Top-pair production at NNLO in QCD

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Based on:

P. Bärnreuther, M. Czakon, A. Mitov `12

✓ The call of the day is top A_{FB} (large deviation from NLO QCD observed D0/CDF)

✓ The main question is: is it due to higher order QCD effects?

- Higher order soft effects probed. No new effects appear (beyond Kuhn & Rodrigo).

Almeida, Sterman, Wogelsang '08
Ahrens et al '11
Manohar, Trott '12
Skands, Webber, Winter '12

- EW effects checked. Not as small as one might naively expect. Can't explain it.

Hollik, Pagani '11

- BLM scales setting does the job?

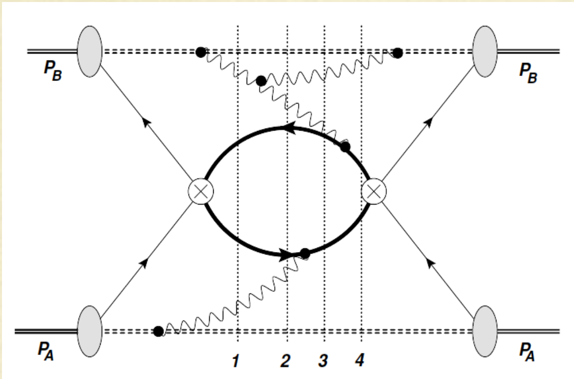
Brodsky, Wu '12

- Higher order QCD corrections? Not yet known; but some indications it might be *yes* (end of talk).

- Final state non-factorizable interactions (*next slide*)?

A_{FB}: non-factorizing contributions?

Motivating question: can A_{FB} be generated (or enhanced) by tT final state interactions?



Work with George Sterman, to appear

Prompted, in turn, by older work in QED

See, for example, Brodsky, Gillespie`68

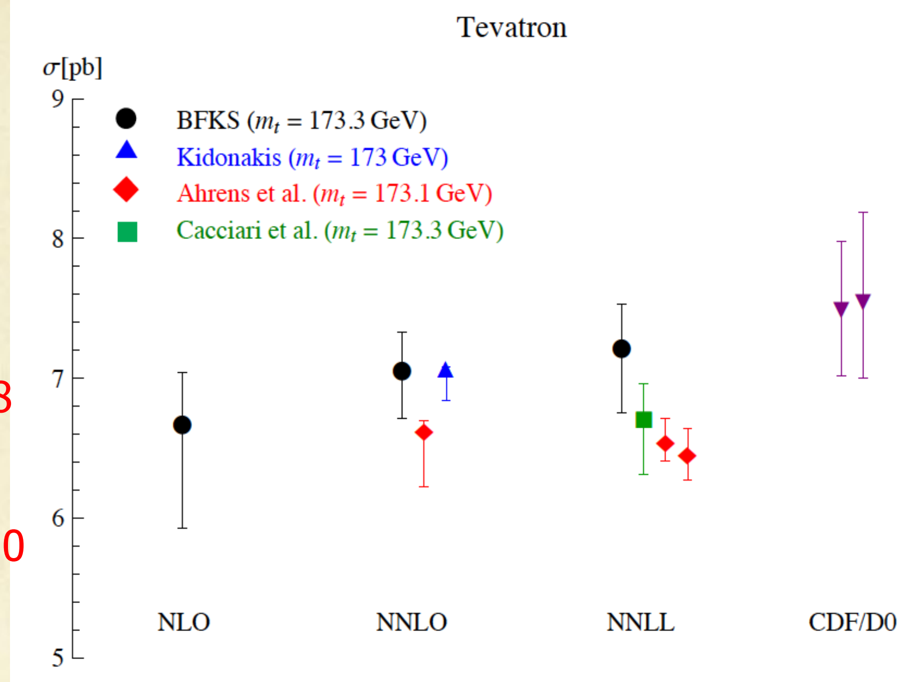
- ✓ We have devised an all-order proof of the cancellations of such interactions
- ✓ The subtle point is: What is the remainder? All depends on observables' definition.
- ✓ For inclusive observables (with conventional factorization) the remainder is small.
- ✓ For observables with rapidity gaps: large corrections are possible.

- ✓ In the rest of the talk I'll discuss the total inclusive cross-section.
- ✓ σ_{TOT} is relevant to A_{FB} in the following ways:
  - ✓ Gives normalizations,
  - ✓ An orthogonal, stringent constraint on SM/data.

Until now σ_{TOT} analyzed in approximate NNLO QCD:

Beneke, Czakon, Falgari, Mitov, Schwinn `09

Beneke, Falgari, Klein, Schwinn `11



... as an extension of the NLO

Nason, Dawson, Ellis '88
Beenakker et al '89

... resumed NLL

Sterman, Kidonakis '97
Bonciani, Catani, Mangano, Nason '98

and now NNLL resummation

Beneke, Falgari, Schwinn '10
Czakon, Mitov, Sterman '10
Ahrens et al '10-'11

EW small (~ 1.5%)

Hollik, Kollar `07
Kuhn, Scharf, Uwer '07
Beenakker, Denner, Hollik, Mertig, Sack, Wackeroth `93

Comparison between various $\text{NNLO}_{\text{approx}}$ groups shows:

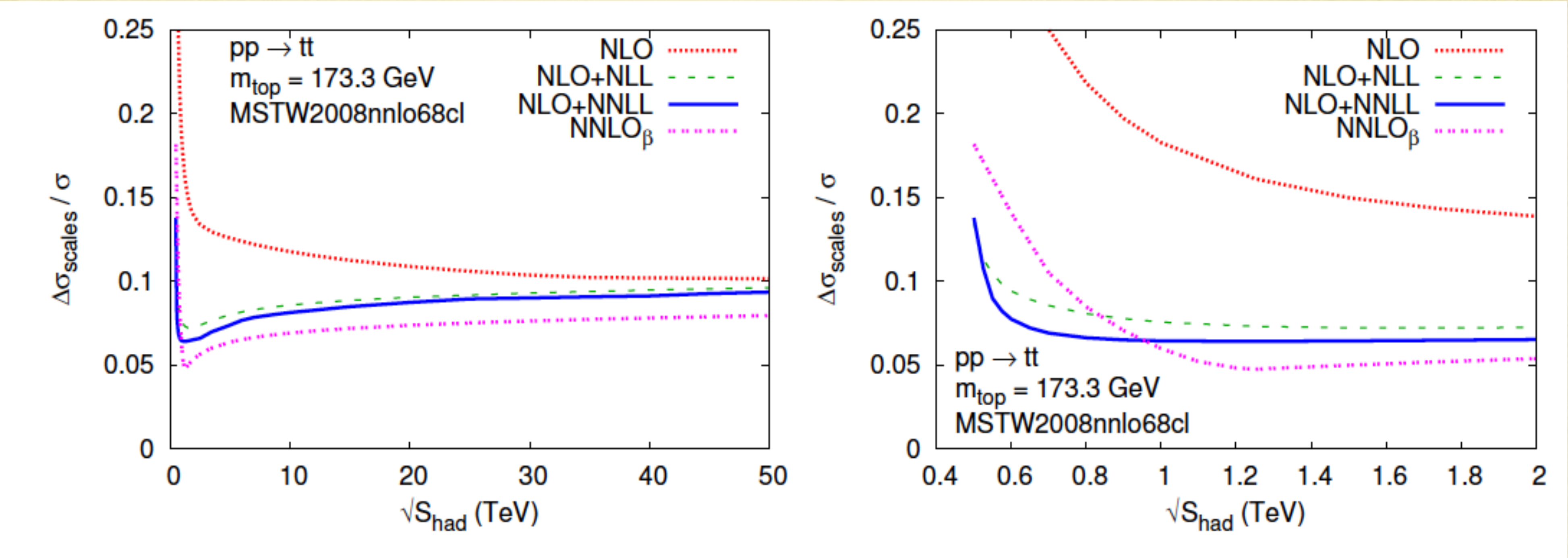
- ✓ Significant differences between various predictions
- ✓ Suggests the true approximate NNLO uncertainty
- ✓ The realistic improvements over NLO+NLL are small (to be expected)

Cacciari, Czakon, Mangano, Mitov, Nason `11

A nice example

Textbook example: by changing the collider energy go into (out of) the threshold region

Cacciari, Czakon, Mangano, Mitov, Nason `11



- Resummed results are better when close to threshold (as expected)
- One can quantify the question: when are we close to threshold? (below 1 TeV or so)
- $\mathrm{NNLO}_{\mathrm{approx}}$ is a subset of the resumed result.
  Has accidentally small scale dependence.

Our answer: compute the full NNLO

Bärnreuther, Czakon, Mitov `12

- So far published qq → tt +X
- Remaining reactions in the works

✓ First ever hadron collider calculation at NNLO with more than 2 colored partons.

✓ First ever NNLO hadron collider calculation with massive fermions.

Structure of the cross-section

$$\sigma = \frac{\alpha_s^2}{m_t^2} \sum_{ij} \int_0^{\beta_{\max}} \mathcal{L}_{ij}(\beta)\hat{\sigma}(\beta)$$

Relative velocity of tT

$$\rho = \frac{4m_t^2}{s} \qquad \beta = \sqrt{1-\rho}$$

- ✓ The partonic cross-section computed numerically in 80 points. Then fitted.
- ✓ Many contributing partonic channels:

Computed. Dominant at Tevatron (~85%)

$$q\bar{q} \longrightarrow t\bar{t}$$
$$q\bar{q} \longrightarrow t\bar{t}g$$
$$q\bar{q} \longrightarrow t\bar{t}gg$$
$$q\bar{q} \longrightarrow t\bar{t}q'\bar{q}' , \qquad q \neq q'$$

$$gg \longrightarrow t\bar{t}$$
$$gg \longrightarrow t\bar{t}g$$
$$gg \longrightarrow t\bar{t}gg$$
$$gg \longrightarrow t\bar{t}q\bar{q}$$

$$qg \longrightarrow t\bar{t}q$$
$$qg \longrightarrow t\bar{t}qg$$
$$qq' \longrightarrow t\bar{t}qq' , \qquad q \neq \bar{q}'$$
$$q\bar{q} \longrightarrow t\bar{t}q\bar{q}$$

All of the same complexity. No more conceptual challenges expected (just lots of CPU)

What goes into the NNLO?

- There are 3 principle contributions:

  ✓ 2-loop virtual corrections (V-V)

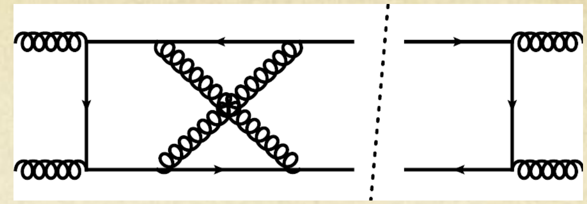
  Czakon `08

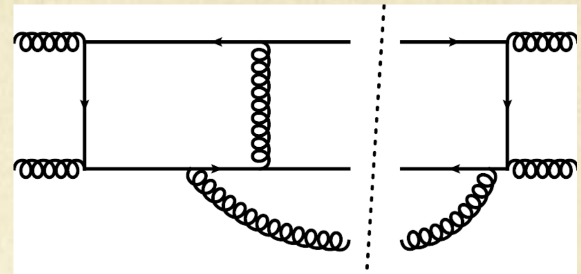
  ✓ 1-loop virtual with one extra parton (R-V)

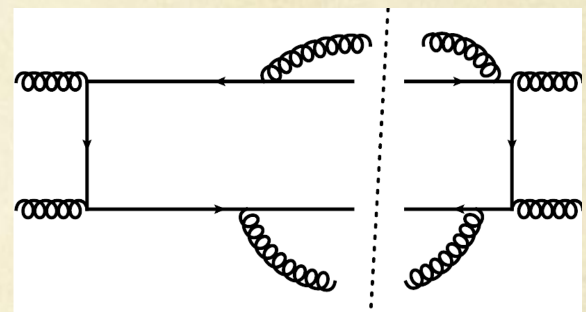
  1-loop amplitude: thanks to S. Dittmaier `07

  ✓ 2 extra emitted partons at tree level (R-R)

  Czakon `11







- And 2 secondary contributions:

  ✓ Collinear subtraction for the initial state

  Known, in principle. Done numerically.

