The Standard Model Gauge-β-Functions at 3-Loop Order

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LoopFest XI in Pittsburgh in 2012

Introduction and Historical Overview

- [Gross, Wilczek '73; Politzer '73]
- [Jones '74]
- [Tarasov, Vladimirov '76]
- [Caswell '74; Egorian, Tarasov '78]
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Outline

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General Remarks for Multi-Loop Calculations

To calculate a given process on a computer, we need

1. a program to create the corresponding Feynman diagrams and amplitudes,
2. a program to reduce the Feynman integrals to “master integrals”,
3. an implementation of these master integrals.

The Framework of the Calculation

QGRAF → Q2E ⟶ Exp → FORM-MINCER-MATAD-Setup

The Framework of the Calculation

1 1

QGRAF → Q2E ⟶ Exp → FORM-MINCER-MATAD-Setup

The Framework of the Calculation

QGRAF [1] → Q2E [1] → Exp → FORM-MINCER-MATAD-Setup [2] [3]

The Framework of the Calculation

FeynArts (1)

↑

FeynRules

The Framework of the Calculation

(1) QGRAF → (1) Q2E ⟶ Exp → (2) (3) FORM-MINCER-MATAD-Setup

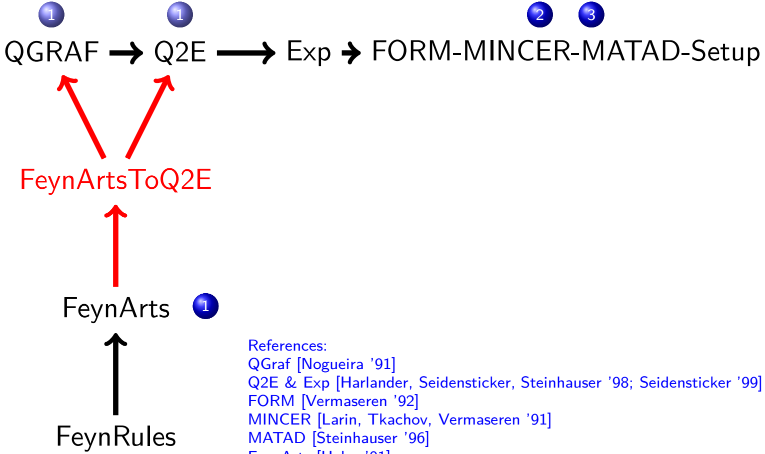
FeynArts (1)

↑

FeynRules

References:
QGraf [Nogueira '91]
Q2E & Exp [Harlander, Seidensticker, Steinhauser '98; Seidensticker '99]
FORM [Vermaseren '92]
MINCER [Larin, Tkachov, Vermaseren '91]
MATAD [Steinhauser '96]
FeynArts [Hahn '01]
FeynRules [Christensen, Duhr '08]

The Framework of the Calculation



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FeynRules [Christensen, Duhr '08]

Outline

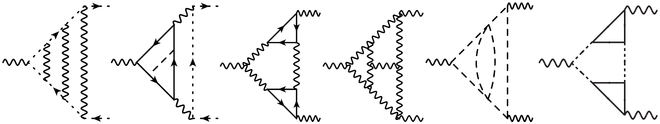
Overview of the Calculation of the β-Functions

- $\overline{\text{MS}}$
- Implementation of the unbroken phase of the SM in FeynRules
- We took into account all 3-loop contributions due to $g_1, g_2, g_3, y_{\text{top}}, y_{\text{bottom}}, y_\tau, \lambda$.
- We performed a direct calculation of the corresponding 3-loop diagrams in Lorenz gauge.
- I. e., we calculated $Z_{g_{1/2/3}}^{\text{3-loop}}$ directly and applied

$$\beta_i = -\left[\epsilon\frac{\alpha_i}{\pi} + 2\frac{\alpha_i}{Z_{g_i}}\sum_{j\neq i}\frac{\partial Z_{g_i}}{\partial\alpha_j}\beta_j\right]\left(1 + 2\frac{\alpha_i}{Z_{g_i}}\frac{\partial Z_{g_i}}{\partial\alpha_i}\right)^{-1}$$

Overview of the Calculation of the Renormalization Constants

- $Z_{g_i} = \frac{Z_{\text{vertex}}}{\prod_k \sqrt{Z_{\text{k, wave function}}}} = \frac{Z_{\text{ghost, Ghost, gauge boson}}}{Z_{\text{ghost}}\sqrt{Z_{\text{gauge boson}}}} = \frac{Z_{\text{triple gauge boson}}}{Z^{3/2}_{\text{gauge boson}}}$
- $\mathcal{O}(10^6)$ diagrams



- One external momentum set to zero.
  $\Rightarrow$ The required master integrals are provided by MINCER.

