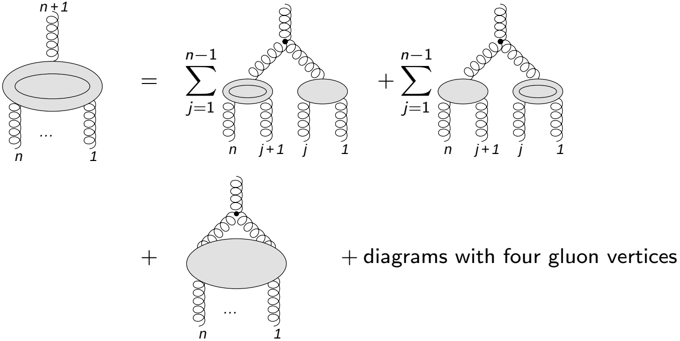
Off-shell recurrence relations(born)

- We use Berends-Giele type recurrence relations for primitive amplitudes.
- Example for the $n-$gluon tree-level amplitude.



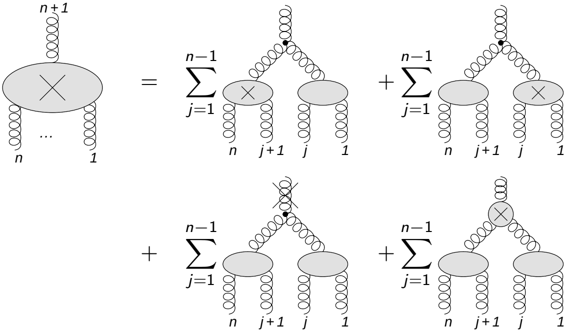
Off-shell recurrence relations(one-loop)

- We use similar recurrence relations for the computation of the one-loop amplitude.



Off-shell recurrence relations(UV)

- UV-subtraction terms are also constructed recursively.



$+$ diagrams with four gluon vertices

Overview of the contour deformation

- Again the one loop integrand

$$\int \frac{d^4k}{(2\pi)^4} G^{(1)}_{bare} = \int \frac{d^4k}{(2\pi)^4} P(k) \prod_{j=1}^{n} \frac{1}{k_j^2 + i\delta}$$

- We deform the integration contour into the complex plane to match Feynman's $+i\delta$ rule.
- Use direct deformation of the loop momenta

$$k \to \tilde{k} = k + i\kappa(k).$$

- After the deformation the integral reads

$$= \int \frac{d^4k}{(2\pi)^4} \left| \frac{\partial \tilde{k}}{\partial k} \right| P(\tilde{k}(k)) \prod_{j=1}^{n} \frac{1}{k_j^2 - \kappa^2 + 2ik_j \cdot \kappa}$$

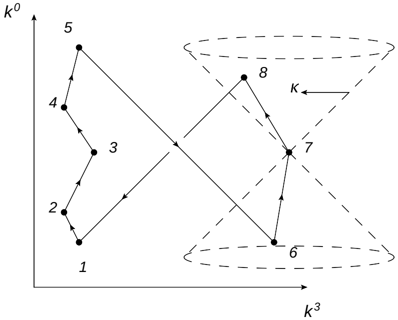
- We have to construct the deformation vector κ such

$$k_j^2 = 0 \quad \to \quad k_j \cdot \kappa \geq 0.$$

- The numeric stability of the Monte Carlo integration depends strongly on the definition of the deformation vector κ.
- At the moment we use a slightly modified algorithm by W. Gong, Z. Nagy and D. Soper to construct the deformation vector.

Overview of the contour deformation

- Illustrating the kinematics of a primitive amplitude with $n = 8$ legs in the loop momenta space.



- The dots correspond to the kinematic variables $q_i = \sum_{j=1}^{i} p_i$.
- The line segments correspond to the external momenta $p_i = q_i - q_{i-1}$.
- $(k - q_i)^2 = 0$ defines a light cone.
- The deformation must direct inside the cone.
- Alongside the line segments this is not possible.

