Scattering Amplitudes with Open Loops

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based on

F. Cascioli, P. Maierhöfer and S.P.
PRL **108** (2012) 111601 [arXiv:1111.5206]

LoopFest XI, University of Pittsburg, May 10, 2012

Outline of the talk

1. **NLO calculations: challenges, techniques**
2. **Treatment of colours, helicities and loop momenta**
3. **Open loops generator**
4. **Implementation and benchmark results**

Multi-particle processes at NLO

Various general techniques to perform NLO calculations

- exist since decades but badly fail when applied to multi-particle processes

Important $2 \to 4$ processes at the LHC (2005/2007 Les Houches priority list)

$$\mathrm{pp} \to \quad \mathrm{t\bar{t}b\bar{b}}, \qquad \mathrm{t\bar{t}}jj, \qquad VV\mathrm{b\bar{b}}, \qquad VVjj, \qquad Vjjj, \qquad \mathrm{b\bar{b}b\bar{b}}$$

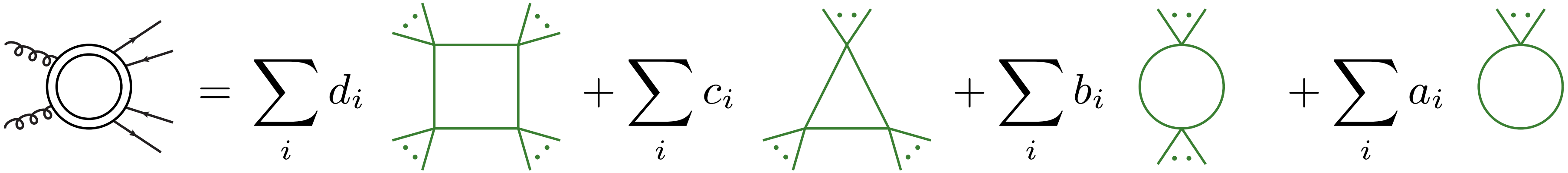
Multi-particle challenges at one loop

- *AUTOMATION* and *FLEXIBILITY:* max applicability/min man power
- *SPEED:* avoid big clusters
- *STABILITY:* avoid Gram determinants

The "NLO revolution"

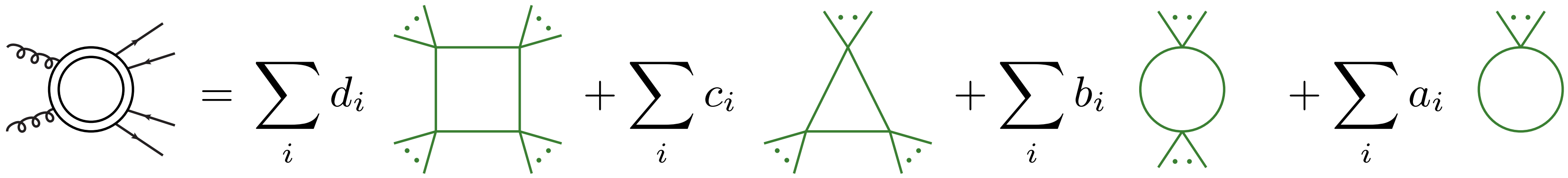
$\mathrm{pp} \to \mathrm{t\bar{t}b\bar{b}}$	[Bredenstein, Denner, Dittmaier, S.P. '09]
	[Bevilacqua, Czakon, Papadopoulos, Pittau, Worek '09]
$\mathrm{pp} \to \mathrm{t\bar{t}}jj$	[Bevilacqua, Czakon, Papadopoulos, Pittau, Worek '10]
$\mathrm{pp} \to \mathrm{WWb\bar{b}}$	[Denner, Dittmaier, Kallweit, S.P. '11]
	[Bevilacqua, Czakon, van Hameren, Papadopoulos, Worek '11]
$\mathrm{pp} \to \mathrm{b\bar{b}b\bar{b}}$	[Greiner, Guffanti, Reiter, Reuter '11]
$\mathrm{pp} \to \mathrm{WW}jj$	[Melia, Melnikov, Rontsch, Zanderighi '10]
	[Greiner, Heinrich, Mastrolia, Ossola, Reiter, Tramontano '12]
$\mathrm{pp} \to \mathrm{W/Z} + 3j$	[Ellis, Melnikov, Zanderighi '09]
	[Berger, Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre '09–'10]
$\mathrm{pp} \to \mathrm{W/Z} + 4j$	[Berger, Bern, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre '11]
$\mathrm{pp} \to 4j$	[Bern, Diana, Dixon, Febres Cordero, Hoeche, Kosower, Ita, Maitre, Ozeren '11]
$\mathrm{pp} \to W\gamma\gamma\, j$	[Campanario, Englert, Rauch, Zeppenfeld '11]
$\mathrm{e^+e^-} \to 7j$	[Becker, Goetz, Reuschle, Schwan, Weinzierl '11]

Main one-loop approaches

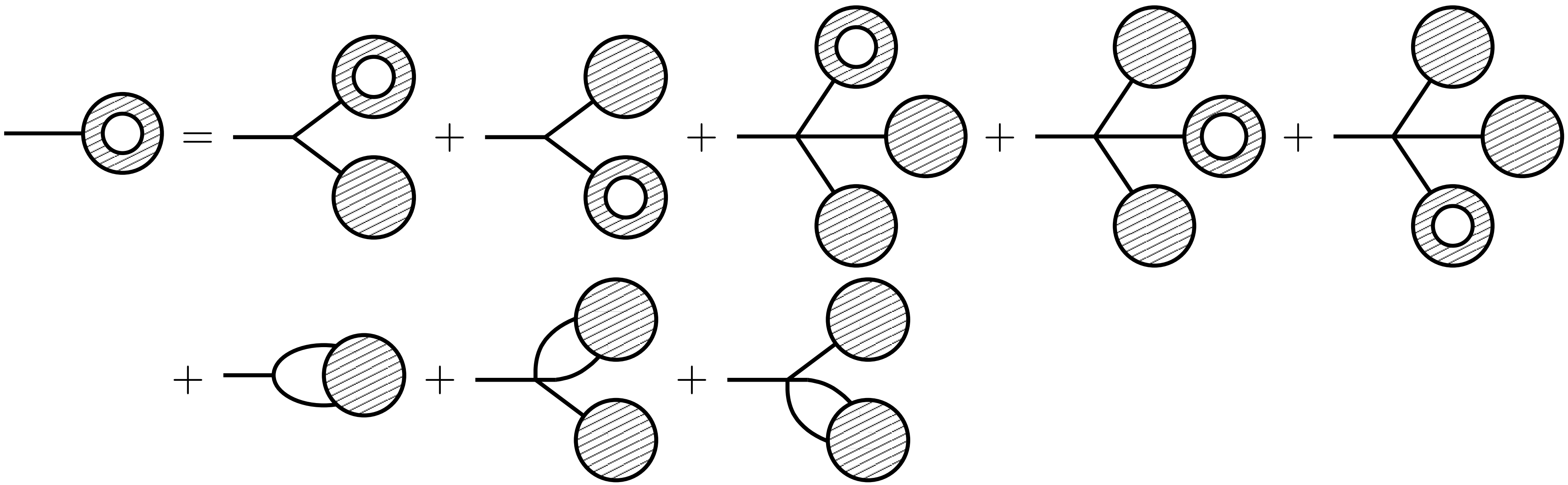


	Standard	On-shell
Reduction to SIs	tensor integrals	on-shell/OPP
Amplitudes	loop diagrams	tree amplitudes
	mostly algebraic	mostly numerical
Max. multiplicity	$2 \to 4$	$2 \to 5$
Speed for $n_{\mathrm{part}} \leq 6$	very high	slower
Automation ($n_{\mathrm{part}} > 5$)	very large codes	highly automatic
Stability	stable expansions	quadruple precision

Main one-loop approaches



	Standard	**Open Loops**	**On-shell**
Reduction to SIs	tensor integrals	tensor int. & OPP	on-shell/OPP
Amplitudes	loop diagrams	open loops	tree amplitudes
	mostly algebraic	numerical recursion	mostly numerical
Max. multiplicity	$2 \to 4$	$2 \to 4(5)$	$2 \to 5$
Speed for $n_{\text{part}} \leq 6$	very high	very high	slower
Automation ($n_{\text{part}} > 5$)	very large codes	highly automatic	highly automatic
Stability	stable expansions	stable with TIs	quadruple precision



A. van Hameren, **JHEP** 0907 (2009) 088

- numerical 1-loop recursion
- off-shell currents and tensor integrals
- only colour-ordered N-gluon amplitudes

faster than on-shell approach for $n_{\mathrm{gluons}} \leq 10$!

Open Loops

- diagrammatic recursion
- fully general
- tensor-integral or OPP reduction

(2) Colours, helicities and loop momenta

(2.1) Diagrams and colour

Tree amplitudes and one-loop corrections as *sums of diagrams*

$$\mathcal{M} = \sum_{d} \mathcal{M}^{(d)}, \qquad \delta\mathcal{M} = \sum_{d'} \delta\mathcal{M}^{(d')} \qquad (\text{up to } 10^4 \text{ loops})$$

Scattering probability densities involve *colour* and *helicity* sums

$$\mathcal{W} = \sum_{\text{hel,col}} |\mathcal{M}|^2, \qquad \delta\mathcal{W} = \sum_{\text{hel,col}} 2\,\text{Re}\,(\mathcal{M}^*\delta\mathcal{M})$$

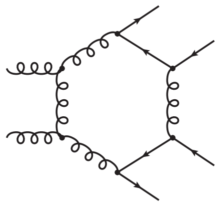
Colour sums at **zero cost** exploiting *factorisation of individual diagrams* into simple *colour factors* and *colour-stripped amplitudes*

$$\mathcal{M}^{(d)} = \mathcal{A}^{(d)}\mathcal{C}^{(d)}, \qquad \delta\mathcal{M}^{(d')} = \delta\mathcal{A}^{(d')}\mathcal{C}^{(d')}$$

Algebraic colour reduction and sums once and for all after diagram generation. Everything else is done numerically.