The Issue of γ_5

- The "naive" regularization prescription $\{\gamma_\mu, \gamma_5\} = 0$ leads to the problematic result

$$\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) = 0 \;(D \neq 4)\,,$$
$$\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) = -4\mathrm{i}\epsilon_{\mu\nu\rho\sigma}\;(D = 4)\,.$$

- Consistent regularization prescriptions are less convenient, however.
- Our solution: Use naive prescription for fermion traces with even number of γ_5 matrices.
- In case of odd number of γ_5 matrices: Show that contributions must cancel *a priori*. This rests strongly on the cancellation of anomalies:

$$\mathrm{tr}\left(T^A\left\{T^B, T^C\right\}\right) = 0.$$

- Naive prescription for γ_5 applicable: [Pickering, Gracey, Jones '01]

Checks

- We have also calculated Z_{g_2} by regarding the $\phi^+\phi^- W_3$ vertex and found agreement.
- We find $Z_{g_{1/2/3}}$ to be gauge parameter independent, local and we find the β-functions to be finite.
- We find agreement with the 2-loop result of [Machacek, Vaughn '83; Fischler, Hill '81; Jones '82; Jack, Osborn '85].
- We find agreement with the 3-loop result of $\mathcal{O}\left(\alpha_3^4\right)$ in SU(3). [Tarasov, Vladimirov, Zharkov '80; Larin, Vermaseren '93]
- We find agreement with the 3-loop result of $\mathcal{O}\left(\alpha_3^3\alpha_{\mathsf{top}}\right)$ in SU(3). [Steinhauser '98]
- We find agreement with [Pickering, Gracey, Jones '01].

The Result

$$\beta_1 = \left(\frac{\alpha_1}{\pi}\right)^2 \Bigg\{ \frac{1}{40} + \frac{n_G}{3} + \frac{\alpha_1}{\pi}\left(\frac{9}{800} + \frac{19 n_G}{240}\right) + \frac{\alpha_2}{\pi}\left(\frac{9}{160} + \frac{3 n_G}{80}\right) + \frac{\alpha_3}{\pi}\frac{11 n_G}{60}$$
$$+ \left(\frac{\alpha_1}{\pi}\right)^2 \left(\frac{489}{512000} - \frac{29 n_G}{2400} - \frac{209 n_G^2}{8640}\right) + \frac{\alpha_1}{\pi}\frac{\alpha_2}{\pi}\left(\frac{783}{51200} - \frac{7 n_G}{6400}\right) - \frac{\alpha_1}{\pi}\frac{\alpha_3}{\pi}\frac{137 n_G}{14400}$$
$$+ \left(\frac{\alpha_2}{\pi}\right)^2 \left(\frac{3401}{20480} + \frac{83 n_G}{1920} - \frac{11 n_G^2}{960}\right) - \frac{\alpha_2}{\pi}\frac{\alpha_3}{\pi}\frac{n_G}{320} + \left(\frac{\alpha_3}{\pi}\right)^2 \left(\frac{275 n_G}{576} - \frac{121 n_G^2}{2160}\right)$$
$$+ n_t \frac{\alpha_t}{\pi}\left[-\frac{17}{160} - \frac{\alpha_1}{\pi}\frac{2827}{51200} - \frac{\alpha_2}{\pi}\frac{471}{2048} - \frac{\alpha_3}{\pi}\frac{29}{320} + \frac{\alpha_t}{\pi}\left(\frac{339}{5120} + \frac{303 n_t}{2560}\right)\right]$$
$$+ \frac{\lambda}{4\pi^2}\left(\frac{\alpha_1}{\pi}\frac{27}{3200} + \frac{\alpha_2}{\pi}\frac{9}{640} - \frac{\lambda}{4\pi^2}\frac{9}{320}\right) \Bigg\},$$

$$\alpha_i = \frac{g_i^2}{4\pi}\ (i = 1, 2, 3), \quad \alpha_t = \frac{y_t^2}{4\pi}$$

