# A subtraction scheme for NNLO hadronic cross sections

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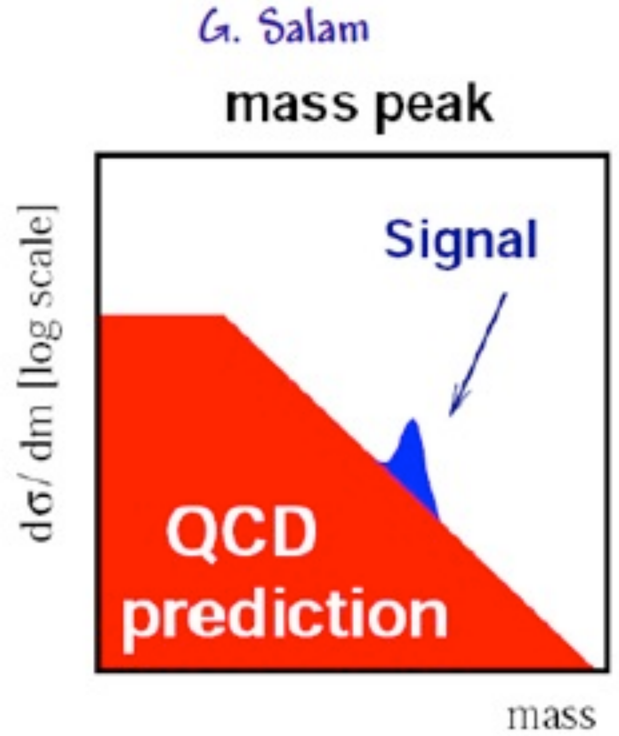
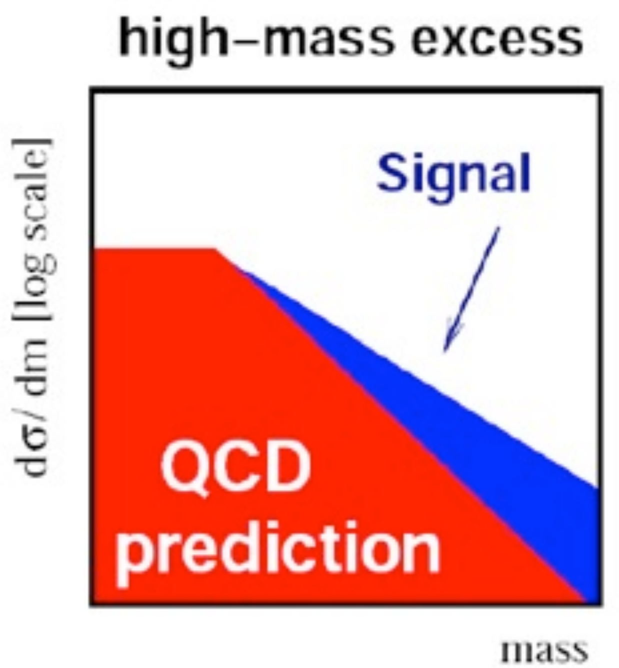
# Outline

- Motivations
- A sector decomposition based subtraction method
- A pedagogical example: differential $Z \to e^+e^-$ @ NNLO
- Extension to higher multiplicity jet observables
- Summary