(2.2) Loop-momentum dependence

Structure of colour-stripped loop diagram



$$= \int \frac{\mathrm{d}^D q \, \mathcal{N}(q)}{D_0 D_1 \ldots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r} \int \frac{\mathrm{d}^D q \; q^{\mu_1} \ldots q^{\mu_r}}{D_0 D_1 \ldots D_{n-1}}$$

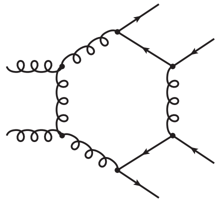
$$D_i = (q + p_i)^2 - m_i^2$$

Treatment of $\mathcal{N}(q)$ polynomial

- separate loop momenta from helicity-dependent coefficients $\mathcal{N}_{\mu_1 \ldots \mu_r}$
- recursive numerical algorithm for $\mathcal{N}_{\mu_1 \ldots \mu_r}$ (see later)

(2.2) Loop-momentum dependence

Structure of colour-stripped loop diagram



$$= \int \frac{\mathrm{d}^D q\, \mathcal{N}(q)}{D_0 D_1 \ldots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r} \int \frac{\mathrm{d}^D q\, q^{\mu_1} \ldots q^{\mu_r}}{D_0 D_1 \ldots D_{n-1}}$$

Usual $\mathcal{N}(q)$ from Feynman rules

$$\mathcal{N}(q) = g_\mathrm{S}^6 \left[\bar{u}_3 \gamma^{\nu_1} (\not{q} + \not{k}_4 + m_\mathrm{t}) \gamma^\alpha v_4\right] \left[\bar{u}_5 \gamma^{\nu_2} (\not{q} + \not{k}_2 - m_\mathrm{b}) \gamma^\alpha v_6\right] g^{\rho_1 \rho_2}$$
$$\times\, \varepsilon_1^{\mu_1} \left[g_{\mu_1 \nu_1} (p_1 - q - k_5)_{\rho_1} + g_{\nu_1 \rho_1} (2q + k_5)_{\mu_1} - g_{\rho_1 \mu_1} (p_1 + q)_{\nu_1}\right]$$
$$\times\, \varepsilon_2^{\mu_2} \left[g_{\mu_2 \nu_2} (p_2 + q + k_1)_{\rho_2} - g_{\nu_2 \rho_2} (2q + k_1)_{\mu_2} - g_{\rho_2 \mu_2} (p_2 - q)_{\nu_2}\right]$$

Separation of q-monomials

$$\mathcal{N}(q) = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r}\, q^{\mu_1} \ldots q^{\mu_r}$$

$$= \sum_{\substack{n_0 \ldots n_3 = 0 \\ n_0 + n_1 + n_2 + n_3 \le R}}^{R} \mathcal{N}_{n_0 \ldots n_3} \underbrace{q^0 \ldots q^0}_{n_0} \ldots \underbrace{q^3 \ldots q^3}_{n_3}$$

Systematic symmetrisation

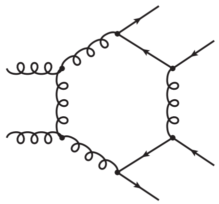
only $\binom{R+4}{4}$ symm. $\mathcal{N}_{n_0 \ldots n_3}$ components

R	0	1	2	3	4	5	6
$\binom{R+4}{4}$	1	5	15	35	70	126	210

6 particles (brace under columns 4–6)

(2.2) Loop-momentum dependence

Structure of colour-stripped loop diagram



$$= \int \frac{\mathrm{d}^D q \, \mathcal{N}(q)}{D_0 D_1 \ldots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r} \underbrace{\int \frac{\mathrm{d}^D q \; q^{\mu_1} \ldots q^{\mu_r}}{D_0 D_1 \ldots D_{n-1}}}_{T^{\mu_1 \cdots \mu_r} =}$$

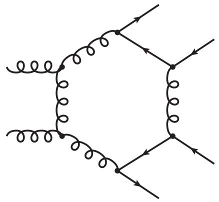
(A) Recursive reduction of **tensor integrals** to scalar integrals [Denner/Dittmaier '05]

$$= \int \mathrm{d}^D q \left[\sum_{i_1} \frac{a_{i_1}^{\mu_1 \cdots \mu_r}}{D_{i_1}} + \sum_{i_1, i_2} \frac{b_{i_1 i_2}^{\mu_1 \cdots \mu_r}}{D_{i_1} D_{i_2}} + \sum_{i_1, i_2, i_3} \frac{c_{i_1 i_2 i_3}^{\mu_1 \cdots \mu_r}}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{i_1, i_2, i_3, i_4} \frac{d_{i_1 i_2 i_3 i_4}^{\mu_1 \cdots \mu_r}}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} \right] + R_1^{\mu_1 \cdots \mu_r}$$

- numerical output with $0 \leq \mu_1 \ldots \mu_r \leq 3$ indices
- R_1 terms $\Longleftrightarrow$ D-dimensional relations between UV-divergent integrals
- avoid instabilities with Gram-determinant (and other) expansions

(2.2) Loop-momentum dependence

Structure of colour-stripped loop diagram

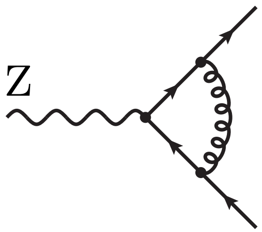


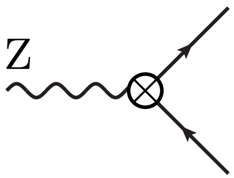
$$= \int \frac{\mathrm{d}^D q \, \mathcal{N}(q)}{D_0 D_1 \dots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \dots \mu_r} \underbrace{\int \frac{\mathrm{d}^D q \, q^{\mu_1} \dots q^{\mu_r}}{D_0 D_1 \dots D_{n-1}}}_{T^{\mu_1 \dots \mu_r} =}$$

Extra rational terms from $3 < \mu_1, \dots, \mu_r \le D - 1$

$$R_2 = \sum_{\mu_1 \dots \mu_r = 0}^{D-1} \mathcal{N}_{\mu_1 \dots \mu_r} \Big|_{D=4-2\varepsilon} T_{\mathrm{UV}}^{\mu_1 \dots \mu_r} - \sum_{\mu_1 \dots \mu_r = 0}^{3} \mathcal{N}_{\mu_1 \dots \mu_r} \Big|_{D=4} T_{\mathrm{UV}}^{\mu_1 \dots \mu_r}$$

From catalogue of 2-, 3- and 4-point 1PI diagrams (depends only on model)

$\Bigg($ 

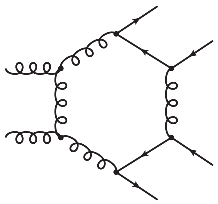
$\Bigg)_{R_2}$ = 

$$= -\frac{g_{\mathrm{S}}^2}{16\pi^2} \frac{N_c^2 - 1}{2N_c} \gamma^\mu (g_{\mathrm{V}}^{\mathrm{Z}} - g_{\mathrm{A}}^{\mathrm{Z}} \gamma_5) \qquad \text{etc.}$$

Draggiotis, Garzelli, Malamos, Papadopoulos, Pittau '09–'11   Shao, Zhang, Chao '11

(2.2) Loop-momentum dependence

Structure of colour-stripped loop diagram



$$= \underbrace{\int \frac{\mathrm{d}^D q\, \mathcal{N}(q)}{D_0 D_1 \ldots D_{n-1}}}_{\delta\mathcal{A}^{(d)} =} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r} \int \frac{\mathrm{d}^D q\; q^{\mu_1} \ldots q^{\mu_r}}{D_0 D_1 \ldots D_{n-1}}$$

(B) Direct OPP reduction at the amplitude level [Ossola, Papadopolous, Pittau '07]

$$= \int \mathrm{d}^D q \left[\sum_{i_1} \frac{a_{i_1}}{D_{i_1}} + \sum_{i_1,i_2} \frac{b_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{i_1,i_2,i_3} \frac{c_{i_1 i_2 i_3}}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{i_1,i_2,i_3,i_4} \frac{d_{i_1 i_2 i_3 i_4}}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} \right] + R_1$$

- **repeated $\mathcal{N}(q)$ evaluations** at on-shell multiple cuts (complex q)
- **very efficient with** $\mathcal{N}(q) = \sum \mathcal{N}_{\mu_1 \ldots \mu_r}\, q^{\mu_1} \ldots q^{\mu_r}$ **construction** [see also tensorial reconstruction Heinrich, Ossola, Reiter, Tramontano '10]
- same R_2 terms
- numerical instabilities require quadruple precision

(2.3) Commuting helicity summation and tensor reduction

Seperation of helicity and loop-momentum dependence very useful

$$\delta\mathcal{M}^{(d')} = \mathcal{C}^{(d')} \int \frac{\mathrm{d}^D q}{D_0 D_1 \dots D_{n-1}} \left\{ \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \dots \mu_r} \, q^{\mu_1} \dots q^{\mu_r} \right\}$$

No CPU expensive helicity-by-helicity reduction: first sum and then reduce only once

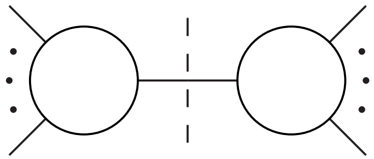
$$\sum_{\mathrm{hel,col}} \left[\mathcal{M}^* \delta\mathcal{M}^{(d')} \right] = \int \frac{\mathrm{d}^D q}{D_0 D_1 \dots D_{n-1}} \left\{ \sum_{r=0}^{R} \sum_{\mathrm{hel,col}} \left[\mathcal{M}^* \mathcal{C}^{(d)} \mathcal{N}_{\mu_1 \dots \mu_r} \right] q^{\mu_1} \dots q^{\mu_r} \right\}$$

Saves lot of CPU, works also with OPP reduction $\Rightarrow$ further strong OPP speed-up!