  ✓ One-loop squared amplitudes (analytic)

  Korner, Merebashvili, Rogal `07

- Glued together in STRIPPER subtraction scheme

  Czakon `10

  (inspired by FKS and Sector Decomposition)

  Frixione, Kunszt, Signer `96
  Binoth, Heinrich `00
  Anastasiou, Melnikov, Petriello '04

How is the calculation organized?

- ✓ Guiding principle: do not try to combine all cuts into a single "finite" integration
  - ✓ To have the flexibility to <u>somehow</u> compute each cut,
  - ✓ Everything is done numerically. And in an independent approach.

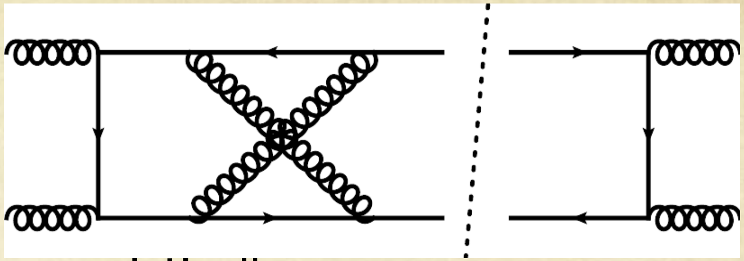
- ✓ The subtraction scheme @ NNLO: STRIPPER subtraction approach

Czakon `10

"the subtraction terms are defined by the phase space, not us"

What's needed for NNLO? V-V

Required are the two loop amplitudes:
qq → QQ and gg → QQ.



- ✓ Their high energy limits and their poles are known analytically. Fermionic corrections, leadig color, too.

Czakon, Mitov, Moch '07
Czakon, Mitov, Sterman '09
Ferroglia, Neubert, Pecjak, Yang '09
Bonciani et al. `08-`09

- ✓ Directly used here: The qq→QQ amplitude is known numerically

Czakon `07

- ✓ Numerical work underway for the gg→QQ

Czakon, Bärnreuther, to appear

What's the future here?

- ✓ Right now this is the biggest (and perhaps only) obstacle for NNLO phenomenology on a mass scale

V-V: how does it work?

$$\mathcal{O}_{VV} \sim \int d^d\Phi_2 \, |M_{2\to 2}|^2(\epsilon)$$

$$|M_{2\to 2}|^2(\epsilon) \sim \sum_{i\leq 4} \frac{M_i}{\epsilon^i}$$

✓ Since the phase space integration is non-singular:

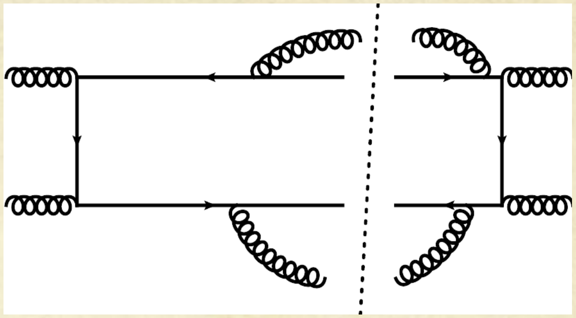
1. Expand phase-space and matrix element in ε
2. Integrate each term separately (i.e. derive 5 results; 4 will cancel)
3. Amplitude is known numerically. But its poles are known analytically.

Ferroglia, Neubert, Pecjak, Yang '09

✓ The poles of any 2-loop amplitude (with masses too) can be predicted

Mitov, Sterman, Sung '09-'10
Ferroglia, Neubert, Pecjak, Yang '09-'10

What's needed for NNLO? R-R



✓ A wonderful result By M. Czakon

Czakon `10-11

✓ The method is general (also to other processes, differential kinematics, etc).

✓ Explicit contribution to the total cross-section given.

✓ Just been verified in an extremely non-trivial problem.

✓ RR: The basic logic is very simple:

Czakon `10

1. Split the phase-space into sectors (algorithmic and process independent)

$$
\begin{aligned}
1 \;=\; & \\
& \left.\begin{array}{l} +\theta_1(k_1)\theta_1(k_2) \\ +\theta_2(k_1)\theta_2(k_2) \end{array}\right\} \text{triple-collinear sector} \\
& \left.\begin{array}{l} +\theta_1(k_1)\theta_2(k_2)(1-\theta_3(k_1,k_2)) \\ +\theta_2(k_1)\theta_1(k_2)(1-\theta_3(k_1,k_2)) \end{array}\right\} \text{double-collinear sector} \\
& +(\theta_1(k_1)\theta_2(k_2)+\theta_2(k_1)\theta_1(k_2))\theta_3(k_1,k_2) \;\}\, \text{single-collinear sector} \,.
\end{aligned}
$$

2. Remap the phase-space integration variables in each sector (algorithmic)

3. The singularities are factored out explicitly (no counterterms needed)

4. $$\mathcal{O}_S = \mathcal{N}\int_0^1 d\zeta d\eta_1 d\eta_2 d\xi_1 d\xi_2\; \theta_1(k_1)\theta_1(k_2)\,\frac{1}{\eta_1^{1-b_1\epsilon}}\frac{1}{\eta_2^{1-b_2\epsilon}}\frac{1}{\xi_1^{1-b_3\epsilon}}\frac{1}{\xi_2^{1-b_4\epsilon}}\int d\Phi_n(Q)\, F_J\, \mathcal{M}_S$$

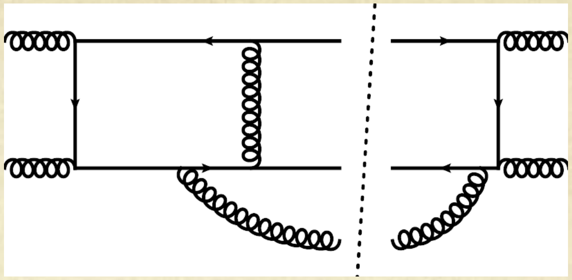
$$\frac{1}{\lambda^{1-b\epsilon}} = \frac{1}{b}\frac{\delta(\lambda)}{\epsilon} + \sum_{n=0}^{\infty}\frac{(b\epsilon)^n}{n!}\left[\frac{\ln^n(\lambda)}{\lambda}\right]_+$$

$$\int_0^1 d\lambda\,\left[\frac{\ln^n(\lambda)}{\lambda}\right]_+ f(\lambda) = \int_0^1 \frac{\ln^n(\lambda)}{\lambda}(f(\lambda)-f(0))$$

All is driven by phase-space

Effective counterterm.
Known from singular limits.