Improving the efficiency

- Efficiency is crucial to apply the method to high multiplicity processes.
- Holomorphic division into sub channels:
  Different contour deformation in each channel;
- Non-holomorphic division into sub channels:
  Different coordinate system in each channel;
- Sampling in the loop momenta space:
  Importance sampling;
- Antithetic variates:
  Reduce oscillations significant.
- Improvement of the UV subtraction terms:
  Better UV behavior;
- For the details please see our most recent publication.

Recent results - $e^+e^- \to jets$

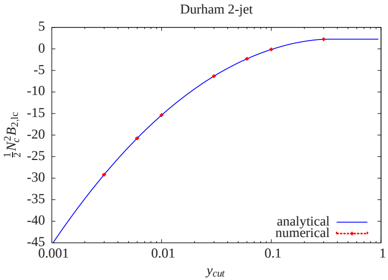
- The cross section for n jets normalised to the LO cross section for $e^+e^- \to$ hadrons.

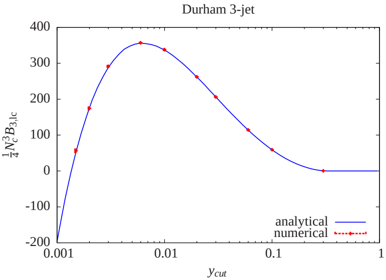
$$\frac{\sigma_{n-jet}}{\sigma_0} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-2} A_n(\mu) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-1} B_n(\mu) + \mathcal{O}(\alpha_s^n).$$

- We expand the NLO perturbative coefficient B_n in $1/N_c$.

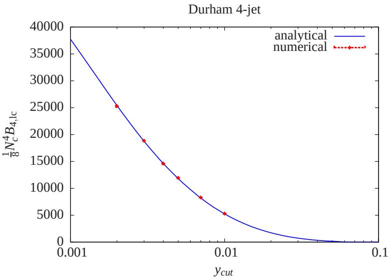
$$B_n = N_c \left(\frac{N_c}{2}\right)^{n-1} \left[B_{n,lc} + \mathcal{O}\left(\frac{1}{N_c}\right)\right]$$

- We calculate the NLO coefficient in leading colour up to $n = 7$ i.e. up to eight-point functions.
- We plot $N_c(N_c/2)^{n-1} B_{n,lc}$ over the resolution parameter y_{cut} in the Durham jet algorithm.





Recent results - $e^+e^- \to jets$



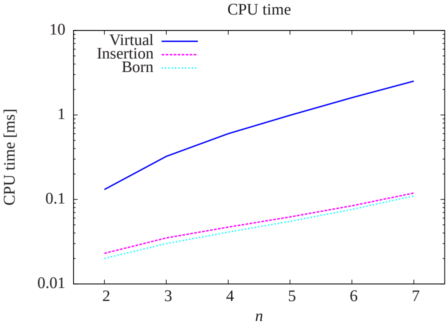
y_{cut}	$\frac{N_c^4}{8} A_{5,lc}$	$\frac{N_c^5}{16} B_{5,lc}$
0.002	$(5.0529 \pm 0.0004) \cdot 10^3$	$(4.275 \pm 0.006) \cdot 10^5$
0.001	$(1.3291 \pm 0.0001) \cdot 10^4$	$(1.050 \pm 0.026) \cdot 10^6$
0.0006	$(2.4764 \pm 0.0002) \cdot 10^4$	$(1.84 \pm 0.15) \cdot 10^6$

y_{cut}	$\frac{N_c^5}{16} A_{6,lc}$	$\frac{N_c^6}{32} B_{6,lc}$
0.001	$(1.1470 \pm 0.0002) \cdot 10^5$	$(1.46 \pm 0.04) \cdot 10^7$
0.0006	$(2.874 \pm 0.002) \cdot 10^5$	$(3.88 \pm 0.18) \cdot 10^7$

y_{cut}	$\frac{N_c^6}{32} A_{7,lc}$	$\frac{N_c^7}{64} B_{7,lc}$
0.0006	$(2.49 \pm 0.08) \cdot 10^6$	$(5.4 \pm 0.3) \cdot 10^8$

Computational performance

- We plot the CPU time required for one evaluation of the Born contribution, the insertion term and the virtual term as a function of the number of external final state partons n.
- The insertion term is almost as cheap as the Born contribution.
- The virtual part has the same scaling behaviour as the Born contribution.
- All three contributions scale asymptotically as n^4.
- The practical limit of our method arise from the fact that the number of evaluations required to reach a certain accuracy increases with n.
- The calculation of the seven-jet rate takes a few days on a cluster with 200 cores.