(3) Tree and open-loops generator

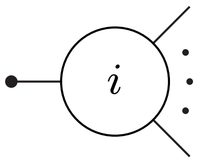
(3.1) Tree generator

Colour-stripped **tree diagrams** are built by merging *sub-trees*



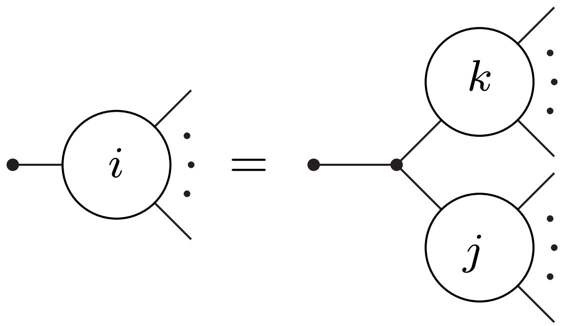
individual tree diagram

Sub-tree amplitudes are handled as *numerical* n-tuples

$w^\beta(i) =$ 

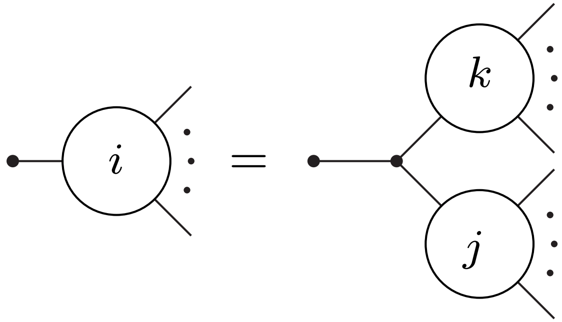
$\beta \leftrightarrow$ off-shell line spin

and *recursively merged* by attaching vertices and propagators



$$w^\beta(i) = \frac{X^\beta_{\gamma\delta}(i,j,k)}{p_i^2 - m_i^2}\, w^\gamma(j)\, w^\delta(k)$$

(sub-tree $=$ individual topology with off-shell line $\neq$ off-shell current)

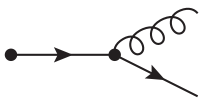


$$w^{\beta}(i) = \frac{X^{\beta}_{\gamma\delta}(i,j,k)}{p_i^2 - m_i^2}\, w^{\gamma}(j)\, w^{\delta}(k)$$

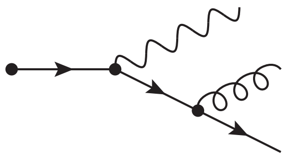
Example

$w_\alpha(1) =$  $= \bar{u}_\alpha(p_1, \lambda_1)$

$w_\mu(2) =$  $= \epsilon^*_\mu(p_2, \lambda_2)$

$w_\beta(12) =$  $= \dfrac{g_{\mathrm{S}}\,[(\not{p}_{12} + m)\gamma^\mu]_{\alpha\beta}}{p_{12}^2 - m^2}\, w_\alpha(1)\, w_\mu(2)$

$w_\nu(3) =$  $= \epsilon^*_\nu(p_3, \lambda_3)$

$w_\gamma(123) =$  $= \dfrac{e\,[(\not{p}_{123} + m)\gamma^\nu(1-\gamma_5)]_{\beta\gamma}}{2\sqrt{2}s_{\mathrm{W}}(p_{123}^2 - m^2)}\, w_\beta(12)\, w_\nu(3)$

etc. . .

. . . until sub-trees cover full set of Feynman diagrams for given process

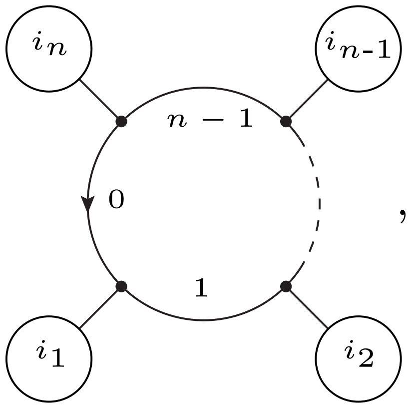
Completely generic and automatic (similar to `Madgraph` and `HELAS`)

- many diagrams *related* by presence of *common sub-trees* $\Rightarrow$ **strong speed-up!**

(3.2) Cut loops and Open loops

A colour-stripped **n-point loop diagram** is an **ordered set of n sub-trees** connected by loop propagators $D_i = (q+p_i)^2 - m_i^2 + \mathrm{i}\varepsilon$

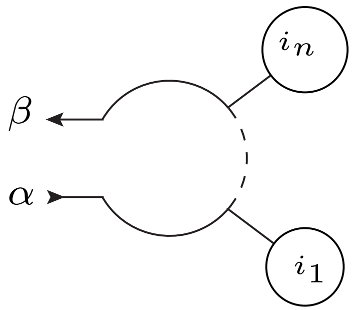
$$\int \frac{\mathrm{d}^D q\, \mathcal{N}(\mathcal{I}_n; q)}{D_0 D_1 \dots D_{n-1}} =$$



$$, \qquad \mathcal{I}_n = \{i_1, \dots, i_n\}$$

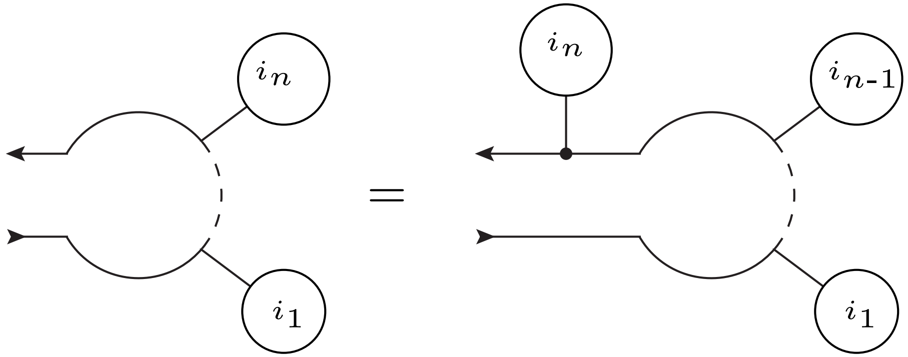
The **OPP-reduction** input $\mathcal{N}(\mathcal{I}_n; q)$ can be obtained by constructing **cut loops** with *tree-like generators*

$$\mathcal{N}_\alpha^\beta(\mathcal{I}_n; q) =$$



$$, \qquad \sum_\alpha \mathcal{N}_\alpha^\alpha(\mathcal{I}_n; q) = \mathcal{N}(\mathcal{I}_n; q)$$

Sub-trees along the loop are recursively attached to each other



Standard cut-loop construction

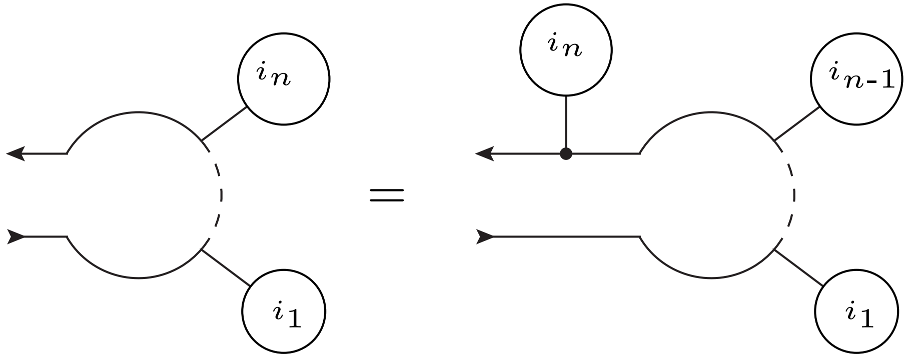
Build n-point cut loops by **merging lower-point cut loops and sub-trees**

$$\mathcal{N}_{\alpha}^{\beta}(\mathcal{I}_n;q) = X^{\beta}_{\gamma\delta}(\mathcal{I}_n, i_n, \mathcal{I}_{n-1})\, \mathcal{N}_{\alpha}^{\gamma}(\mathcal{I}_{n-1};q)\, w^{\delta}(i_n)$$

applying **conventional tree recursion**

- very high one-loop automation level in `Helac-NLO` (off-shell-current recursion), `MadLoop` (Feynman diagrams)
- CPU expensive OPP evaluations at multiple q-values since *tree algorithms conceived for fixed momenta*

Nature of loop amplitudes requires loop-momentum *functional* dependence!