What's needed for NNLO? R-V



✓ Counterterms all known (i.e. all singular limits)

Bern, Del Duca, Kilgore, Schmidt '98-99
Catani, Grazzini '00
Bierenbaum, Czakon, Mitov '11

The finite piece of the one loop amplitude computed with a private code of Stefan Dittmaier.

Extremely fast code!

A great help!

Many thanks!

✓ RV: Similar to RR.

1. Less singular regions (one soft and/or one collinear)
2. Remap the phase-space integration variables in each sector (algorithmic)
3. The singularities are factored out explicitly (no counterterms needed)
4. Apply the usual identities:

$$\frac{1}{\lambda^{1-b\epsilon}} = \frac{1}{b}\frac{\delta(\lambda)}{\epsilon} + \sum_{n=0}^{\infty}\frac{(b\epsilon)^n}{n!}\left[\frac{\ln^n(\lambda)}{\lambda}\right]_+$$

$$\int_0^1 d\lambda \left[\frac{\ln^n(\lambda)}{\lambda}\right]_+ f(\lambda) = \int_0^1 \frac{\ln^n(\lambda)}{\lambda}(f(\lambda) - f(0))$$

Effective counterterm.
Known from singular limits.

5. The matrix elements are now divergent – no problem: expand and integrate
6. Counterterms more complicated, but known analytically.

Results @ parton level

Partonic cross-section through NNLO:

$$\sigma_{ij}\left(\beta, \frac{\mu^2}{m^2}\right) = \frac{\alpha_S^2}{m^2}\left\{\sigma_{ij}^{(0)} + \alpha_S\left[\sigma_{ij}^{(1)} + L\,\sigma_{ij}^{(1,1)}\right] + \alpha_S^2\left[\sigma_{ij}^{(2)} + L\,\sigma_{ij}^{(2,1)} + L^2\sigma_{ij}^{(2,2)}\right] + \mathcal{O}(\alpha_S^3)\right\},$$

The NNLO term:

$$\sigma_{q\bar{q}}^{(2)}(\beta) = F_0(\beta) + F_1(\beta)N_L + F_2(\beta)N_L^2$$

Numeric

Analytic

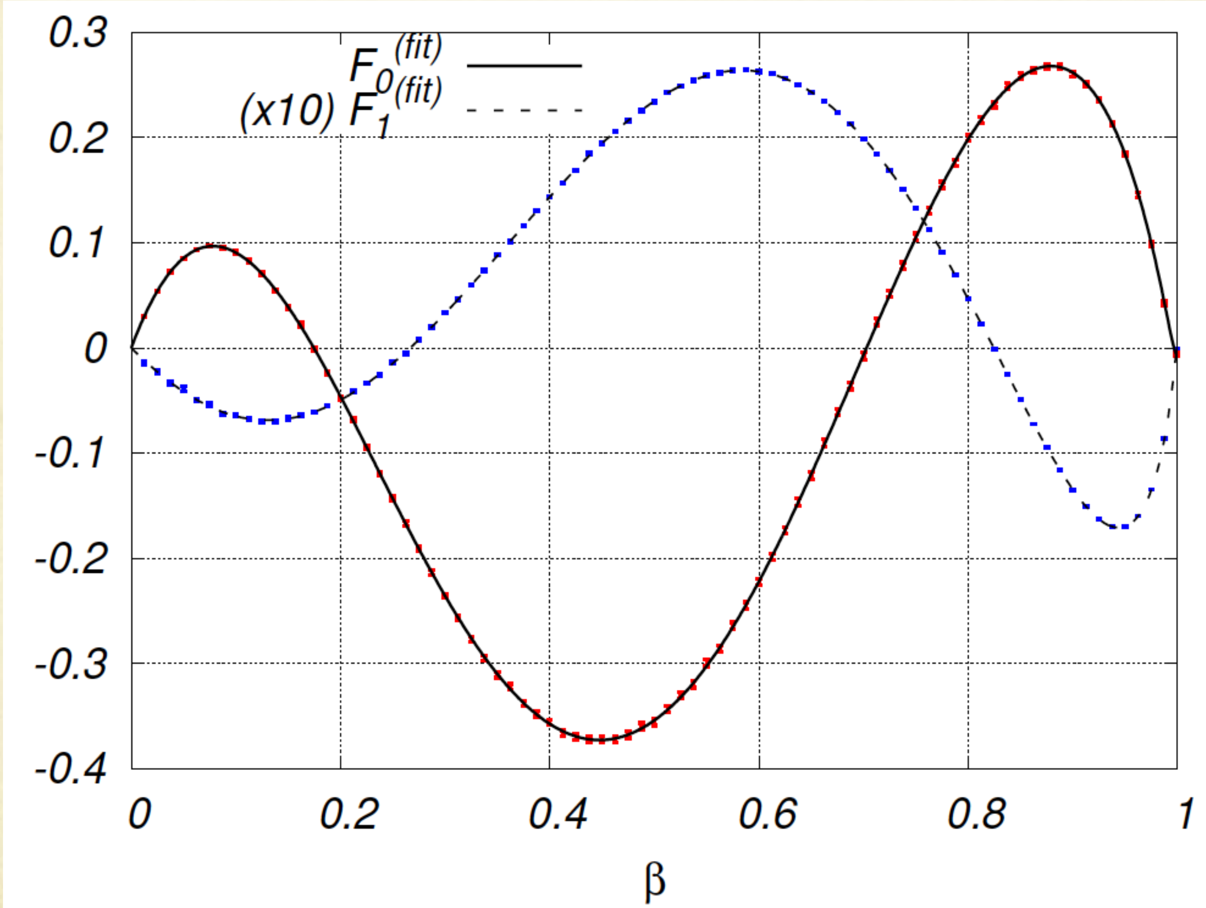
$$F_i \equiv F_i^{(\beta)} + F_i^{(\mathrm{fit})}, i = 0, 1$$