The Result

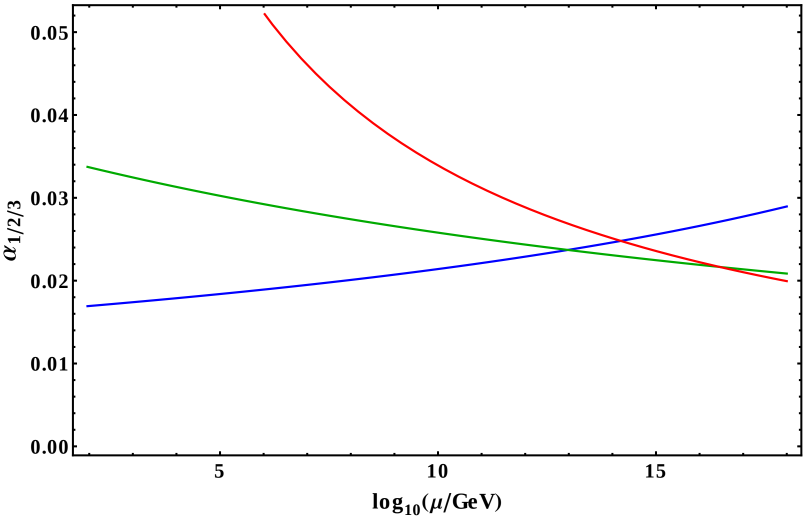
$$
\begin{aligned}
\beta_2 = {} & \left(\frac{\alpha_2}{\pi}\right)^2 \Bigg\{ -\frac{43}{24} + \frac{n_G}{3} + \frac{\alpha_1}{\pi}\left(\frac{3}{160} + \frac{n_G}{80}\right) + \frac{\alpha_2}{\pi}\left(-\frac{259}{96} + \frac{49 n_G}{48}\right) + \frac{\alpha_3}{\pi}\frac{n_G}{4} \\
& + \left(\frac{\alpha_1}{\pi}\right)^2 \left(\frac{163}{102400} - \frac{7 n_G}{960} - \frac{11 n_G^2}{2880}\right) + \frac{\alpha_1}{\pi}\frac{\alpha_2}{\pi}\left(\frac{561}{10240} + \frac{13 n_G}{1280}\right) - \frac{\alpha_1}{\pi}\frac{\alpha_3}{\pi}\frac{n_G}{960} \\
& + \left(\frac{\alpha_2}{\pi}\right)^2 \left(-\frac{667111}{110592} + \frac{1603 n_G}{432} - \frac{415 n_G^2}{1728}\right) + \frac{\alpha_2}{\pi}\frac{\alpha_3}{\pi}\frac{13 n_G}{64} + \left(\frac{\alpha_3}{\pi}\right)^2 \left(\frac{125 n_G}{192} - \frac{11 n_G^2}{144}\right) \\
& + n_t \frac{\alpha_t}{\pi}\left[-\frac{3}{32} - \frac{\alpha_1}{\pi}\frac{593}{10240} - \frac{\alpha_2}{\pi}\frac{729}{2048} - \frac{\alpha_3}{\pi}\frac{7}{64} + \frac{\alpha_t}{\pi}\left(\frac{57}{1024} + \frac{45 n_t}{512}\right)\right] \\
& + \frac{\lambda}{4\pi^2}\left(\frac{\alpha_1}{\pi}\frac{3}{640} + \frac{\alpha_2}{\pi}\frac{3}{128} - \frac{\lambda}{4\pi^2}\frac{3}{64}\right) \Bigg\}, \\
\alpha_i = {} & \frac{g_i^2}{4\pi}\ (i = 1, 2, 3), \quad \alpha_t = \frac{y_t^2}{4\pi}
\end{aligned}
$$