Open-loops construction

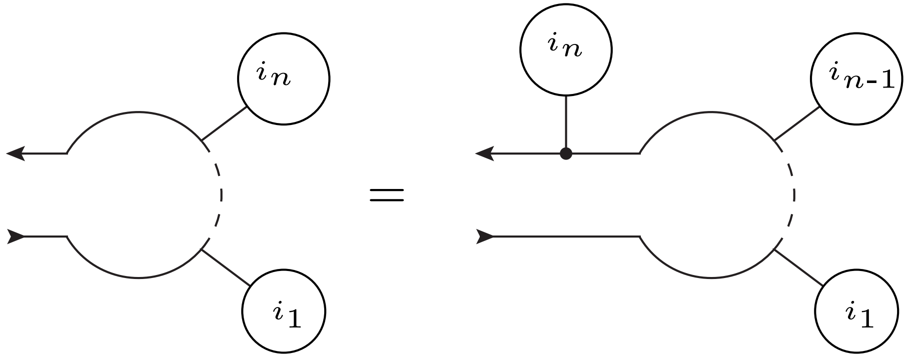
Handle building blocks of recursion as ***polynomials*** **in the loop momentum** q

$$\underbrace{\mathcal{N}^{\beta}_{\alpha}(\mathcal{I}_n;q)}_{\sum_{r=0}^{n}\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_n)\,q^{\mu_1}\ldots q^{\mu_r}} = \underbrace{X^{\beta}_{\gamma\delta}(\mathcal{I}_n,i_n,\mathcal{I}_{n-1})}_{Y^{\beta}_{\gamma\delta}+q^{\nu}\,Z^{\beta}_{\nu;\gamma\delta}}\;\underbrace{\mathcal{N}^{\gamma}_{\alpha}(\mathcal{I}_{n-1};q)}_{\sum_{r=0}^{n-1}\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_{n-1})\,q^{\mu_1}\ldots q^{\mu_r}}\;w^{\delta}(i_n)$$

and construct polynomial coefficients with

$$\underbrace{\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_n)}_{\text{“open loop”}} = \left[Y^{\beta}_{\gamma\delta}\,\mathcal{N}^{\gamma}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_{n-1}) + Z^{\beta}_{\mu_1;\gamma\delta}\,\mathcal{N}^{\gamma}_{\mu_2\ldots\mu_r;\alpha}(\mathcal{I}_{n-1})\right]\;w^{\delta}(i_n)$$

encodes full q-dependence of cut loop



Open-loops construction

Handle building blocks of recursion as *polynomials* in the loop momentum q

$$\underbrace{\mathcal{N}^{\beta}_{\alpha}(\mathcal{I}_n;q)}_{\sum_{r=0}^{n}\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_n)\,q^{\mu_1}\ldots q^{\mu_r}} = \underbrace{X^{\beta}_{\gamma\delta}(\mathcal{I}_n,i_n,\mathcal{I}_{n-1})}_{Y^{\beta}_{\gamma\delta}+q^{\nu}\,Z^{\beta}_{\nu;\gamma\delta}}\;\underbrace{\mathcal{N}^{\gamma}_{\alpha}(\mathcal{I}_{n-1};q)}_{\sum_{r=0}^{n-1}\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_{n-1})\,q^{\mu_1}\ldots q^{\mu_r}}\;w^{\delta}(i_n)$$

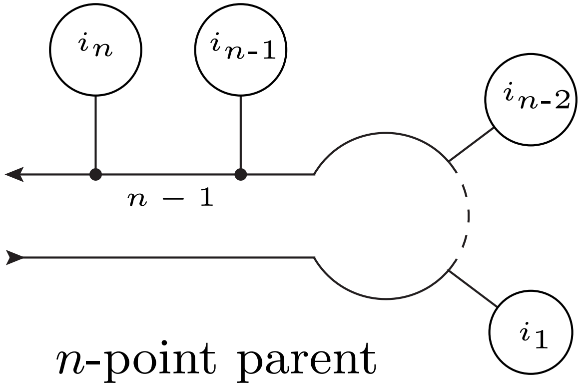
and construct polynomial coefficients with

$$\mathcal{N}^{\beta}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_n) = \left[Y^{\beta}_{\gamma\delta}\,\mathcal{N}^{\gamma}_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_{n-1}) + Z^{\beta}_{\mu_1;\gamma\delta}\,\mathcal{N}^{\gamma}_{\mu_2\ldots\mu_r;\alpha}(\mathcal{I}_{n-1})\right]\,w^{\delta}(i_n)$$

- fast OPP reduction of $\mathcal{N}(q)$ or tensor integrals with $\mathcal{N}_{\mu_1\ldots\mu_r}$
- combines *tree-recursion* and *tensor-integral* language
- simple concept but entirely new generator (fully flexible and automatic)

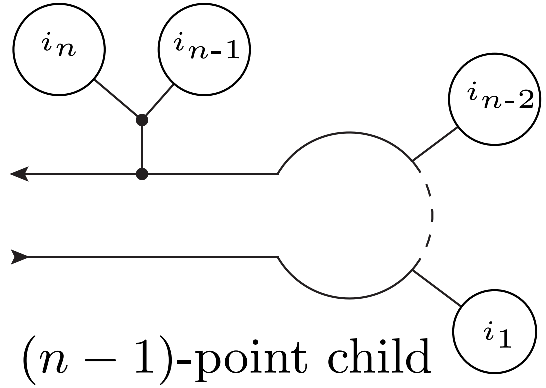
(3.3) Recursive construction of higher-point open loops

Construct n-point *"parent diagrams"* from pre-computed parts of $(n-1)$-point *"child diagrams"* using **pinch relations**



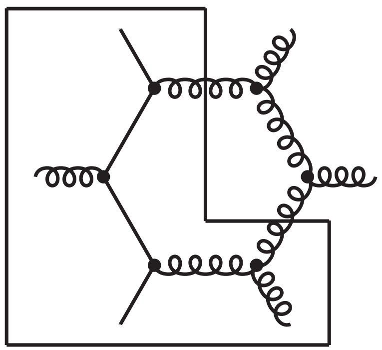
n-point parent

recycle $\mathcal{I}_{n-2}$ open loop
$\Longleftarrow$

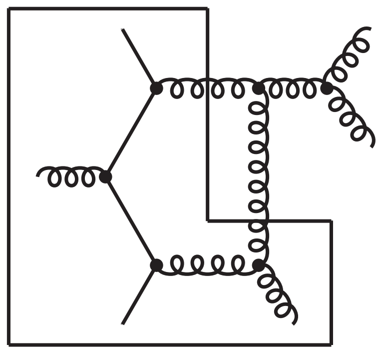


$(n-1)$-point child

Example



6-point parent

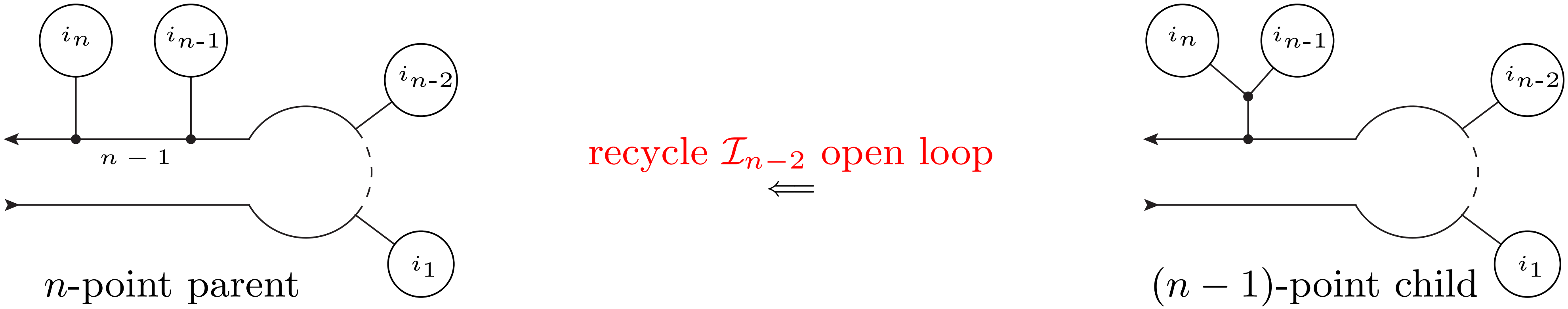


5-point child

Complicated diagrams require only "last missing piece" (always works in QCD!)

(3.3) Recursive construction of higher-point open loops

Construct n-point *"parent diagrams"* from pre-computed parts of $(n-1)$-point *"child diagrams"* using **pinch relations**



Pinch-invariant cut rule

Recycling of $\mathcal{I}_{n-2}$ guaranteed by invariance of cut wrt $i_n \oplus i_{n-1}$ merging

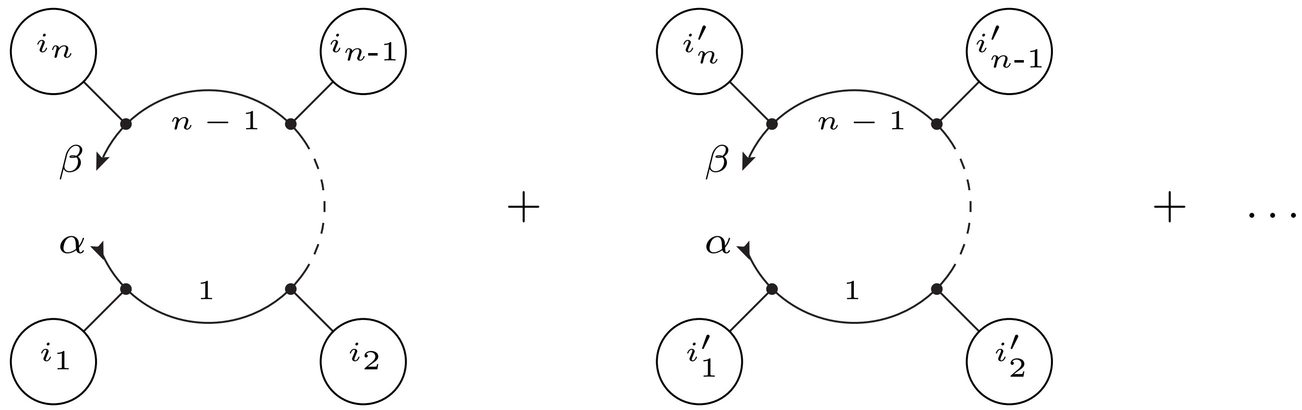
- ordering function $i_k \to \mathcal{S}(i_k) > 0$ with $\mathcal{S}(i_k \oplus i_l) > \max\{\mathcal{S}(i_k), \mathcal{S}(i_l)\}$
- choice of contiguous sub-trees i_1, i_n (cut position and direction) such that

$$\mathcal{S}(i_k) > \mathcal{S}(i_1) \quad \forall \quad k > 1, \qquad \mathcal{S}(i_n) > \mathcal{S}(i_2)$$