# Collider Searches



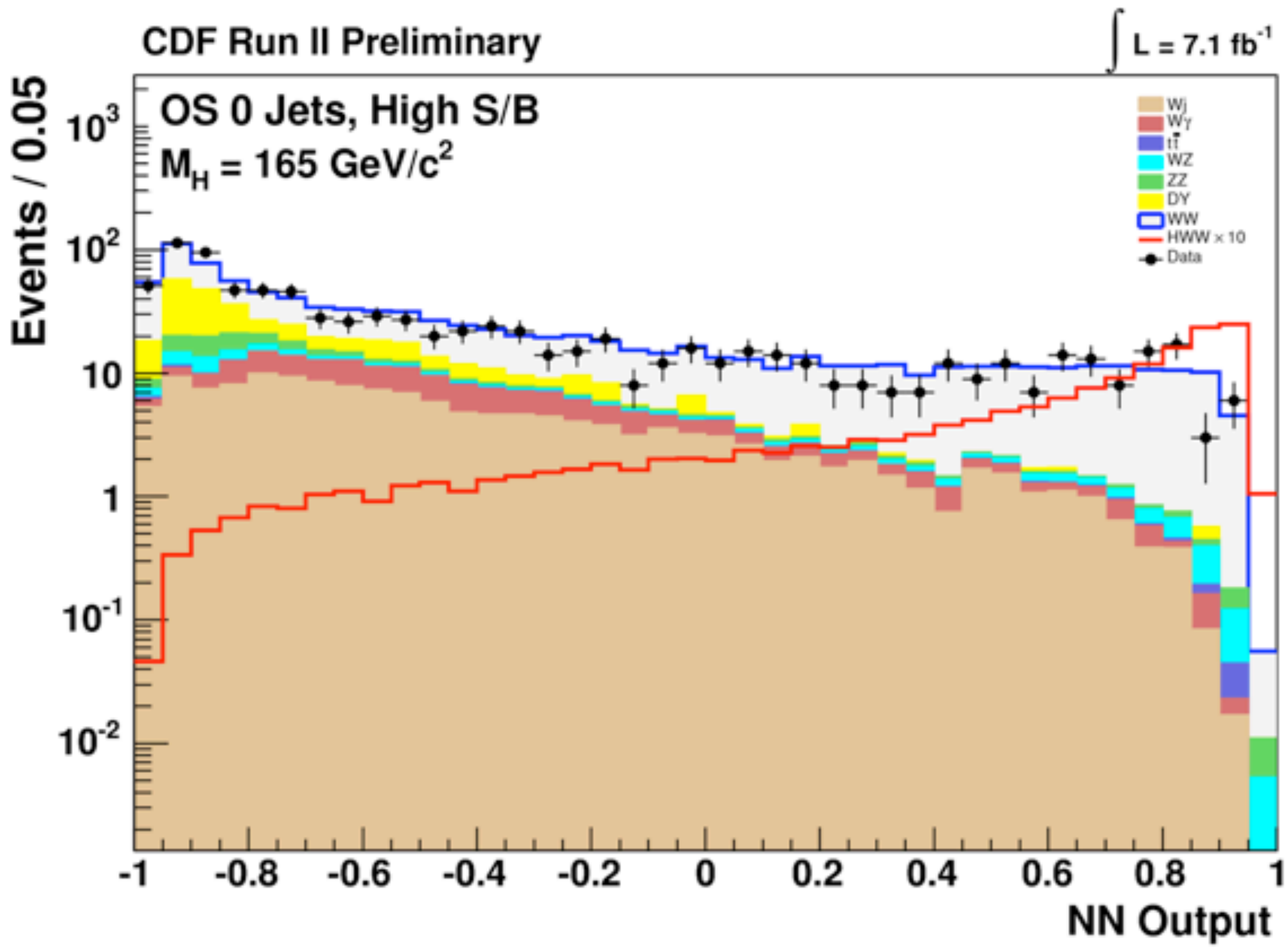




Discoveries are not easy at the LHC, don't always get a resonance peak or a sharp kinematic structure

Examples: $H \rightarrow WW$, SUSY in missing energy plus jets

# Collider Searches

Higgs searches require combining many kinematic variables to see a slight excess over background

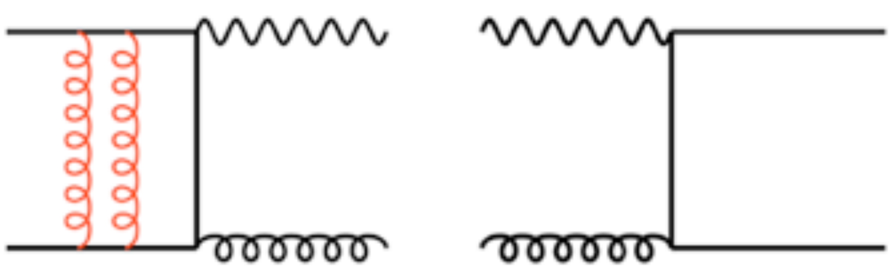



Need accurate predictions for signal and background to correctly design the nural network

# NNLO Differential Cross Sections

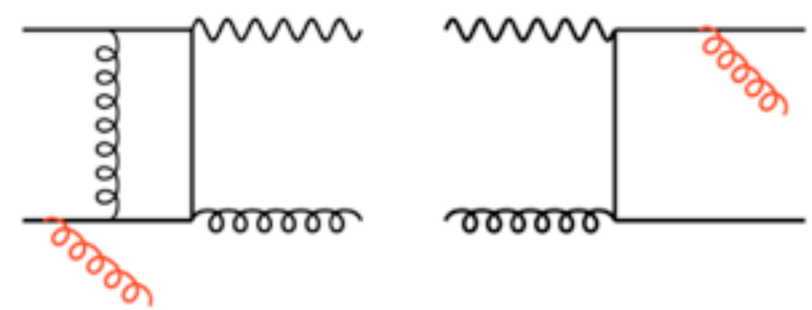
- Need the following ingredients for a NNLO cross section

2-loop matrix elements, m partons



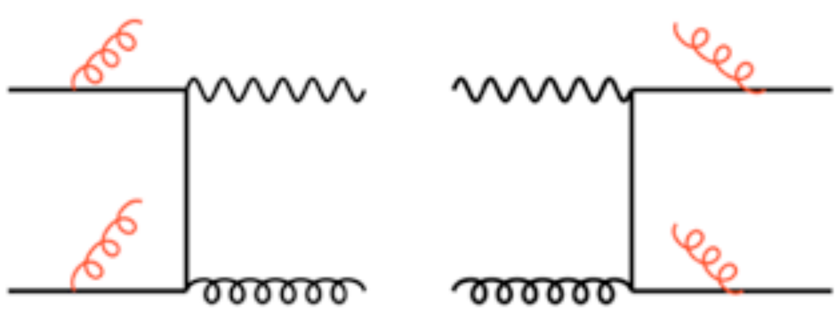
- Explicit IR poles from loop integrals

1-loop matrix elements, m+1 partons



- Explicit IR poles from loops
- Implicit IR poles from single unresolved radiation