Summary and outlook

Summary

- In this talk the extension of the subtraction method to the virtual corrections was presented.
- The major ingredients...
  - ... the subtraction terms,
  - ... the recurrence relations,
  - ... and a suitable contour deformation

  was presented.
- We demonstrated the functionality of the algorithm on the process $e^+e^- \to jets$.

Outlook

- LHC physics.

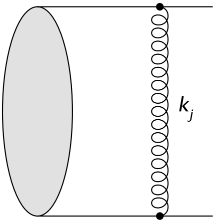
Thank you for your attention!

ltr: Daniel Götz, Sebastian Becker, Stefan Weinzierl, Christopher Schwan, Christian Reuschle.

Definition of the soft singularity

- Propagator j is soft and
- propagator j corresponds to a gluon and
- the external particles j and $j+1$ are on-shell.

$$k_j \to 0 \quad \text{and} \quad p_j^2 = 0 \quad \text{and} \quad p_{j+1}^2 = 0 \quad \Rightarrow \quad k_{j-1}^2 = k_j^2 = k_{j+1}^2 = 0$$



Derivation of the soft subtraction term

- For each gluon in the loop we define the soft subtraction function

$$S_{j,soft}(\mathfrak{G}) = \frac{\lim\limits_{k_j \to 0} \left\{ k_{j-1}^2 k_j^2 k_{j+1}^2 F(\mathfrak{G}, k) \right\}}{k_{j-1}^2 k_j^2 k_{j+1}^2}$$

- The sum of the soft subtraction function over all one-loop diagrams is proportional to the tree-level amplitude $A_j^{(0)}$.
- To get the full soft subtraction term we have to sum over all gluons in the loop,

$$G_{soft}^{(1)} = i \sum_{j \in I_g} \frac{4 p_j \cdot p_{j+1}}{k_{j-1}^2 k_j^2 k_{j+1}^2} A_j^{(0)}$$

- The integrated soft subtraction term yields the expected pole-structure.

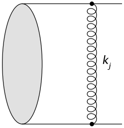
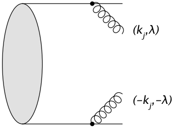
$$S_\epsilon^{-1} \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} G_{soft}^{(1)} = -\frac{1}{(4\pi)^2} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \sum_{j \in I_g} \frac{2}{\epsilon^2} \left(\frac{-2 p_j \cdot p_{j+1}}{\mu^2} \right)^{-\epsilon} A_j^{(0)} + \mathcal{O}(\epsilon).$$

Derivation of the soft subtraction term

- In the soft limit we replace the metric tensor $g_{\mu\nu}$ of propagator j by a polarisation sum and gauge terms.

$$g_{\mu\nu} = \sum_{\lambda} \epsilon^{\mu}_{\lambda}(k_j, n)\epsilon^{\nu}_{-\lambda}(k_j, n) - 2\frac{k_j^{\mu} n^{\nu} - k_j^{\nu} n^{\mu}}{2k_j \cdot n}$$

where n^{μ} is a light like reference vector.

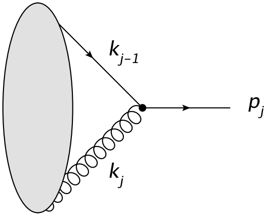
$\lim\limits_{k_j \to 0} \sum\limits_{\mathfrak{G}}$  $= \lim\limits_{k_j \to 0} \sum\limits_{\mathfrak{G}}$ 

- The terms proportional to $k_j^{\mu} n^{\nu}$ and $k_j^{\nu} n^{\mu}$ vanish due gauge invariance.
- The two "inserted" gluons lead in the soft limit to a tree-level amplitude, where these gluons are absent, times a eikonal factor $4p_j \cdot p_{j+1}$.