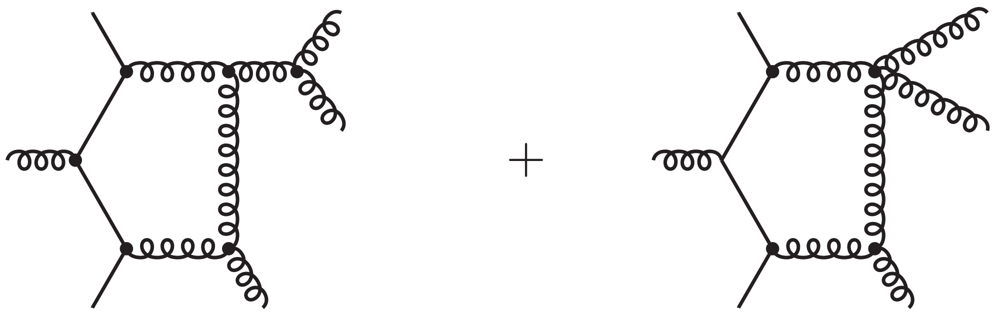
Additional rule (gluon cut priority) permits to avoid "childless" diagrams in QCD
$\Rightarrow$ **optimal efficiency**

Tensor integral or OPP reduction only once per 1-loop topology

Diagrams with identical 1-loop topology are interfered with Born, summed over col/hel, combined in a **single open loop** ⇒ **CPU expensive reduction only once**



Example



(4) Implementation and benchmark results

OpenLoops Implementation

Full automation of 1-loop QCD corrections to SM processes

- process-definition file $\Rightarrow$ `Fortran 90` tree & 1-loop matrix elements
- R_2 and UV counterterms, IR subtraction terms (CS I-operator)
- Cheks: UV/IR cancellations, Ward identities, cut loops $\equiv$ open loops

Internal structure of the program

(1)	Topologies, field insertions	`FeynArts`
(2)	Skeleton of recursion, colour, code generation	`OpenLoops` (`Mathematica`)
(3)	Merging of sub-trees and open loops	`OpenLoops` (`Fortran 90`)
(4)	Tensor-integral reduction	`COLLIER` (Denner/Dittmaier/Hofer)
	OPP reduction	`CutTools 1.6.5 / Samurai 1.0`

Speed and stability studies

Four families of $2 \to 2, 3, 4$ reactions with $n = 0, 1, 2$ gluons

- $\mathrm{u\bar{u}} \to \mathrm{t\bar{t}} + n\mathrm{g}$
- $\mathrm{u\bar{u}} \to \mathrm{W^+W^-} + n\mathrm{g}$
- $\mathrm{u\bar{d}} \to \mathrm{W^+g} + n\mathrm{g}$
- $\mathrm{gg} \to \mathrm{t\bar{t}} + n\mathrm{g}$

Airm of technical studies

- performance for non-trivial processes of Les Houches priority list
- scaling with particle multiplicity

Standard processor and compiler

- single `Intel i5-750` core (not a cluster!)
- `ifort 10.1` (similar performance with `gfortran`)

Automation and flexibility

Process	size [MB]	t_{code} [s]
$u\bar{u} \to t\bar{t}$	0.1	2.2
$u\bar{u} \to W^+W^-$	0.1	7.2
$u\bar{d} \to W^+g$	0.1	4.2
$gg \to t\bar{t}$	0.2	5.4
$u\bar{u} \to t\bar{t}g$	0.4	12.8
$u\bar{u} \to W^+W^-g$	0.4	39.8
$u\bar{d} \to W^+gg$	0.5	22.9
$gg \to t\bar{t}g$	1.2	52.9
$u\bar{u} \to t\bar{t}gg$	3.6 (200)*	236 ($\sim 10^6$)*
$u\bar{u} \to W^+W^-gg$	2.5 (1000)*	381.7 ($\sim 10^6$)*
$u\bar{d} \to W^+ggg$	4.2	366.2
$gg \to t\bar{t}gg$	16.0	3005

Size of matrix-element routines

- 100 kB to few MB (object files)

Code generation/compilation time

- few seconds to minutes
  (lot of room for improvement)

Improvement for $2 \to 4$ processes

- $\mathcal{O}(10^2$–$10^3)$ code compression!
- $\mathcal{O}(10^3)$ code-generation speed-up!

*pp $\to$ t$\bar{\text{t}}$b$\bar{\text{b}}$ & WWb$\bar{\text{b}}$ (Bredenstein, Denner, Dittmaier, Kallweit and S.P. '09–'11)

Speed of one-loop amplitudes with tensor integrals

Process	$t_{\mathrm{pol}}^{\mathrm{TI}}$ [ms]	n_{hel}	$t_{\mathrm{unpol}}^{\mathrm{TI}}$ [ms]
$u\bar{u} \to t\bar{t}$	0.25	2	0.27
$u\bar{u} \to W^+W^-$	0.25	2	0.28
$u\bar{d} \to W^+g$	0.39	2	0.43
$gg \to t\bar{t}$	0.89	4	1.16
$u\bar{u} \to t\bar{t}g$	3.5	4	4.2
$u\bar{u} \to W^+W^-g$	2.7	4	3.6
$u\bar{d} \to W^+gg$	5.3	4	6.7
$gg \to t\bar{t}g$	13.6	8	23.4
$u\bar{u} \to t\bar{t}gg$	56.2	8	88.4 (180)*
$u\bar{u} \to W^+W^-gg$	65.6	8	96.4 (180)*
$u\bar{d} \to W^+ggg$	134.5	8	190.5
$gg \to t\bar{t}gg$	335.0	16	725.0

(W/t decays to massless fermions)

Timings including col/hel sums

- $2 \to 2$: $t_{\mathrm{unpol}} \lesssim 1\,\mathrm{ms/point}$
- $2 \to 4$: $t_{\mathrm{unpol}} \lesssim 0.1\text{–}1\,\mathrm{s/point}$

unprecedented speed!

Efficient helicity summation

- for $2 \to 4$ processes saves factor

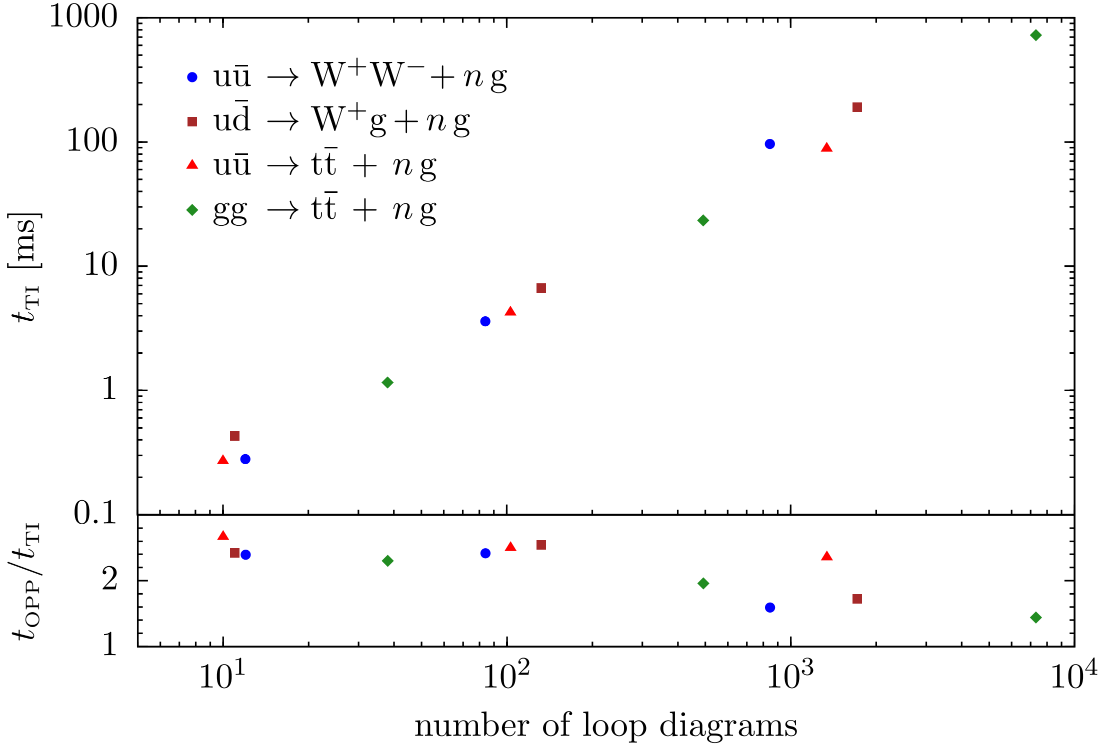
$$\frac{n_{\mathrm{hel}} t_{\mathrm{pol}}}{t_{\mathrm{unpol}}} \simeq 5\text{–}7$$

*$pp \to t\bar{t}b\bar{b}$ & $WWb\bar{b}$ (Bredenstein, Denner, Dittmaier, Kallweit and S.P '09/'10)

Scaling with diagram/particle number & OPP vs TIs

Colour/helicity summed timings with tensor integrals

- linear scaling with $n_{\rm diag}$ up to $n_{\rm diag} = \mathcal{O}(10^4)$ $\Rightarrow$ $n_{\rm diag} = \mathcal{O}(10^5)$ feasible
- $t_{2\to2} : t_{2\to3} : t_{2\to4} \lesssim 1 : 20 : 600$ $\Rightarrow$ $2 \to 5$ feasible