Tree level matrix elements, m+2 partons



Implicit IR poles from double unresolved radiation

- IR singularities cancel in the sum of real and virtual corrections and mass factorisation counterterm but only after phase space integration for real radiations

# NNLO Real Corrections and the IR singularities Problem

Integration of squared matrix elements over phase space of the final state particles includes regions where matrix elements develop soft and collinear singularities

Need a method to extract the singularities from the real-emission corrections that allows for differential observables

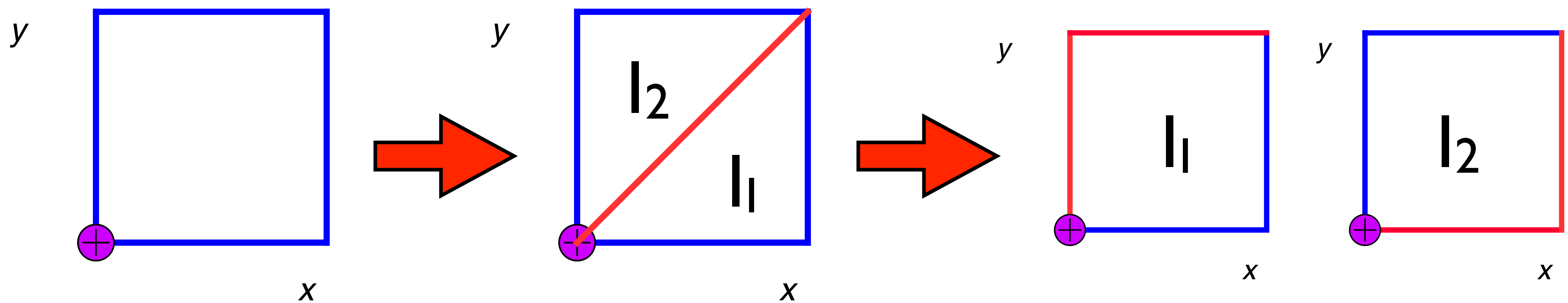
Various methods exist to deal with IR singularities

- Phase space slicing
- Sector Decomposition: this talk
- Subtraction based methods: A. Gehrmann-de Ridder's talk

Yet no method was successfully used to get NNLO cross section for a 2 ➡ 2 process until a few weeks ago (Baernrauter, Czakon, Mitov: inclusive ttbar@NNLO in the qqbar channel)

# Sector decomposition: past and present

- Original idea by Binoth, Heinrich; Anastasiou, Melnikov, Petriello
- 'Entangled singularities' occur at NNLO; deal with as shown in the example below




$$I = \int_0^1 dx dy \frac{x^\epsilon y^\epsilon}{(x+y)^2}$$

$$I_1 = \int_0^1 dx \int_0^x dy \frac{x^\epsilon y^\epsilon}{(x+y)^2}$$

$$I_2 = \int_0^1 dy \int_0^y dx \frac{x^\epsilon y^\epsilon}{(x+y)^2}$$

$$I_1 = \int_0^1 dx dy \frac{x^{-1+2\epsilon} y^\epsilon}{(1+y)^2}$$

$$I_2 = \int_0^1 dx dy \frac{y^{-1+2\epsilon} x^\epsilon}{(1+x)^2}$$

# Sector decomposition: past and present

Several NNLO cross sections successfully calculated using sector decomposition in its original version:

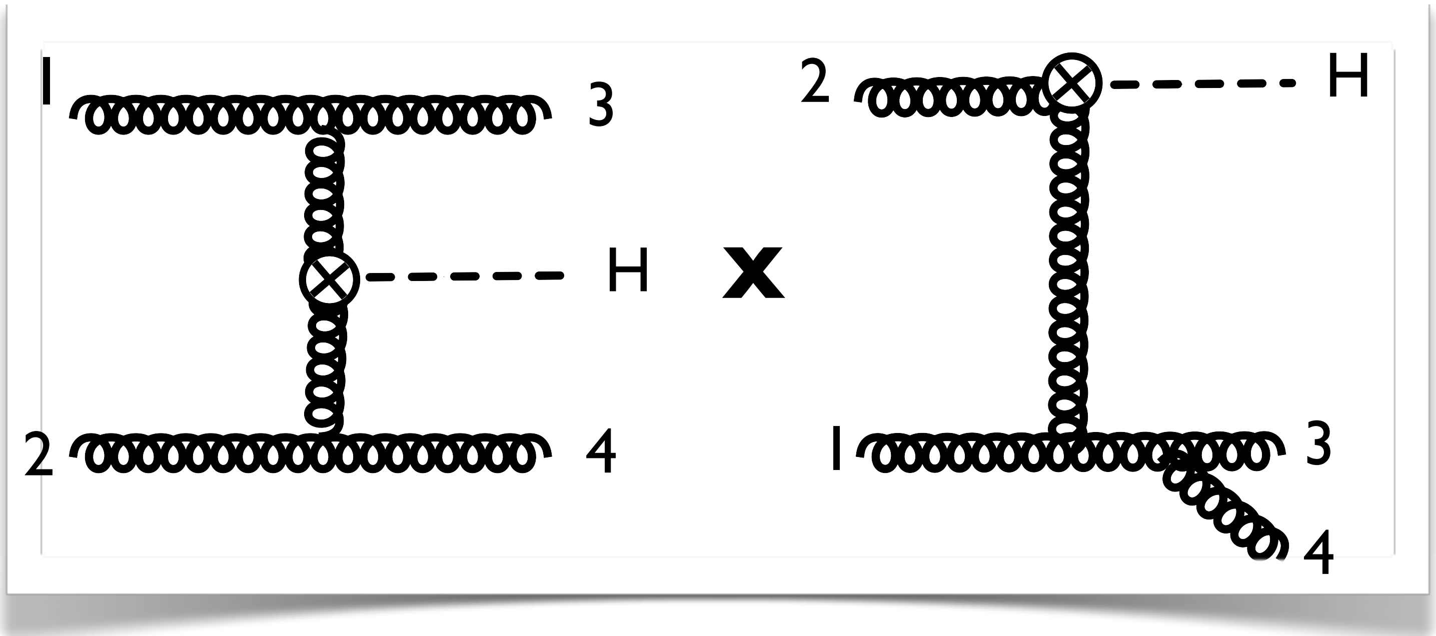
- ee ➡ 2 jets   Anastasiou, Melnikov, Petriello
- Fully differential Higgs production cross section   Anastasiou, Melnikov, Petriello
- Fully differential W and Z production cross section   Melnikov, Petriello
- NNLO QED corrections to the electron energy spectrum in muon decay   Anastasiou, Melnikov, Petriello

Drawback of the original idea:

no initial partitioning of phase space to separate collinear singularities, instead attempted to find suitable phase space parametrisation for each diagram topology based on its denominators

# Sector decomposition: past and present

- Details of the drawback: Higgs production as an example  Phys.Rev.Lett. 93 (2004) 262002




• invariants that occur in this topology : s13, s24, s134, s34. These contain collinear singularities $p_1||p_3$, $p_2||p_4$, $p_3||p_4$, $p_1||p_3||p_4$

• original idea attempted to find a parametrisation that disentangles singularities from all the occurring invariants in one topology

• can only have: p1||p3 & p2||p4 or p1||p3||p4. Not all invariants above can have collinear singularities simultaneously

- Despite its initial success, no 2 ➞ 2 cross sections were calculated for over a decade!

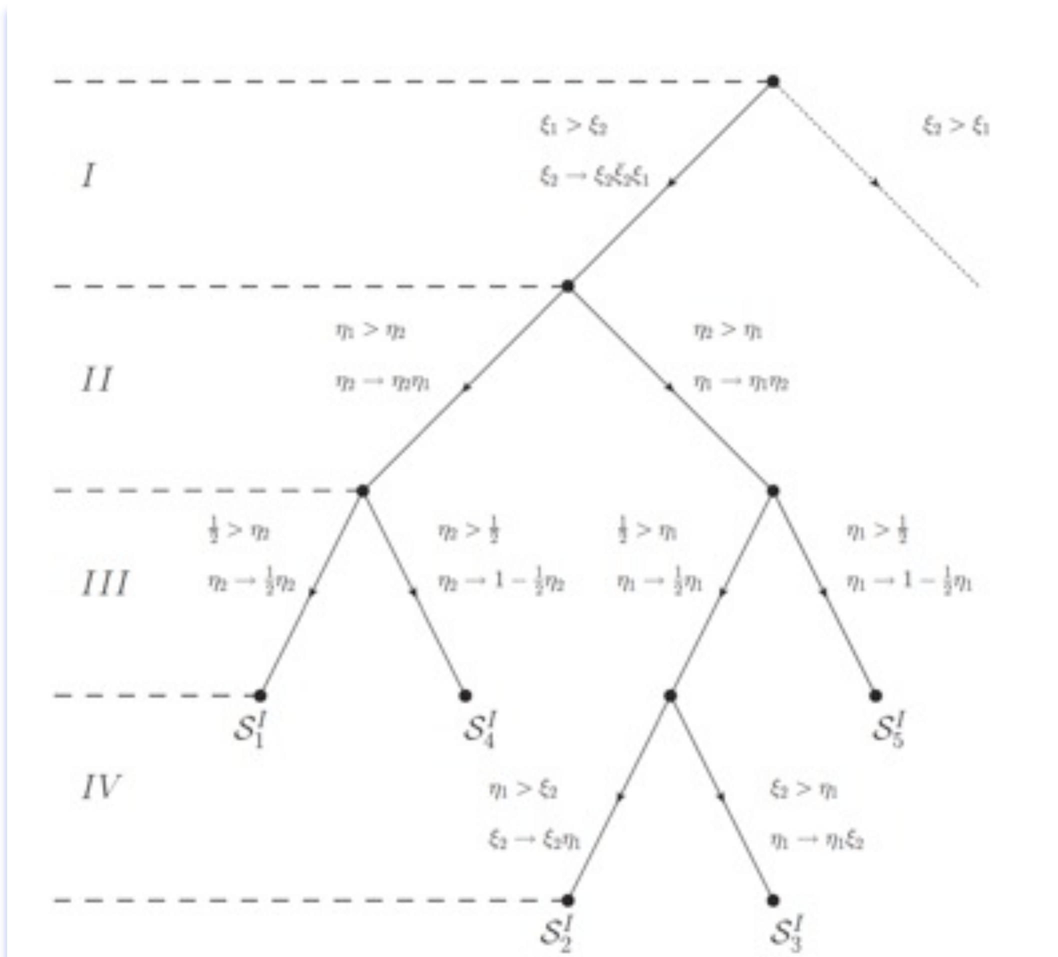
# Sector decomposition: past and present