The known threshold approximation

Beneke et al `09

Notable features:

- ✓ Small numerical errors
- ✓ Agrees with limits

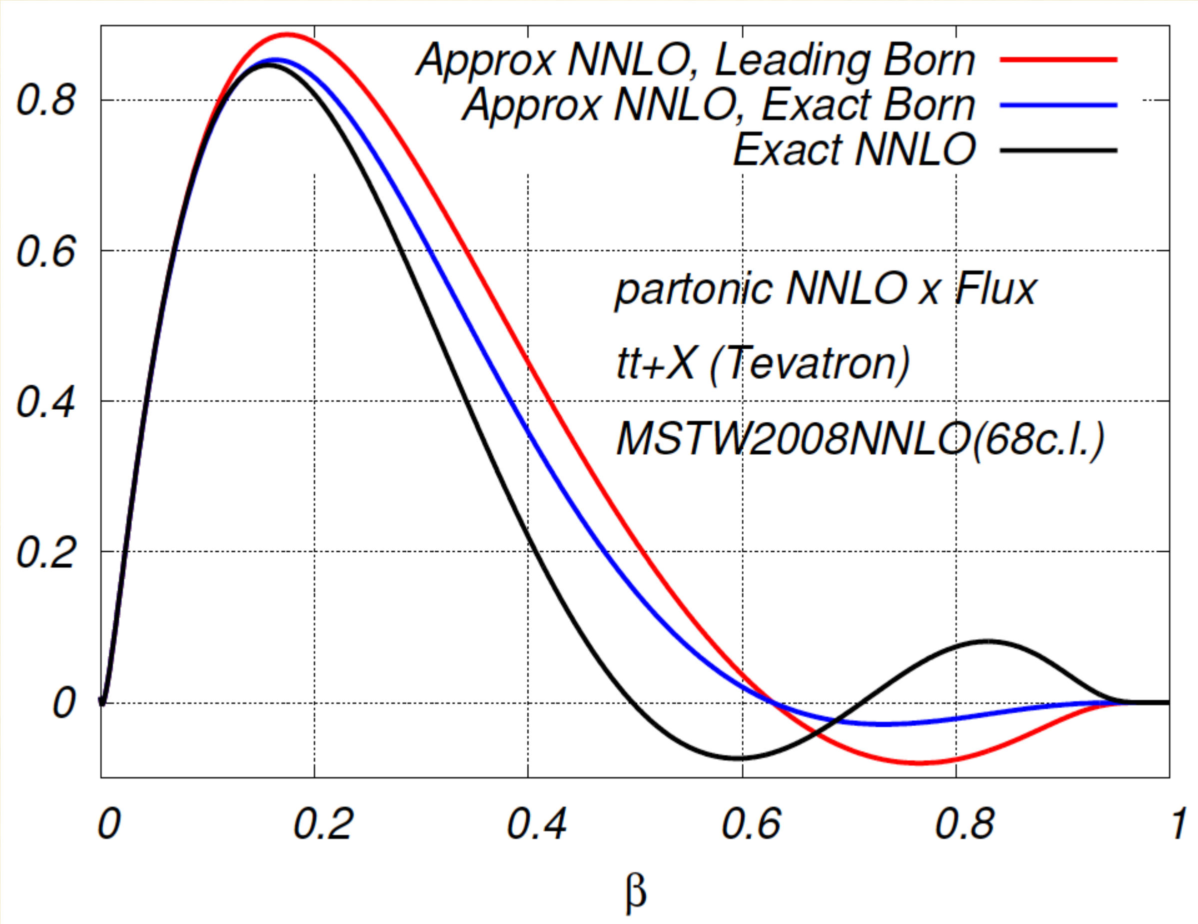


P. Bärnreuther et al arXiv:1204.5201

What happens once we add the flux?

$$\sigma = \frac{\alpha_s^2}{m_t^2} \sum_{ij} \int_0^{\beta_{\max}} \mathcal{L}_{ij}(\beta) \hat{\sigma}(\beta)$$

P. Bärnreuther et al arXiv:1204.5201



- ✓ Approximate NNLO is a an OK approximation at parton level
- ✓ There are non-trivial cancellations; the integrated numbers are closer to the exact ones than one might anticipate
- ✓ The power corrections to the Leading Born term have important effect

Here are the numbers for the Tevatron:

P. Bärnreuther et al arXiv:1204.5201

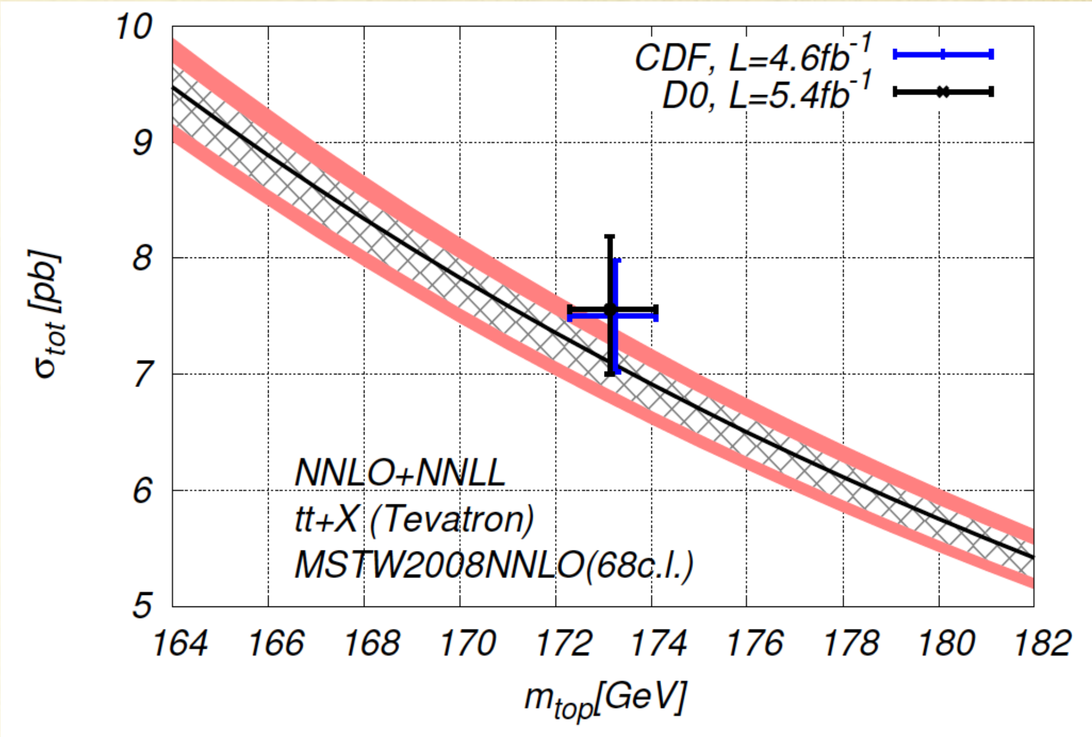
- ✓ Independent F/R scales
- ✓ MSTW2008NNLO
- ✓ mt=173.3

NNLO

$$\sigma_{\rm tot}^{\rm NNLO} = 7.005 \ {}^{+0.202\,(2.9\%)}_{-0.310\,(4.4\%)}\ [\text{scales}]\ {}^{+0.170\,(2.4\%)}_{-0.122\,(1.7\%)}\ [\text{pdf}]$$

$$\sigma_{\rm tot}^{\rm res} = 7.067 \ {}^{+0.143\,(2.0\%)}_{-0.232\,(3.3\%)}\ [\text{scales}]\ {}^{+0.186\,(2.6\%)}_{-0.122\,(1.7\%)}\ [\text{pdf}]$$

Best prediction at NNLO+NNLL



- ✓ Two loop hard matching coefficient extracted and included
- ✓ Very week dependence on unknown parameters (sub 1%): gg NNLO, A, etc.
- ✓ 50% scales reduction compared to the NLO+NNLL analysis of

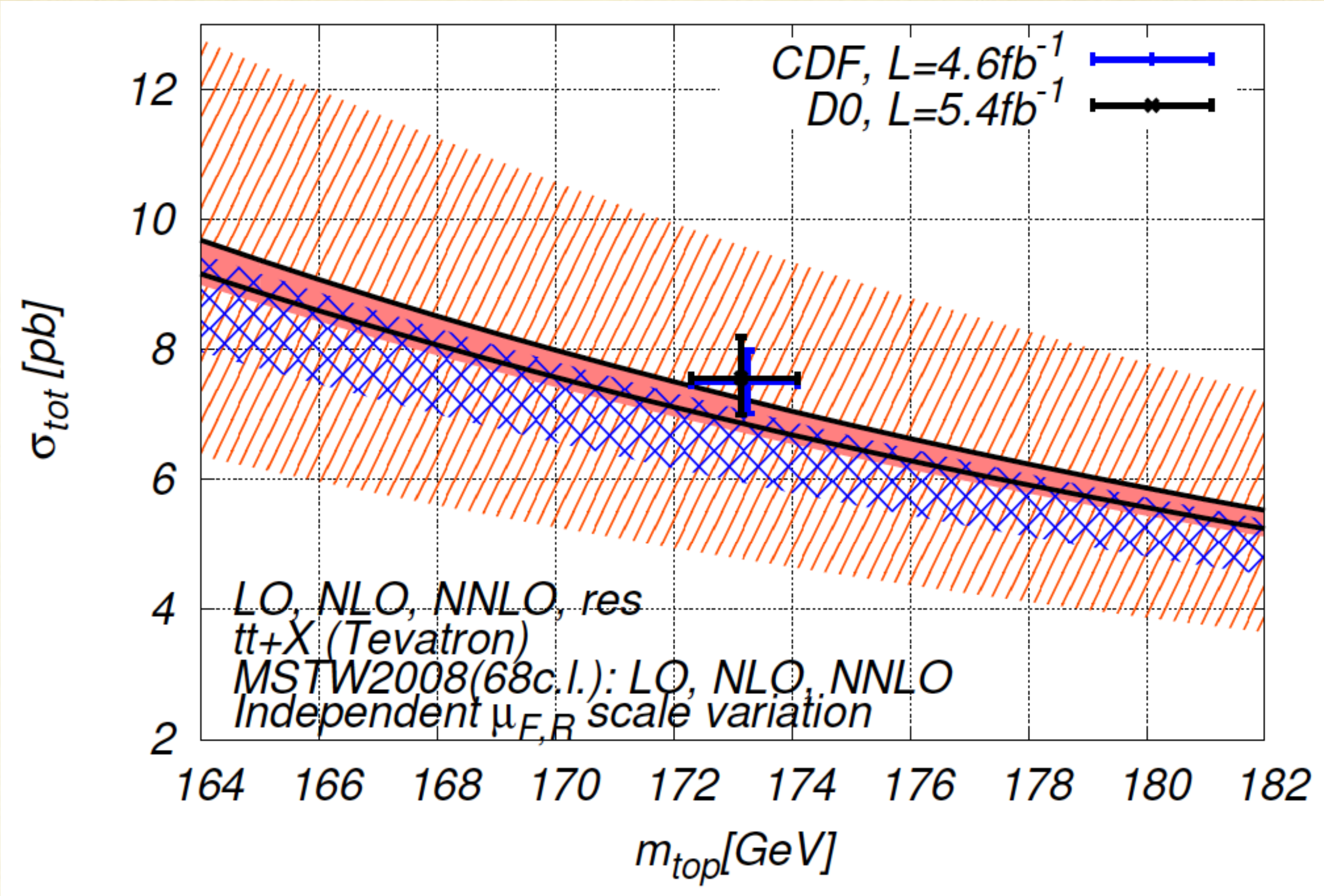
Cacciari, Czakon, Mangano, Mitov, Nason '11