OPP reduction

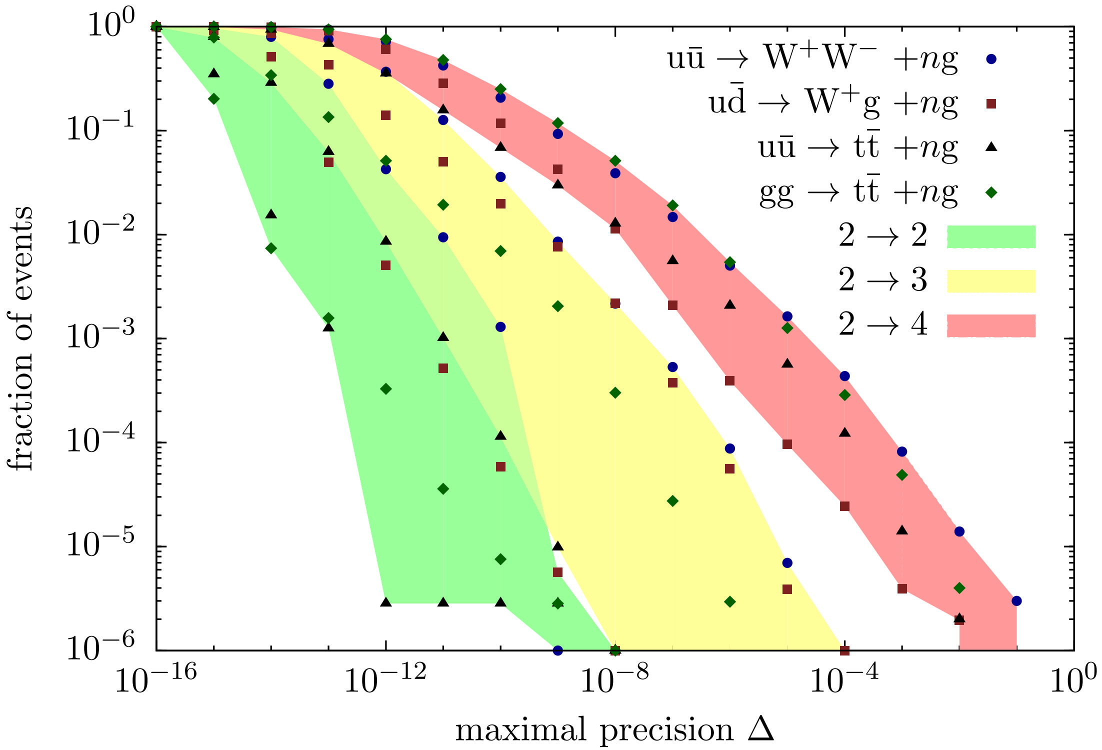
- $t_{\rm OPP}/t_{\rm TI} \simeq 1$ thanks to open loops!
- `CutTools 1.6.5` (double precision); similar speed with `Samurai 1.0`

Numerical stability with tensor integrals (double precision)

Study stability of 12 processes via rescaling of dimensionful parameters

$$\{p_i, m_j, \mu\} \to \{\xi p_i, \xi m_j, \xi \mu\}\,, \qquad \delta\mathcal{W} \to \delta\mathcal{W}' = \xi^K \delta\mathcal{W}, \qquad \Delta_\mathrm{S} = \frac{\xi^{-K}\delta\mathcal{W}' - \delta\mathcal{W}}{\delta\mathcal{W}}$$

Samples of 10^6 **homog. dist. points** ($\sqrt{\hat{s}} = 1\,\mathrm{TeV}$, $p_\mathrm{T} > 50\,\mathrm{GeV}$, $\Delta R_{ij} > 0.5$)



Average number of digits

- 11-15

Digits of best 99.9% (99.99%) **points**

$2 \to 2$ $-\log_{10}(\Delta_\mathrm{S}) > 10\,(9)$

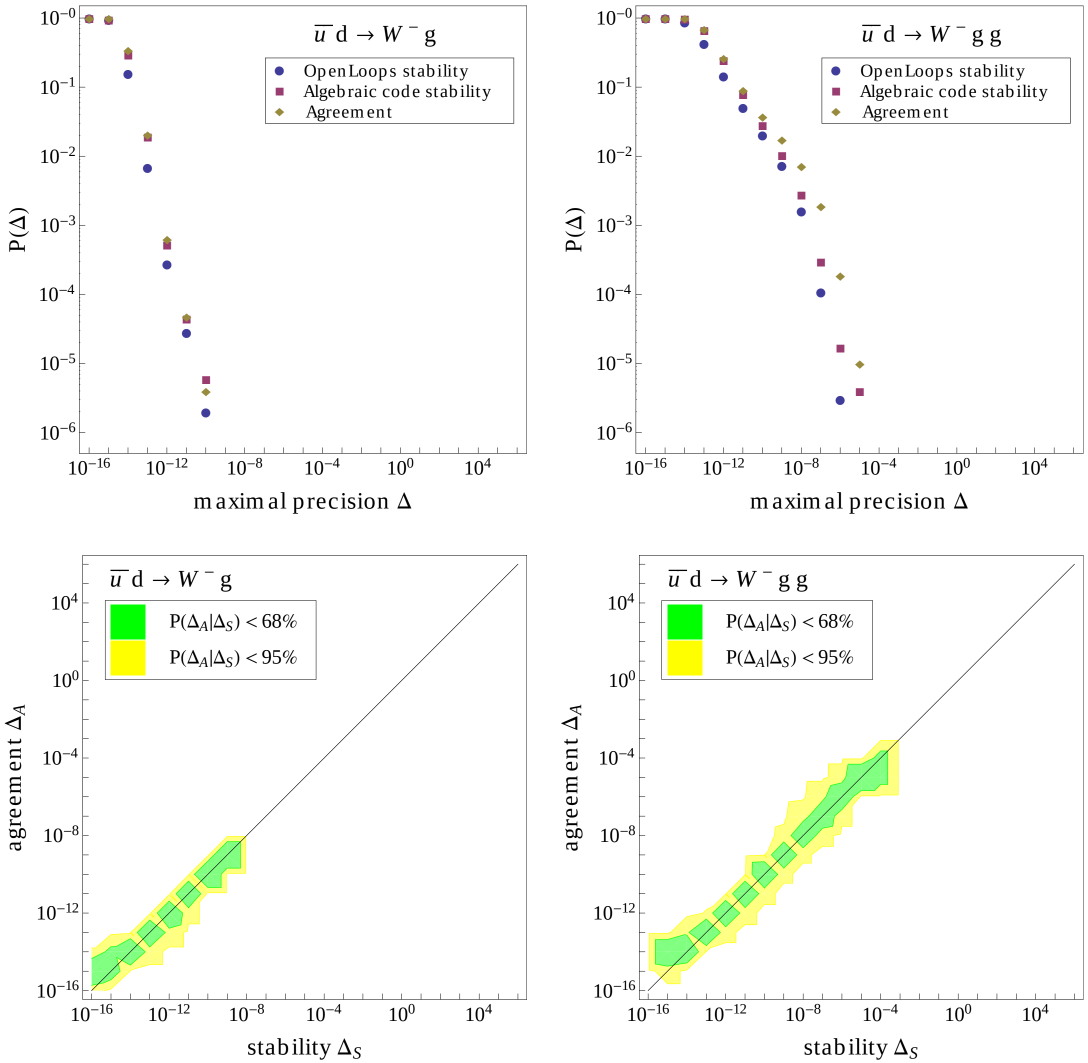
$2 \to 3$ $-\log_{10}(\Delta_\mathrm{S}) > 7\,(6)$

$2 \to 4$ $-\log_{10}(\Delta_\mathrm{S}) > 5\,(3)$

i.e. probability of < 5 digits $\simeq 10^{-3}$!

Stability issues increse with n_part but remain **well under control for** $n_\mathrm{part} \leq 6$

Example of ongoing OpenLoops Validation ($\bar{u}d \to W + ng$)



Agreement of OpenLoops with in-house "algebraic" 1-loop generator

$$\Delta_A = \frac{\delta\mathcal{W}_{\texttt{OpenLoops}}}{\delta\mathcal{W}_{\text{algebraic}}} - 1$$

compared to **intrinsic stability** Δ_S of individual codes

Probabilities $P(\Delta_A)$ **and** $P(\Delta_S)$ **consistent within ± 1 digit**

Conditional probability $P(\Delta_A|\Delta_S)$ of agreement Δ_A given stability Δ_S: **1-2 digit correlation at 95% CL**

Speed of OpenLoops $\Rightarrow$ extensive high-statistics studies $\Rightarrow$ solid stability understanding

Summary and conclusions

New 1-loop generator

- numerical, recursive, diagrammatic
- open loops (loop-momentum polynomials) instead of trees with fixed kinematics
- interfaced with tensor-integral and OPP reduction in a natural way

Automation and flexibility

- full automation: input file $\Rightarrow$ very compact code within seconds/minutes
- SM processes with QCD corrections, up to $\mathcal{O}(10^4\text{–}10^5)$ diagrams

Speed and Stability

- very fast (100–1000 ms/point in $2 \to 4$), also with OPP thanks to open loops!
- tensor integrals very stable (OPP to be studied)

We are looking forward to apply `OpenLoops` to phenomenology

BACKUP SLIDES

Colour reduction and colour sums

Reduction to colour basis $\{\mathcal{C}_i\}$, Born colour vector

$$\mathcal{C}^{(d)} = \sum_i c_i^{(d)}\,\mathcal{C}_i, \qquad \mathcal{M} = \sum_i \mathcal{M}_i\,\mathcal{C}_i, \qquad \mathcal{M}_i = \sum_d c_i^{(d)}\mathcal{A}^{(d)}$$

Colour sums encoded into colour-interference matrix $\mathcal{K}_{ij}$ (once per process)