A successful new framework based on a combination of the sector decomposition with the FKS (Frixione, Kunszt, Signer) idea was proposed in arXiv:1005.0274 [hep-ph], M. Czakon

- @ NNLO the elementary building block is the double unresolved phase space where two unresolved particles can become collinear to one or two hard directions

- partition the phase space such that in each partition only a subset of particles leads to singularities, and only one triple collinear or one double collinear singularity can occur

- the partitioning is done using energies and angles of the unresolved particles w.r.t. the hard parton(s) emitting them




$\eta \sim$ angles

$\xi \sim$ energies

# The new Sector decomposition

- Main difference w.r.t. other schemes dealing with double real radiation:
  - subtraction terms constructed from known soft and collinear limits of tree and one-loop scattering amplitudes (Catani, Campbell, Glover, Grazzini, Kosower, Uwer, ...)
  - no analytic integration is required for the subtraction terms

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- Idea of arXiv:1005.0274 [hep-ph], M. Czakon in a nutshell:

pre-partitioning of the phase space leads to a phase-space parameterization applicable to NNLO real-radiation corrections for any process, regardless of multiplicity

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pre-partitioning of the phase space leads to a phase-space parameterization applicable to NNLO real-radiation corrections for any process, regardless of multiplicity

- Our work in arXiv:1111.7041 [hep-ph] (R. B., Melnikov, Petriello):

  • show explicitly how this framework allows the extraction of singularities by applying it to a simple example: NNLO QED corrections to differential $Z \rightarrow e^+e^-$

  • a new phase-space parameterization suitable for the double-collinear partition of $1 \rightarrow 2$ decays

  • discuss the details of computing the real-virtual corrections with this method

# $Z \to e^+e^-\gamma\gamma$ with the new sector decomposition

- We study the process $Z(p_Z) \to e^+(p_+) + e^-(p_-) + \gamma(p_1) + \gamma(p_2)$

- The starting point is the partitioning: $1 = \delta_{12}^{--} + \delta_{12}^{++} + \delta_{12}^{-+} + \delta_{12}^{+-}$

has only $p_1 \| p_2 \| p_-$ and $p_1, p_2$ soft. Don't care how ugly $s_{1+}$ and $s_{2+}$ are

has only $p_1 \| p_+$ & $p_2 \| p_-$ and $p_1, p_2$ soft. Don't care how ugly $s_{1-}$ and $s_{2+}$ are

- Using the energies and angles of the electron and photons we get the following invariants

$$s_{-1} = 2\, E_-\, M_z\, \xi_1\, \eta_1$$
$$s_{-2} = 2\, E_-\, M_z\, \xi_2\, \eta_2$$
$$s_{12} = M_z^2\, \xi_1\, \xi_2\, \frac{(\eta_1 - \eta_2)^2}{N_1(x_3, x_4, x_5)}$$

The dangerous invariant is when both photons are emitted from p_

$$s_{-12} = 2\, M_z\, (E_-\, \xi_1\, \eta_1 + E_-\, \xi_2\, \eta_2 + \frac{M_z}{2}\, \xi_1\, \xi_2\, (\eta_1 - \eta_2)^2 / \bar{N}_1(x_3, x_4, x_5))$$

- The entangled singularities as $xi_1$, $xi_2$, $eta_1$, $eta_2$ vanish lead to the tree, and to the variable changes in each sector