Definition of the collinear singularity

- Propagator $j-1$ is collinear to propagator j and
- propagator j or propagator $j-1$ corresponds to a gluon and
- the external particle j is massless and on-shell.

$$k_{j-1} || k_j \quad \text{and} \quad m_j = 0 \quad \text{and} \quad p_j^2 = 0 \quad \Rightarrow \quad k_{j-1}^2 = k_j^2 = 0$$



Derivation of the collinear subtraction term

- For each gluon in the loop we define the collinear subtraction function

$$S_{j,coll}(\mathfrak{G}) = \frac{\lim\limits_{k_{j-1}\|k_j} \left\{ k_{j-1}^2 k_j^2 F(\mathfrak{G}, k) \right\}}{k_{j-1}^2 k_j^2} - \text{soft double counting}$$

- The sum of the collinear subtraction function over all one-loop diagrams is proportional to the tree level amplitude $A_j^{(0)}$.
- We have to sum over all gluons in the loop,

$$G_{coll}^{(1)} = i \sum_{j \in I_g} (-2) \left(\frac{S_j g_{UV}(k_{j-1}^2, k_j^2)}{k_{j-1}^2 k_j^2} + \frac{S_{j+1} g_{UV}(k_j^2, k_{j+1}^2)}{k_j^2 k_{j+1}^2} \right) A_j^{(0)}.$$

$$S_q = 1,\ S_g = \frac{1}{2}, \quad \lim_{k_{j-1}||k_j} g_{UV}(k_{j-1}^2, k_j^2) = 1, \quad \lim_{k \to \infty} g_{UV}(k_{j-1}^2, k_j^2) = \mathcal{O}\left(\frac{1}{k}\right).$$

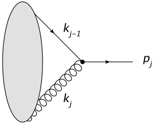
- The integrated collinear subtraction terms yields the expected pole structure:

$$S_\epsilon^{-1} \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} G_{coll}^{(1)} = -\frac{1}{(4\pi)^2} \frac{e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \sum_{j \in I_g} (S_j + S_{j+1}) \left(\frac{\mu_{UV}^2}{\mu^2} \right)^{-\epsilon} \frac{2}{\epsilon} A_j^{(0)} + \mathcal{O}(\epsilon).$$

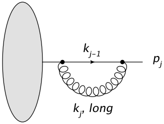
Derivation of the collinear subtraction term

- Only diagrams with collinear $q \to qg$ or $g \to gg$ splitting lead to a divergence after integration.
- As an example, the $q \to qg$ splitting.

$$\lim_{k_{j-1}\|k_j} \sum_{\mathfrak{G}}$$



$$= -\lim_{k_{j-1}\|k_j} \sum_{\tilde{\mathfrak{G}}}$$



The sum of the left side is almost gauge invariant, only the self energies of external legs are missing.

- The self-energy insertions on the external lines introduce a spurious $1/p_j^2$-singularity. We define $p_j = k_{j-1} - k_j$ slightly off shell by introducing the Sudakov parametrisation.

$$k_{j-1} = xp + k_\perp - \frac{k_\perp^2}{x}\frac{n}{(2p\cdot n)}, \quad -k_j = (1-x)p - k_\perp - \frac{k_\perp^2}{(1-x)}\frac{n}{(2p\cdot n)}.$$

- The singular parts of the self-energies are proportional to

$$P^{long}_{q\to qg} = -\frac{2}{2k_{j-1}\cdot k_j}\left(-\frac{2}{1-x}+2\right)\not{p}$$

- The terms with $2/(1-x)$ correspond to the soft singularities.

Contour deformation for massive QCD

- We define the massive propagator

$$D_i \quad = \quad (k - q_i)^2 - m_i^2$$

and deform the loop-momenta into the complex plane

$$\tilde{k} \quad = \quad k + i\kappa(k).$$

- We make the Ansatz for the deformation vector

$$\kappa \quad = \quad -\sum_{i \leq j} c_{ij}(k - v(k, q_i, q_j))$$

with the vectors $v \in \mathbb{R}^4$ and the coefficient $c_{ij} \in [0, 1]$.