The Result

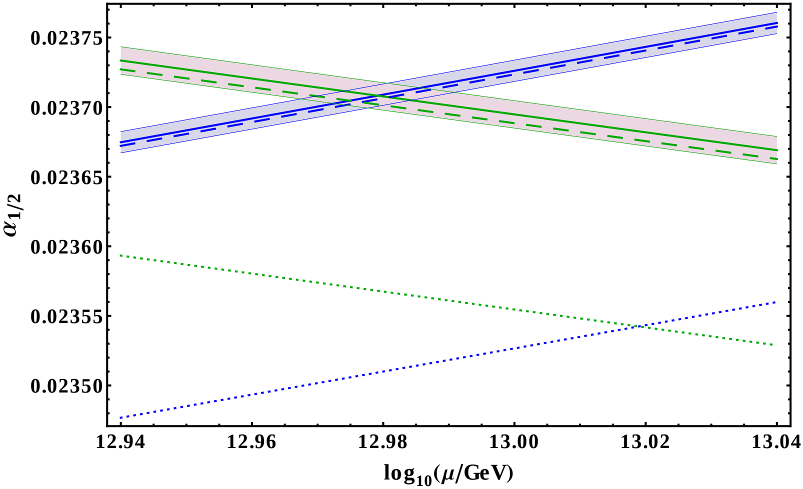
$$\beta_3 = \left(\frac{\alpha_3}{\pi}\right)^2 \Bigg\{ -\frac{11}{4} + \frac{n_G}{3} + \frac{\alpha_1}{\pi}\frac{11n_G}{480} + \frac{\alpha_2}{\pi}\frac{3n_G}{32} + \frac{\alpha_3}{\pi}\left(-\frac{51}{8} + \frac{19n_G}{12}\right)$$
$$+ \left(\frac{\alpha_1}{\pi}\right)^2\left(-\frac{13n_G}{7680} - \frac{121n_G^2}{17280}\right) - \frac{\alpha_1}{\pi}\frac{\alpha_2}{\pi}\frac{n_G}{2560} + \frac{\alpha_1}{\pi}\frac{\alpha_3}{\pi}\frac{77n_G}{2880} + \left(\frac{\alpha_2}{\pi}\right)^2\left(\frac{241n_G}{1536} - \frac{11n_G^2}{384}\right)$$
$$+ \frac{\alpha_2}{\pi}\frac{\alpha_3}{\pi}\frac{7n_G}{64} + \left(\frac{\alpha_3}{\pi}\right)^2\left(-\frac{2857}{128} + \frac{5033n_G}{576} - \frac{325n_G^2}{864}\right)$$
$$+ n_t\frac{\alpha_t}{\pi}\left[-\frac{1}{8} - \frac{\alpha_1}{\pi}\frac{101}{2560} - \frac{\alpha_2}{\pi}\frac{93}{512} - \frac{\alpha_3}{\pi}\frac{5}{8} + \frac{\alpha_t}{\pi}\left(\frac{9}{128} + \frac{21n_t}{128}\right)\right]\Bigg\},$$

$$\alpha_i = \frac{g_i^2}{4\pi}\ (i = 1, 2, 3), \quad \alpha_t = \frac{y_t^2}{4\pi}$$

Running of Standard Model Gauge Couplings



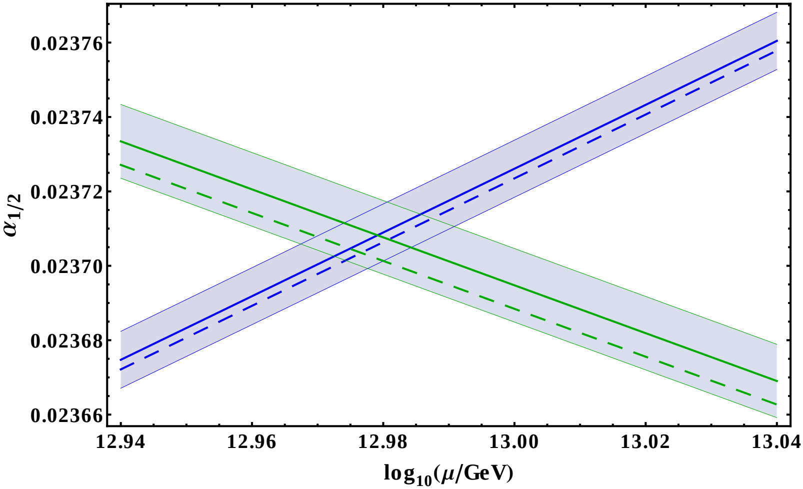
Numerical Impact of Three-Loop Corrections I



Dotted: 1-Loop   Dashed: 2-Loop   Solid: 3-Loop

Error bands: experimental uncertainty

Numerical Impact of Three-Loop Corrections II



Dashed: 2-Loop Solid: 3-Loop

Error bands: experimental uncertainty

Summary

- We created an interface which connects the programs FeynRules and FeynArts with our FORM-MINCER-MATAD-Setup.
- We calculated $\beta^{\overline{\text{MS}}}_{g_{1/2/3}}$ in the SM to 3-loop order.
- We successfully applied several internal and external checks.
- For g_1 and g_2, the numerical shift is of the order of the experimental uncertainty.
- Phys. Rev. Lett. 108 (2012) 151602 [arXiv:1201.5868 [hep-ph]]

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