# Triple collinear partition $\delta_{12}^{--}$

- Sector decomposition tells us to do the following variable changes to disentangle singularities in the triple collinear partition $\delta_{12}^{--}$ :

1. $S_1^{--}$, where $\xi_1 = x_1$, $\xi_2 = x_{\max} x_2 x_1$, $\eta_1 = x_3$, $\eta_2 = x_4 x_3$, $\kappa = x_5$;
2. $S_2^{--}$, where $\xi_1 = x_1$, $\xi_2 = x_{\max} x_2 x_4 x_1$, $\eta_1 = x_3 x_4$, $\eta_2 = x_3$, $\kappa = x_5$;
3. $S_3^{--}$, where $\xi_1 = x_1$, $\xi_2 = x_{\max} x_2 x_1$, $\eta_1 = x_2 x_3 x_4$, $\eta_2 = x_3$, $\kappa = x_5$.

- Take sector $S_1$-- as an example. Energies and angles take a simple form in terms of $x_i$

$$E_1 = \frac{m_Z}{2} x_1, \quad E_2 = \frac{m_Z}{2} x_1 x_2 x_{\max},$$

$$\cos\theta_1 = 1 - 2x_3, \quad \cos\theta_2 = 1 - 2x_3 x_4.$$

$\kappa$ is related to the azimuthal angle between the two photons

# Sector $S_1^{--}$ of the triple collinear partition $\delta_{12}^{--}$

- we have reduced our calculation to needing the following objects:

$$\int \underline{\mathrm{dLips}}_{S_1}^{--} F_1(x_1, x_2, x_3, x_4, x_5)$$

with

regular functions of $x_i$

$$\underline{\mathrm{dLips}}_{S_1}^{--} = \mathrm{dNorm}\ \mathrm{PS}_w\ (\mathrm{PS})^{-\epsilon} \times \frac{dx_1}{x_1^{1+4\epsilon}} \frac{dx_2}{x_2^{1+2\epsilon}} \frac{dx_3}{x_3^{1+2\epsilon}} \frac{dx_4}{x_4^{1+\epsilon}} \frac{d\kappa}{\pi(\kappa(1-\kappa))^{1/2}}$$

expandable in plus distributions

and

$$F_1(\{x_{i=1..5}\}) = \left[x_1^4 x_2^2 x_3^2 x_4 m_Z^2 \delta_{12}^{--}\right] |\mathcal{M}_{Z \to e^+ e^- \gamma\gamma}|^2$$

lets look at some of the singularities that can occur

# Sector S1-- of the triple collinear partition $\delta_{12}^{--}$

$$E_1 = \frac{m_Z}{2} x_1, \quad E_2 = \frac{m_Z}{2} x_1 x_2 x_{\max},$$

$$\cos\theta_1 = 1 - 2x_3, \quad \cos\theta_2 = 1 - 2x_3 x_4.$$

- what happens if $x_1 = 0$ ? → $E_1 = E_2 = 0$ → double soft limit

the QED eikonal current factorises completely

$$|\mathcal{M}_{Z\to e^+e^-\gamma\gamma}|^2 \to e^4 J_1 J_2 |\mathcal{M}_{Z\to e^-e^+}|^2$$

with

$$J_i = \frac{2p_- \cdot p_+}{(p_- \cdot p_i)(p_+ \cdot p_i)}$$

derive the following formula

$$F_1|_{x_1=0} = \frac{16e^4}{m_Z^2} |\mathcal{M}_{Z\to e^-e^+}|^2$$

easy to calculate numerically

# Sector $S_{1}$-- of the triple collinear partition $\delta_{12}^{--}$

$$E_1 = \frac{m_Z}{2}x_1, \quad E_2 = \frac{m_Z}{2}x_1 x_2 x_{\max},$$

$$\cos\theta_1 = 1 - 2x_3, \quad \cos\theta_2 = 1 - 2x_3 x_4.$$

- what happens if $x_2 = 0$ & $x_3 = 0$ ? ➡ $E_2 = 0$ & $p_1 \parallel p_-$ ➡ soft-collinear limit

the QED eikonal current factorises in two steps:

soft factorisation of $\gamma_2$ $\quad |\mathcal{M}_{Z\to e^+e^-\gamma_1\gamma_2}|^2 \to e^2 J_2 |\mathcal{M}_{Z\to e^+e^-\gamma_1}|^2$

collinear factorisation of $\gamma_1$ $\quad |\mathcal{M}_{Z\to e^+e^-\gamma_1}|^2 \approx \frac{2e^2}{s_{1e}} P_{e\gamma}(\epsilon, z) |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2$

derive the following formula

$$F_1|_{x_2=0,x_3=0} = \frac{16e^4 x_1}{m_Z E_- x_{\max}^2 \Delta_{12}} P_{e\gamma}(\epsilon, z) \times |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2.$$

easy to calculate numerically

# Double collinear partition $\delta_{12}^{-+}$

• New feature w.r.t. triple collinear partition: all final state particles participate in the singular structure. Difficult to find a parametrisation that makes all collinear singularities nice