- The imaginary part of the propagator reads with this Ansatz:

$$\mathrm{Im}\left(\tilde{D}_l\right) \quad = \quad -2\sum_{i \leq j} c_{ij}(k - q_l) \cdot (k - v)$$

- To avoid a wrong sign of the imaginary part the coefficients c_{ij} have to fulfil the condition

$$\{D_l \; = \; 0 \quad \text{and} - (k - q_l) \cdot (k - v) \; < \; 0\} \quad \Rightarrow \quad c_{ij} \; = \; 0$$

Contour deformation for massive QCD

- We define

$$c_{ij} = \prod_{l=1}^{n} \max\left\{h_\delta(k-q_l, m_l^2), h_\theta(k-q_l, k-v)\right\}$$

with the smooth functions h_δ and h_θ:

$$\begin{aligned} D_l &= 0 &\Rightarrow\quad h_\delta(k-q_l, m_l^2) &= 0 \\ -(k-q_l)\cdot(k-v) &< 0 &\Rightarrow\quad h_\theta(k-q_l, k-v) &= 0 \end{aligned}$$

- In detail

$$h_\delta(u, m^2) = \begin{cases} \dfrac{\left(|u^0|-\sqrt{\vec{u}^2+m^2}\right)^2}{\left(|u^0|-\sqrt{\vec{u}^2+m^2}\right)^2+M_1^2} & : \quad m^2 > 0 \\ \dfrac{\left(\sqrt{(u^0)^2-m^2}-|\vec{u}|\right)^2}{\left(\sqrt{(u^0)^2-m^2}-|\vec{u}|\right)^2+M_1^2} & : \quad m^2 < 0 \end{cases}$$

With M_1 a parameter depending on the typical energy scale of the process.

- The second function is given by

$$h_\theta(u,v) \quad = \quad h_\delta\left(\frac{u+v}{2}, \frac{(u-v)^2}{4}\right)\,\theta\,(-u\cdot v)$$

- To understand this function we rewrite the scalar product.

$$(k-q_j)\cdot(k-v) \quad = \quad \left(k-\frac{v+q_j}{2}\right)^2 - \left(\frac{v-q_j}{2}\right)^2$$

- This looks again like a massive propagator.
- The forbidden region is the interior of the mass shells.

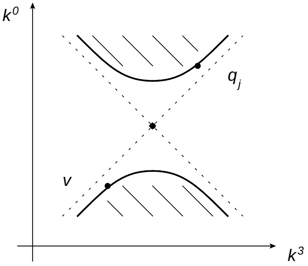
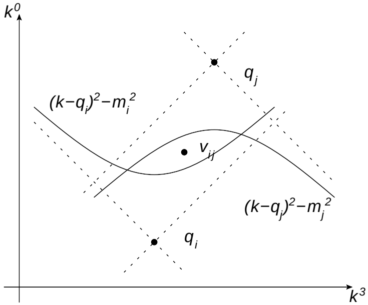


Figure: Region in the loop momenta space were $-(k-v)\cdot(k-q_i) < 0$;

- The deformation vector κ is a smooth function in the loop momenta and never yields to a wrong imaginary part.
- One have to ensure that for every singular surface a vector $v(k, q_i, q_j)$ exists such we deform correctly.
- The vector $-(k - v)$ deforms correctly if the vector v lies inside the surface.