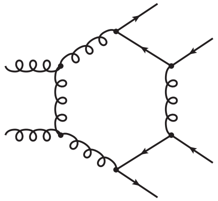
$$\sum_{\text{col}} |\mathcal{M}|^2 = \sum_{i,j} \mathcal{M}_i^* \underbrace{\left(\sum_{\text{col}} \mathcal{C}_i^* \mathcal{C}_j\right)}_{\mathcal{K}_{ij}} \mathcal{M}_j, \qquad \tilde{\mathcal{M}}_j := \sum_{\text{col}} \mathcal{M}^* \mathcal{C}_j = \sum_i \mathcal{M}_i^* \mathcal{K}_{ij}$$

Colour sums for loop-Born interference

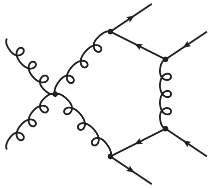
$$\sum_{\text{col}} 2\,\mathrm{Re}\,(\mathcal{M}^*\delta\mathcal{M}) = \sum_{d'} 2\,\mathrm{Re}\Bigg\{ \underbrace{\left(\sum_{\text{col}} \mathcal{M}^* \mathcal{C}^{(d')}\right)}_{\sum_j c_j^{(d')}\tilde{\mathcal{M}}_j} \delta\mathcal{A}^{(d')} \Bigg\}$$

Example: $gg \to t\bar{t}b\bar{b}$

Factorisation of colour structures (3 contributions per quartic vertex)



$$= \mathcal{A}^{(d)} f^{a_1bd} f^{a_2cd} \left(T^bT^e\right)_{i_3i_4} \left(T^cT^e\right)_{i_5i_6}$$



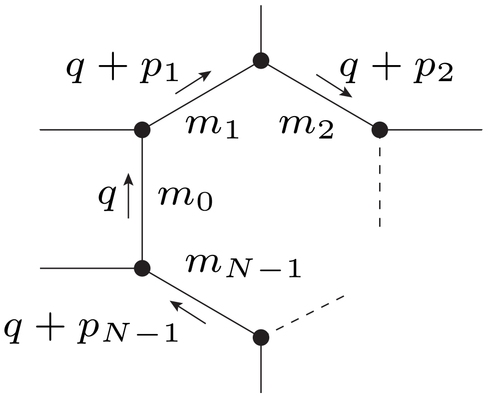
$$= \left[\mathcal{A}^{(d_1)} f^{a_1bd} f^{a_2cd} + \mathcal{A}^{(d_2)} f^{a_1cd} f^{a_2bd} + f^{a_1a_2d} f^{bcd} \mathcal{A}^{(d_3)}\right] \left(T^bT^e\right)_{i_3i_4} \left(T^cT^e\right)_{i_5i_6}$$

Fully automatic colour reduction with well-known SU(N) identities

$$f^{abc}T^c = -\mathrm{i}[T^a, T^b], \qquad T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_\mathrm{c}}\delta_{ij}\delta_{kl}\right), \qquad \text{etc.}$$

Colour interference and summation once and for all at the level of colour basis

$$\begin{aligned}
\mathcal{C}_1 &= T^{a_1}_{i_3i_4}T^{a_2}_{i_5i_6}, \quad \mathcal{C}_2 = \delta_{i_3i_4}\left(T^{a_1}T^{a_2}\right)_{i_5i_6}, \quad \mathcal{C}_3 = \delta_{i_3i_4}\left(T^{a_2}T^{a_1}\right)_{i_5i_6},\\
\mathcal{C}_4 &= T^{a_1}_{i_5i_6}T^{a_2}_{i_3i_4}, \quad \mathcal{C}_5 = \delta_{i_5i_6}\left(T^{a_1}T^{a_2}\right)_{i_3i_4}, \quad \mathcal{C}_6 = \delta_{i_5i_6}\left(T^{a_2}T^{a_1}\right)_{i_3i_4},\\
\mathcal{C}_7 &= T^{a_1}_{i_3i_6}T^{a_2}_{i_5i_4}, \quad \mathcal{C}_8 = \delta_{i_3i_6}\left(T^{a_1}T^{a_2}\right)_{i_5i_4}, \quad \mathcal{C}_9 = \delta_{i_3i_6}\left(T^{a_2}T^{a_1}\right)_{i_5i_4},\\
\mathcal{C}_{10} &= T^{a_1}_{i_5i_4}T^{a_2}_{i_3i_6}, \quad \mathcal{C}_{11} = \delta_{i_5i_4}\left(T^{a_1}T^{a_2}\right)_{i_3i_6}, \quad \mathcal{C}_{12} = \delta_{i_5i_4}\left(T^{a_2}T^{a_1}\right)_{i_3i_6},\\
\mathcal{C}_{13} &= \delta^{a_1a_2}\delta_{i_3i_4}\delta_{i_5i_6}, \quad \mathcal{C}_{14} = \delta^{a_1a_2}\delta_{i_3i_6}\delta_{i_5i_4}.
\end{aligned}$$



$$T_N^{\mu_1\ldots\mu_P} = \int \mathrm{d}^D q \frac{q^{\mu_1}\ldots q^{\mu_P}}{D_0\ldots D_{N-1}}, \qquad D_i = (q+p_i)^2 - m_i^2 + \mathrm{i}\varepsilon$$

$$T_{1,2,3,4,\ldots} = A, B, C, D, \ldots$$

Recursive reduction to scalar integrals ($N \leq 4,\ P = 0$)

Collection of methods developed for $e^+e^- \to 4f$ [Denner/Dittmaier '05]

(A) **Space-time 4-dimensionality ($N \geq 5$)** — Melrose '65 Denner/Dittmaier '02–'05, Binoth/Guillet/Heinrich/Pilon/Schubert '05

(B) **Lorentz invariance ($N \leq 4$)** — Passarino/Veltman '79

(C) **Expansions to cure $\det(Z)$-instabilities** — Denner/Dittmaier '05

(A) Space-time 4-dimensionality for $N \geq 5$ [Binoth et al. '05; Denner, Dittmaier '05]

Linear dependence in $D=4$ of *hexagon* external momenta $p_1, \ldots, p_5$ yields

$$\mathcal{F}^\mu = \begin{vmatrix} q^\mu & 2qp_1 & \ldots & 2qp_5 \\ p_1^\mu & 2p_1p_1 & \ldots & 2p_1p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_4^\mu & 2p_4p_1 & \ldots & 2p_4p_5 \\ 0 & f_1 & \ldots & f_5 \end{vmatrix} = 0, \qquad 2qp_i = D_i - D_0 - f_i$$

Rank-$(P+1)$ hexagon $\to$ **rank-P pentagon reduction** expanding along 1^{st} row

$$\int \frac{q^{\mu_1} \ldots q^{\mu_P}}{D_0 D_1 \cdots D_5} \mathcal{F}^\mu = \begin{vmatrix} F^{\mu\mu_1\cdots\mu_P} & \Delta E^{\mu_1\cdots\mu_P}(1) & \ldots & \Delta E^{\mu_1\cdots\mu_P}(5) \\ p_1^\mu & 2p_1p_1 & \ldots & 2p_1p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_4^\mu & 2p_4p_1 & \ldots & 2p_4p_5 \\ 0 & f_1 & \ldots & f_5 \end{vmatrix} = \mathcal{O}(D-4)$$

- qualitatively similar reduction for all $N \geq 5$
- simultaneous N, P reduction until $N \leq 4$ and $P \leq 4$

(B) Tensor integrals with $N = 4, 3$ [Passarino/Veltman '79; Denner '93]

Rank reduction via contractions

$$2p_i^\mu q_\mu = -f_i + D_i - D_0, \qquad g^{\mu\nu} q_\mu q_\nu = m_0^2 + D_0$$

Covariant decomposition

$$T_N^{\mu_1 \dots \mu_P} = \sum_{i_1 \dots i_P = 0}^{N-1} T^{(P)}_{i_1 \dots i_P} \{g \dots p\}^{\mu_1 \dots \mu_P}_{i_1 \dots i_P}$$

General solution

$$2(D + P - N - 1)\, T^{(P)}_{00 i_3 \dots i_P} = \sum_{k=1}^{N-1} f_k\, T^{(P-1)}_{k i_3 \dots i_P} + 2m_0^2\, T^{(P-2)}_{i_3 \dots i_P} + \text{lower-point}$$

$$\sum_{n=1}^{N-1} Z_{mn}\, T^{(P)}_{n i_2 \dots i_P} = -2 \sum_{r=2}^{P} \delta_{m i_r}\, T^{(P)}_{00 i_2 \dots \hat{i}_r \dots i_P} - f_m\, T^{(P-1)}_{i_2 \dots i_P} + \text{lower-point}$$

- R_1 **rational terms** from **catalogue of UV residues**

$$(D-4)\, T^{(P)}_{00 i_3 \dots i_P} = R^{(P)}_{00 i_3 \dots i_P}$$

- **unstable Gram-matrix inversion** in $\det(Z) \to 0$ regions

$$Z_{kl} = 2p_k p_l, \qquad (Z)^{-1}_{kl} = \frac{\tilde{Z}_{lk}}{\det(Z)}$$

$$\tilde{X}_{0j} T^{(P)}_{i_1\ldots i_P} = \det(Z)\, T^{(P+1)}_{ji_1\ldots i_P} + 2\sum_{n=1}^{N-1} \tilde{Z}_{jn} \sum_{r=1}^{P} \delta_{ni_r} T^{(P+1)}_{00i_1\ldots\hat{i}_r\ldots i_P} + \text{ lower-point}$$

$$2\tilde{Z}_{kl} T^{(P+1)}_{00i_2\ldots i_P} = \Big\{ -\det(Z)\, T^{(P+1)}_{kli_2\ldots i_P} + 2m_0 \tilde{Z}_{kl} T^{(P-1)}_{i_2\ldots i_P} + \sum_{n,m=1}^{N-1} \Big[f_n f_m T^{(P-1)}_{i_2\ldots i_P} + 2\sum_{r=2}^{P} (f_n \delta_{mi_r} + f_m \delta_{ni_r}) \times T^{(P)}_{00i_2\ldots\hat{i}_r\ldots i_P} + 4 \sum_{\substack{r,s=2 \\ r\neq s}}^{P} \delta_{ni_r}\delta_{mi_s} T^{(P+1)}_{0000i_2\ldots\hat{i}_r\ldots\hat{i}_s\ldots i_P} \Big] \tilde{\tilde{Z}}_{(kn)(lm)} + \text{ lower-point} \Big\} \Big(D+1+P-N+\sum_{r=2}^{P} \bar{\delta}_{i_r 0}\Big)^{-1}$$