• Our approach to tackle this problem was to use an iterated Catani-Seymour parametrisation:

First step: treat photon $\gamma_1$ as emitted, the electron as emitter and the positron as spectator. The 3 ➡ 2 momentum mapping was derived by Catani & Seymour (1997) and is determined by momentum conservation:

$$\gamma_1 + p_- + p_+ = \tilde{p}_{1-} + \tilde{p}_+$$

Second step: apply a similar mapping to the reduced momenta of the reduced reaction

$$Z \to \tilde{p}_{1-} + \tilde{p}_+ + p_2$$

treat photon $\gamma_2$ as emitted, $\tilde{e}^+$ as emitter and $\tilde{e}_{1-}$ as spectator. New momenta satisfy momentum conservation:

$$\tilde{p}_{1-} + \tilde{p}_+ + p_2 = \tilde{p}_{+2} + \tilde{\tilde{p}}_{1-}$$

• Now follow same steps as before: write down the invariants in the squared matrix element and derive the needed variable changes that disentangle the singularities. Four sectors are found in this case.

# Sector $S_4^{-+}$ of the double collinear partition $\delta_{12}^{-+}$

- Need to perform the following integration numerically over all regions of phase space:

$$\int dLips_{S_4}^{-+} F_4(x_1, x_2, x_3, x_4, x_5)$$

with

$$dLips_{S_4}^{-+} = \mathrm{dNorm}\, PS_w\, PS^{-\epsilon} \times \frac{dx_1}{x_1^{1+2\epsilon}} \frac{dx_2}{x_2^{1+\epsilon}} \frac{dx_3}{x_3^{1+2\epsilon}} \frac{dx_4}{x_4^{1+\epsilon}} dx_5$$

and

$$F_4(\{x_{i=1,..5}\}) = x_1^2 x_2 x_3^2 x_4 \delta_{12}^{-+} |M_{Z\to e^+e^-\gamma_1\gamma_2}|^2$$

# Sector $S_4$-+ of the double collinear partition $\delta_{12}^{-+}$

$$E_1 = \frac{m_Z}{2} x_1 \Omega_1, \; E_2 = \frac{m_Z}{2} x_3 \Omega_2$$

finite functions

- what happens if $x_1 = 0$ & $x_3 = 0$ ? ⇨ $E_1 = E_2 = 0$ ⇨ double soft limit

  the QED eikonal current factorises completely and we get

$$F_4|_{x_1=0, x_3=0} = \frac{16 e^4}{m_Z^2 \Omega_1 \Omega_2} |\mathcal{M}_{Z \to e^+ e^-}|^2$$

- other limits, collinear or soft-collinear, are studied in a similar way

# The real-virtual corrections

- Partitioning is simple, photon collinear either to the electron or positron

$$Z \to e^- e^+ \gamma_1 \quad \Longrightarrow \quad 1 = \delta_1^- + \delta_1^+$$

- A subtlety occurs here: in contrast to tree-level amplitudes, one-loop amplitudes are not rational functions of energies and angles

$$2\mathrm{Re}\left(\mathcal{M}^{(1)}_{Z\to e^+e^-\gamma}\mathcal{M}^{(0)*}_{Z\to e^+e^-\gamma}\right) \sim F_1 + F_2\,(s_{e1})^{-\varepsilon}$$

- Soft limit:

$$\mathcal{M}^{(0,1))}_{Z\to e^+e^-\gamma} \to e\left(\frac{p_-\cdot\epsilon_1}{p_-\cdot p_1} - \frac{p_+\cdot\epsilon_1}{p_+\cdot p_1}\right)\mathcal{M}^{(0,1)}_{Z\to e^+e^-}$$

▷ In QED, no fractional power, therefore no F2 term

$$F_1(0,\eta_1) = \frac{4\,e^2}{m_Z^2}\left(\mathcal{M}^{(1)}_{Z\to e^+e^-}\mathcal{M}^{(0)*}_{Z\to e^+e^-}\right)$$

# The real-virtual corrections

- Collinear limit:
  - Factorisation happens in terms of splitting amplitudes

$$
\begin{aligned}
\mathcal{M}^{(0)}_{Z\to e^- e^+ \gamma_1} &\to \text{Split}^{(0)}_{e^*_\lambda \to e_- \gamma} \mathcal{M}^{(0)}_{Z\to e^- e^+}, \\
\mathcal{M}^{(1)}_{Z\to e^- e^+ \gamma_1} &\to \text{Split}^{(0)}_{e^*_\lambda \to e_- \gamma} \mathcal{M}^{(1)}_{Z\to e^- e^+} \\
&\quad + \text{Split}^{(1)}_{e^*_\lambda \to e_- \gamma} \mathcal{M}^{(0)}_{Z\to e^- e^+}.
\end{aligned}
$$