Mapping of the variables for I_{int}

- The variables $(\rho, \zeta, \theta, \phi)$ we generate as follows:

$$
\begin{aligned}
\rho &= \ln\left(1 + \frac{\mu_0}{|p|}\tan\frac{\pi}{2}u_0\right) \\
\zeta &= \pi u_1 \\
\theta &= \begin{cases} \arccos\left[(1+\epsilon)\left(\frac{1+\epsilon}{\epsilon}\right)^{-2u_2} - \epsilon\right] & 0 \le u_2 < \frac{1}{2}, \\ \arccos\left[\epsilon - (1+\epsilon)\left(\frac{1+\epsilon}{\epsilon}\right)^{-2(1-u_2)}\right] & \frac{1}{2} \le u_2 \le 1, \end{cases} \\
\phi &= 2\pi u_3
\end{aligned}
$$

with $(u_0, u_1, u_2, u_3) \in [0, 1]$ random numbers and

$$
\epsilon = \sinh\rho \sin\zeta .
$$

Contour deformation and mapping for I_{ext}

- In I_{ext} only the UV propagator, $\bar{k} - \mu_{UV}^2$ appear.
- The contour deformation is rather simple.

$$k = \tilde{k} + \imath\kappa, \qquad \kappa^\mu = g_{\mu\nu}\left(\tilde{k}^\nu - Q^\nu\right).$$

- We have

$$\bar{k}^2 - \mu_{UV}^2 = 2\imath(\tilde{k} - Q)\circ(\tilde{k} - Q) - \mu_{UV}^2$$

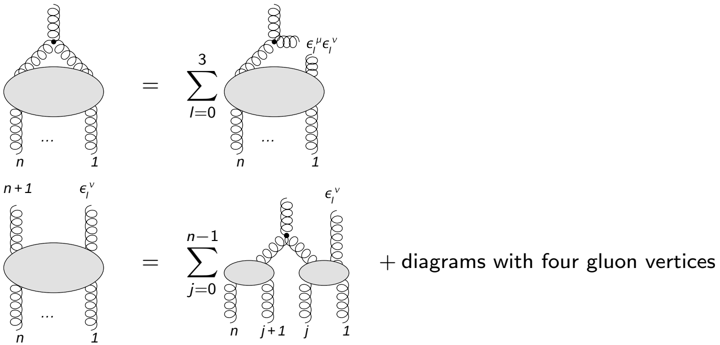
- The sampling for $\bar{k}_{real} = \tilde{k} - Q$ is

$$\bar{k}_{real} = k_E \begin{pmatrix} \cos\zeta \\ \sin\zeta\sin\theta\sin\phi \\ \sin\zeta\sin\theta\cos\phi \\ \sin\zeta\cos\theta \end{pmatrix}, \qquad \begin{aligned} k_E &= \mu_1\sqrt{\tan\tfrac{\pi}{2}u_0} \\ \zeta &= \arccos(1-2u_1) \\ \theta &= \arccos(1-2u_2) \\ \phi &= 2\pi u_3 \end{aligned}$$

with $u_0, u_1, u_2, u_3 \in [0,1]$ random numbers.

Off-shell recurrence relations(one-loop)

- We cut a gluon line by using $\sum_{l=0}^{3} \epsilon_l^{\mu} \epsilon_l^{\nu} = g^{\mu\nu}$.



- The recursion starts with $n = 0$ i.e. no external gluon is left at the r.h.s and is given by $-\frac{\imath \epsilon_l^{\nu}}{k_{n+1}^2}$.
- Ghost loops are calculated similarly and closed fermion loops do not appear in the leading colour approximation.

Splitting in I_{ext} and I_{int}

- We define the function

$$f_{UV}(k) = \prod_{j=1}^{n} \frac{k_j^2 - m_j^2}{\bar{k}^2 - \mu_{UV}^2}.$$

- We spilt the integration into an exterior and an interior region, $I = I_{ext} + I_{int}$, with

$$I_{ext} = \int \frac{d^4k}{(2\pi)^4} f_{UV}(k) \frac{N(k)}{\prod_{j=1}^{n}\left(k_j^2 - m_j^2\right)}$$

$$I_{int} = \int \frac{d^4k}{(2\pi)^4} (1 - f_{UV}(k)) \frac{N(k)}{\prod_{j=1}^{n}\left(k_j^2 - m_j^2\right)}$$

- The pole structure of I_{ext} is very simple.
- We can choose Q in $\bar{k} = k - Q$ such that $(1 - f_{UV}(k))$ drops off with an extra power of $1/|k|$ for $k \to \infty$.
- Because f_{UV} is a meromorphic[1] function we can choose different integration contours for I_{ext} and I_{int}.