6.72 + 3.6% - 6.1%

Good perturbative convergence:

- ✓ Independent F/R scales
- ✓ mt=173.3

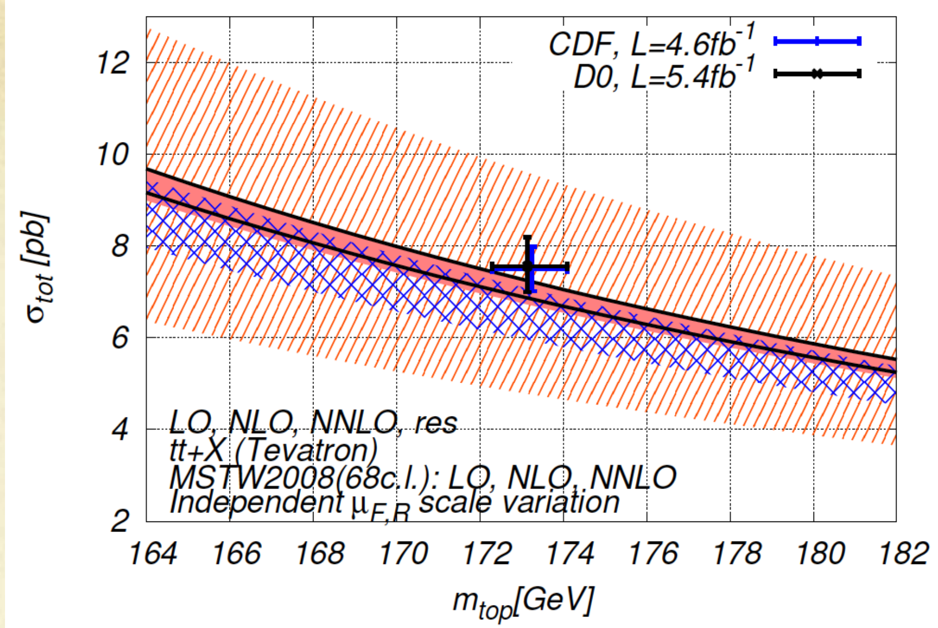
P. Bärnreuther et al arXiv:1204.5201



- ✓ Good overlap of various orders (LO, NLO, NNLO).
- ✓ Suggests our (restricted) independent scale variation is good

K-factors:

- Tevatron
- Fixed Order only



Same K-factors with NLO pdf

σ^{NNLO} (NNLO pdf) =

$$\underbrace{\underbrace{\underbrace{5.22059}_{(\alpha_S)^2} + \underbrace{1.23417}_{(\alpha_S)^3}}_{K_{NLO/LO}=1.24} + \underbrace{0.548064}_{(\alpha_S)^4}}_{K_{NNLO/NLO}=1.08}$$

$K_{NNLO/NLO}=1.08$ $\quad$ ($K_{NNLO+NNLL/NLO}=1.09$)

σ^{NNLO} (LO, NLO, NNLO pdf's) =

$$\underbrace{\underbrace{\underbrace{6.61926}_{(\alpha_S)^2} \; ; \; \underbrace{6.68123}_{(\alpha_S)^2+(\alpha_S)^3}}_{K_{NLO/LO}=1.01} \; ; \; \underbrace{7.00531}_{(\alpha_S)^2 \ldots (\alpha_S)^4}}_{K_{NNLO/NLO}=1.05}$$

K-factors alone not totally adequate without taking uncertainties into account

Implementation and numbers

✓ We have also prepared the tools for top physics:

Top++ : a C++ program for the calculation of the total cross-section:

✓ Includes:

Czakon, Mitov `11

- Fixed Order:
  - LO,NLO,
  - $\text{NNLO}_{\text{approx}}$ (gg),
  - exact NNLO (qqbar)

- Resummation (in Mellin space; full NNLL already there).

✓ Very user friendly.

✓ Developments:

- ✓ ver. 1.1: $\text{NNLO}_{\text{approx}}$ +NNLL (Released)
- ✓ ver. 1.2: NNLO(qqbar) + NNLL. Complete Tevatron pheno. (Released)
- ✓ ver. 2.0: Full NNLO + NNLL (Sometime this year)

Summary and Conclusions

✓ Computed the NNLO to qqbar -> ttbar

Significantly improved precision; right now O(½) from the experimental one at Tevatron

✓ Future work:

- Compute the remaining partonic reactions
- Compute the forward-backward asymmetry
- Compute differential distributions
- Add top decay

✓ Compute many more processes: dijets, W+jet, H+jet, etc @ NNLO

Facing the future, the stumbling block seems to be the availability of 2-loop amplitudes

✓ Our work is a strong motivation for new developments in this direction, too.

Backup

Structure of the cross-section close to threshold:

✓ Use soft-gluon expansion (from resummation)

Czakon, Mitov, Sterman '09
Beneke, Falgari, Schwinn '09

✓ Extract 2-loop Coulombic terms (from, say, e+e- → tT)

Beneke, Czakon, Falgari, Mitov, Schwinn '09

$$
\begin{aligned}
\sigma_{ij,\mathbf{I}}(\beta,\mu,m) \;=\; & \sigma^{(0)}_{ij,\mathbf{I}}\Bigg\{1+\frac{\alpha_s(\mu^2)}{4\pi}\left[\sigma^{(1,0)}_{ij,\mathbf{I}}+\sigma^{(1,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)\right] \\
& +\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\left[\sigma^{(2,0)}_{ij,\mathbf{I}}+\sigma^{(2,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)+\sigma^{(2,2)}_{ij,\mathbf{I}}\ln^2\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha_s^3)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{(2)}_{q\bar{q}} \;=\; & \frac{3.60774}{\beta^2}+\frac{1}{\beta}\Big(-140.368\ln^2\beta+32.106\ln\beta+3.95105\Big) \\
& +910.222\ln^4\beta-1315.53\ln^3\beta+592.292\ln^2\beta+528.557\ln\beta+C^{(2)}_{q\bar{q}}\,,
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{(2)}_{gg} \;=\; & \frac{68.5471}{\beta^2}+\frac{1}{\beta}\Big(496.3\ln^2\beta+321.137\ln\beta-8.62261\Big) \\
& +4608\ln^4\beta-1894.91\ln^3\beta-912.349\ln^2\beta+2456.74\ln\beta+C^{(2)}_{gg}\,,
\end{aligned}
$$

LL

NLL

NNLL

Hard matching

How is the threshold resummation done?