  - Splitting amplitudes are defined through standard matrix elements and were computed by Kosower & Uwer (1999). Need to rewrite them in terms of splitting functions

$$
\begin{aligned}
\text{Split}^{(0)} &= -\frac{\bar{u}_a \not{\epsilon}_b u_{e^*}}{s_{ab}}, \\
\text{Split}^{(1)} &= -2\Big(r_3(z)\text{Split}^{(0)} - r_4(z)\text{Split}^{(2)}\Big) \\
\text{Split}^{(2)} &= \frac{2\bar{u}_a \not{k}_b u_{e^*}(k_a\cdot\epsilon_b)}{s_{ab}^2}
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\text{Split}^{(0)} \times \text{Split}^{(0)} &\to \frac{2}{s_{ab}} P_{e\gamma}(\epsilon, z), \\
\text{Split}^{(0)} \times \text{Split}^{(2)} &\to -\frac{2}{s_{ab}} \frac{z(1+z)}{1-z}
\end{aligned}
$$

# The real-virtual corrections

- From form of splitting amplitudes, can separate into $F_1$ and $F_2$ term

$$F_1|_{\eta_1=0} = \frac{\xi_1 P_{e\gamma}(\epsilon, z)}{E_- m_Z} \mathrm{Re}\left(2\mathcal{M}^{(0)*}_{Z\to e^+e^-}\mathcal{M}^{(1)}_{Z\to e^+e^-}\right)$$

$$F_2(\xi_1, 0) = -2\mathrm{Re}\left[\mathcal{M}^{(0)}_{Z\to e^+e^-}\mathcal{M}^{(0)}_{Z\to e^+e^-}(-z)^{-\epsilon}\right] \times \frac{2\xi_1}{E_- m_Z}\left(P_{e\gamma}(\epsilon, z)\tilde{r}_3(z) + \frac{z(1+z)}{1-z}\tilde{r}_4(z)\right),$$

Easily computed numerically

# Numerics

- Result for Z-decay

$$\Gamma_{Z\to e^+e^-} = \Gamma^{(0)}_{Z\to e^+e^-}\left(1+\frac{3}{4}\frac{\alpha}{\pi}+\left(\frac{\alpha}{\pi}\right)^2\delta^{(2)}\right) \quad \text{with} \quad \delta^{(2)} = \delta^{(2)}_{RR}+\delta^{(2)}_{RV}+\delta^{(2)}_{VV}$$

The results for the real-virtual and double real corrections based on the soft and collinear limits of the relevant matrix elements, as well as the known virtual-virtual correction which we have cross-checked:

$$\delta^{(2)}_{RR} = \frac{0.5}{\epsilon^4}+\frac{1.5}{\epsilon^3}-\frac{1.726}{\epsilon^2}-\frac{14.12}{\epsilon}-24.40$$

$$\delta^{(2)}_{RV} = -\frac{1}{\epsilon^4}-\frac{3}{\epsilon^3}+\frac{3.179}{\epsilon^2}+\frac{22.84}{\epsilon}+32.97$$

$$\delta^{(2)}_{VV} = \frac{0.5}{\epsilon^4}+\frac{1.5}{\epsilon^3}-\frac{1.4548}{\epsilon^2}-\frac{8.806}{\epsilon}-8.8058$$

in full agreement with an analytic computation based on the optical theorem

# Extension to higher multiplicities

- Method and parametrisation can be extended without difficulty to higher-multiplicity final states. Consider the real-real correction $Z \to e^+(p_+)\, e^-(p_-)\gamma(p_1)\gamma(p_2)\gamma(p_3)$. Partition the phase space, consider a triple-collinear partition with $p_3$ hard, $p_1 || p_2 || p_+$:

$$\frac{d\sigma}{d\mathcal{O}_0} = \int \mathrm{dLips}_{e^+e^-\gamma_1\gamma_2\gamma_3} |\mathcal{M}|^2 \delta\left(\mathcal{O} - \mathcal{O}_0\right) \frac{\delta_{12,+}}{D}$$

- For the phase space, decompose as:

$$\begin{aligned} \mathrm{dLips}_{e^+e^-\gamma_1\gamma_2\gamma_3} &= ds_{+-12}[dp_3][dp_{+-12}]\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2} \\ &\times \ \delta^{(d)}(p_Z - p_3 - p_{+-12}) \end{aligned}$$

- Can recycle the same parameterization as for $Z \to e^+e^-\gamma\gamma$; $p_3$ doesn't participate in singularity structure, its form can be arbitrarily complicated

# Summary

• Described in detail a subtraction scheme that enables the calculation of fully differential cross sections at NNLO

• The method combines sector decomposition with known soft and collinear limits of tree and one-loop scattering amplitudes to get the subtraction terms. No analytic integration is required for them

• Can recycle the same parameterization for lower multiplicity jet observables to get higher multiplicity ones

• Presented differential $Z \to e^+e^-$ as a simple example to describe the method. Applications of these ideas to more phenomenologically interesting QCD processes is ongoing.