[1] Wikipedia:In complex analysis, a meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all D except a set of isolated points, which are poles for the function.

The mapping of I_{int}

- The collinear regions are defined by the line segments $k = q_i + xp_{i+1}$ and are important for the numerics.
- We split the original integral into several channels, such that for each line segment corresponds to a separate channel.
- w_i not a holomorphic function; We have to use the same contour in each channel.

$$I = \sum_{i=1}^{n} \int \frac{d^4k}{(2\pi)^4} w_i(k) f(k), \qquad w_i \geq 0, \quad \sum_{i=1}^{n} w_i = 1.$$

- The weights w_i defined by

$$w_i = \frac{\left(\frac{1}{|(k-q_i)^2||(k-q_{i+1})^2|}\right)^2}{\sum_{j=1}^{n}\left(\frac{1}{|(k-q_j)^2||(k-q_{j+1})^2|}\right)^2}$$

- The weights have the properties that

$$\begin{aligned} w_i &= 1 \quad \text{if} \quad k = q_i + xp_{i+1} \\ w_i &= 0 \quad \text{if} \quad k = q_j + xp_{j+1} \quad \text{if} \quad i \neq j \end{aligned}$$

The mapping of I_{int}

- By choose the mapping for a given channel wisely one can improve the numerical performance of the Monte Carlo.
- First we write an external momenta in spherical coordinates.

$$p_i = ||p_i||_{euc} R_3^{(i)} \cdot R_2^{(i)} \cdot R_1^{(i)} \cdot \hat{e}_0, \quad \hat{e}_0 = (1,0,0,0).$$

- The mapping for the ith channel is given by

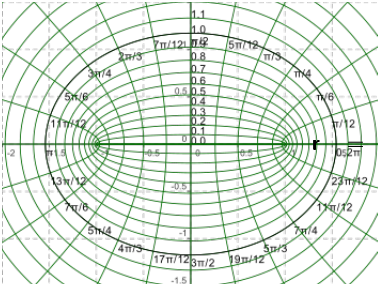
$$k = q_i + \frac{||p_{i+1}||_{euc}}{2} R_3^{(i+1)} \cdot R_2^{(i+1)} \cdot R_1^{(i+1)} \cdot \begin{pmatrix} \cosh\rho\cos\zeta + 1 \\ \sinh\rho\sin\zeta\cos\theta \\ \sinh\rho\sin\zeta\sin\theta\cos\phi \\ \sinh\rho\sin\zeta\sin\theta\sin\phi \end{pmatrix}$$

- with

$$\rho \in [0,\infty), \quad \zeta \in [0,\pi], \quad \theta \in [0,\pi], \quad \phi \in [0,2\pi].$$

The mapping of I_{int}

- Here we introduced elliptic coordinates.(Picture from Wikipedia)



$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} \cosh u \cos v \\ \sinh u \sin v \end{pmatrix}$$

- For $\rho = 0$ the loop momentum is on the critical line segment,

$$k = q_i + \underbrace{\frac{1}{2}(1 + \cos\zeta)}_{\in[0,1]} p_{i+1}$$

- This improves the Monte Carlo, because the VEGAS algorithm can offer significant improvements only as far as the integrand's characteristic regions are aligned with the coordinate axes.
- We evaluate the integrand at ϕ and $(\phi + \pi) \mod (2\pi)$, and at θ and $\pi - \theta$ to average out periodic behaviour of the integrand.

Improved UV-subtraction

- We observed large oscillations in the UV region whenever an external invariant approaches the jet resolution parameter.
- The leading contribution in the UV limit is of the order $1/|k|^5$.
- To improve the situation we subtracting out the order $1/|k|^5$ and $1/|k|^6$ terms in the propagator- and three-particle vertex corrections.
- We also modify the IR subtraction terms such that they fall off like $1/|k|^7$.
- We evaluate the integrand always at the point $\bar{k}$ and $-\bar{k}$ together. Terms which scale with an odd power of $|k|$ in the UV region drop out.