(C) $\det(Z)$**-expansion for** *tensor* **integrals with** $N = 4, 3$ [Denner/Dittmaier '05]

Iterative expansion with K adapted to $\det(Z)$ and target precision

$$T^{(P)} = T^{(P)}_K + \mathcal{O}\left[\det(Z)^{K+1}\right]$$

Various alternative methods: further expansions ($\tilde{Z}_{kl}$ or $\tilde{X}_{0j} = -\sum_k \tilde{Z}_{jk} f_k$ small); modified set of MIs ($T_0 \to T_{00\ldots 00}$); solutions of PV-identities with $\det(Z) \to \det(Y)$

General & robuts solution to instability problems (important for $2 \to 4$!)

Rational parts

$$K_{i_1\ldots i_P}(D) \underbrace{T^{(N)}_{i_1\ldots i_P}}_{\frac{R_1}{(D-4)}+\frac{R_1}{(D-4)}+\frac{R_2}{(D-4)^2}+\text{finite part}} \Rightarrow K'_{i_1\ldots i_P}(4)\,(R_1+R_1)+\frac{1}{2}K''_{i_1\ldots i_P}(4)\,R_2+\ldots$$

When tensor integrals are combined with their D-dimensional coefficients

- UV and IR poles require $(D-4)$ expansions (performed algebraically)
- this produces **rational terms** proportional to the pole residues

Rational terms of **IR origin**

- require the heaviest algebraic work but **cancel in any** *unrenormalized* **QCD amplitude** (proven in App. A of arXiv:0807.1248)
- can thus be neglected from the beginning

Rational terms of **UV origin**

- extracted automatically by means of a catalogue of UV residues R_1
- after the relevent $(D-4)$-expansions we can continue the calculation in $D=4$

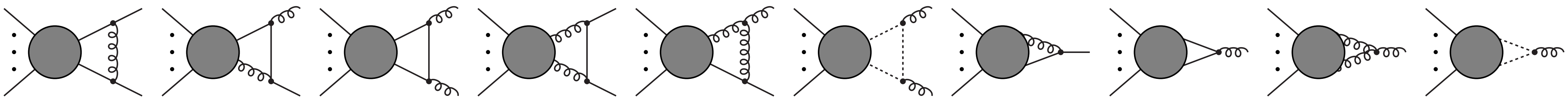
Cancellation of rational terms of IR origin: sketch of the proof (arXiv:0807.1248)

Rational terms originate from D**-dependent** $g^{\mu\nu}$**-contractions** of type $g_{\nu\lambda}\Gamma^{\nu\lambda}$

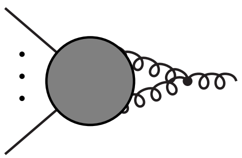
$$g_{\nu\lambda}\, g^{\nu\lambda} = D, \qquad g_{\nu\lambda}\, \gamma^{\nu} \not{p} \gamma^{\lambda} = (2-D)\not{p}, \ldots$$

(1) **The tensor-reduction is free from IR rational terms** since in the soft and collinear regions ($q^\mu \to x p^\mu$) the tensor integrals cannot produce $g^{\mu\nu}$

(2) **All possible diagrams involving IR-divergent integrals**



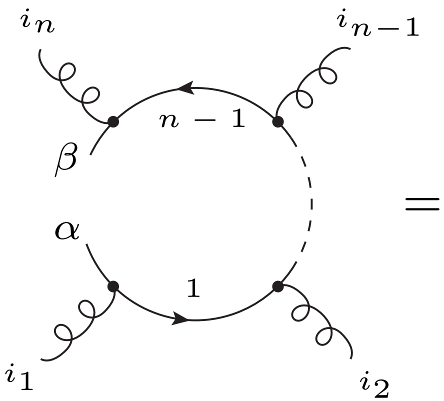
can be cast into a form where $g_{\nu\lambda}\Gamma^{\nu\lambda}$ **contractions cancel in IR regions**



$$= \int \frac{\mathrm{d}^D q}{q^2 (q+p)^2} \underbrace{\epsilon^{\mu *}(p)\,(2q+p)_\mu}_{\to 0 \text{ in soft/coll. regions}} \; g_{\nu\lambda}\,\Gamma^{\nu\lambda}(q) + \ldots$$

Example of Open Loops construction: fermion loop

$$\mathcal{N}_{\alpha}^{\beta}(\mathcal{I}_n;q) \;=$$



$$= g_{\mathrm{S}}\left[(\not{q} + \not{p}_n + m)\gamma^{\nu}\right]_{\beta\gamma}\, \mathcal{N}_{\alpha}^{\gamma}(\mathcal{I}_{n-1};q)\, \varepsilon_{\nu}^{*}(p_n,\lambda_n)$$

- n-point open-loop coefficients of rank $r = 0, 1, \ldots, n$

$$\begin{aligned}
\mathcal{N}_{;\alpha}^{\beta}(\mathcal{I}_n) &= g_{\mathrm{S}}\left[(\not{p}_n + m)\gamma^{\nu}\right]_{\beta\gamma}\, \mathcal{N}_{;\alpha}^{\gamma}(\mathcal{I}_{n-1})\, \varepsilon_{\nu}^{*}(p_n,\lambda_n) \\
\mathcal{N}_{\mu_1;\alpha}^{\beta}(\mathcal{I}_n) &= g_{\mathrm{S}}\left\{\left[(\not{p}_n + m)\gamma^{\nu}\right]_{\beta\gamma}\, \mathcal{N}_{\mu_1;\alpha}^{\gamma}(\mathcal{I}_{n-1}) + \left[\gamma_{\mu_1}\gamma^{\nu}\right]_{\beta\gamma}\, \mathcal{N}_{;\alpha}^{\gamma}(\mathcal{I}_{n-1})\right]\, \varepsilon_{\nu}^{*}(p_n,\lambda_n)
\end{aligned}$$

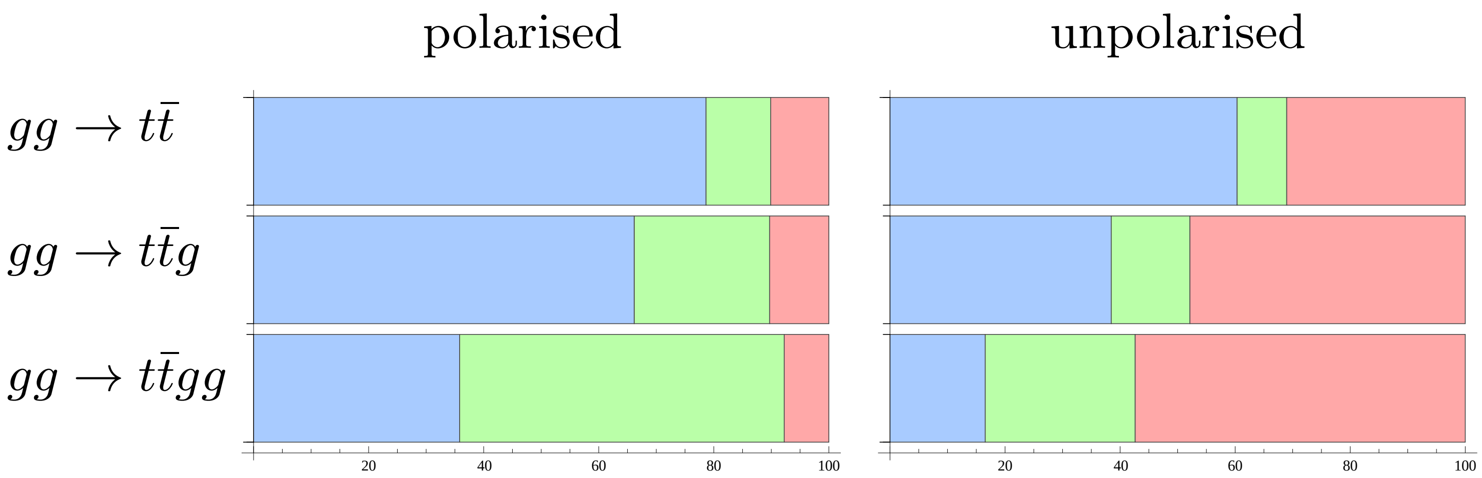
etc.

- initial condition for 0-point rank-0 open loop

$$\mathcal{N}_{;\alpha}^{\gamma}(\mathcal{I}_0) = \delta_{\alpha}^{\gamma}$$

- rank, i.e. complexity, increases with n $\quad\Rightarrow\quad$ symmetrised $\mu_1 \ldots \mu_r$ components!

Detailed CPU budget (using tensor integrals)



runtime fractions for scalar integrals, tensor reduction, open-loop coefficients

- cost of coefficients for single helicity very low ($\lesssim 10\%$) wrt n_{hel}-independent remnant (90%) $\Rightarrow$ helicity-summed coefficients only 30–60%
- total CPU time not far from scalar-integral cost (absolute minimum)