The resummation of soft gluons is driven mostly by kinematics:

Sterman '87
Catani, Trentadue '89

- Only soft emissions possible due to phase space suppression (hence kinematics)
- That's all there is for almost all "standard" processes: Higgs, Drell-Yan, DIS, e^+e^-

Key: the number of hard colored partons < 4

In top pair production (hadron colliders) new feature arises:

Color correlations due to soft exchanges (n>=4)

Non-trivial color algebra in this case.

The top cross-section: NNLL resummation

Factorization of the partonic cross-section close to threshold:

Kidonakis, Sterman '97
Czakon, Mitov, Sterman '09

$$\omega_P\left(N,\hat{\eta},\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right) = J_1(N,\alpha_s(\mu^2))\ldots J_k(N,M/\mu,m/\mu,\alpha_s(\mu^2))$$
$$\times \mathrm{Tr}\left[\mathbf{H}^P\left(\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right)\mathbf{S}^P\left(\frac{N^2\mu^2}{M^2},\frac{M^2}{m^2},\alpha_s(\mu^2)\right)\right] + \mathcal{O}(1/N)$$

N – the usual Mellin dual to the kinematical variable
that defines the threshold kinematics:

$$\sigma(N) = \int_0^1 dz z^{N-1}\sigma(z)$$

$z = Q^2/s$ ← Drell-Yan

$z = 4m^2/s$ ← t-tbar total X-section

$z = M_{t\bar{t}}^2/s$ ← t-tbar – pair invariant mass

J's – jet functions (different from the ones in amplitudes)

S,H – Soft/Hard functions. Also different.

The top cross-section: NNLL resummation

Specifically, for top-pair production we have:

$$\sigma^P(N,m^2,\mu^2) = \sigma^P_{\rm Born}(N)\ \left[J^P_{\rm in}(N,m^2,\mu^2)\right]^2\ \left[J_{\rm incl}(N,m^2,\mu^2)\right]^2\ {\rm Tr}\left[\hat{\mathbf{H}}^P(m^2,\mu^2)\mathbf{S}^P(N,m^2,\mu^2)\right] + \mathcal{O}(1/N)$$

where:

- $J^P_{\rm in}$ – is the Drell-Yan/Higgs cross-section
- $J_{\rm incl}$ – observable dependent function (i.e. depends on the final state)

$$J_{\rm incl}(N,m^2,\mu^2) = \exp\left\{\frac{1}{2}\int_0^1 dx\frac{x^{N-1}-1}{1-x}\ \Gamma_{\rm incl}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right\}$$

$$\Gamma_{\rm incl} = \frac{\alpha_s(\mu^2)}{\pi}\ C_F\left[-1-\ln\left(\frac{m^2}{\mu^2}\right)\right] + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left[\frac{K}{2}C_F\left(-1-\ln\left(\frac{m^2}{\mu^2}\right)\right) - \frac{\zeta_3-1}{2}C_F C_A\right]$$

Defines the poles of the massive QCD formfactor in the small-mass limit.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04
Gluza, Mitov, Moch, Riemann '09
Mitov, Moch '06

The top cross-section: NNLL resummation

Here is the result for the Soft function:

$$
\begin{aligned}
\mathbf{S}\left(\frac{N^2\mu^2}{M^2}, \beta_i\cdot\beta_j, \alpha_s(\mu^2)\right)\Bigg|_{\mu=M} &= \overline{\mathcal{P}}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S^{\dagger}\left(\beta_i\cdot\beta_j,\alpha_s\left(\mu'^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/\bar{N}^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j,\alpha_s\left(\mu'^2\right)\right)\right\} \\
&= \overline{\mathcal{P}}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S^{\dagger}\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/N^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\}
\end{aligned}
$$

Note: the Soft function satisfies RGE with the same anomalous dimension matrix as the Soft function of the underlying amplitude!

Therefore: knowing the singularities of an amplitude, allows resummation of soft logs in observables!

The top cross-section: NNLL resummation

We also need to specify a boundary condition for the soft function:

$$\mathbf{S}\left(1, \beta_i \cdot \beta_j, \alpha_s\left(M^2/N^2\right)\right) = \mathbf{S}^{(0)} + \frac{\alpha_s\left(M^2/N^2\right)}{\pi}\,\mathbf{S}^{(1)}\left(1, \beta_i \cdot \beta_j\right) + \ldots$$

For two-loop resummation we need it only at one loop
(since its contribution at two loops is only through the running coupling).

For example, for the total t-tbar cross-section in gg-reaction it reads:

$$\begin{aligned}
\mathbf{S}\left(1, \alpha_s\left(Q^2/N^2\right)\right) &= \mathbf{S}^{(0)}\left[1 + \frac{\alpha_s\left(Q^2/N^2\right)}{\pi}\,C_A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \ldots\right] \\
&= \mathbf{S}^{(0)}\left[1 + C_A\frac{\alpha_s\left(\mu^2\right)}{\pi}\left\{1 + \frac{\alpha_s\left(\mu^2\right)}{\pi}\,\frac{\beta_0}{4}\ln\left(\frac{N^2\mu^2}{Q^2}\right)\right\}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \ldots\right]
\end{aligned}$$

Can be derived by calculating the one-loop eikonal cross-section.

The top cross-section: NNLL resummation

Combining everything we get the following result for the resummed total t-tbar cross-section:

Hard function.
Known at 1 loop; 2 loops for qqbar

Czakon, Mitov '08
Hagiwara, Sumino, Yokoya '08

Bärnreuther, Czakon, Mitov `12

$$\frac{\sigma^P(N,m^2,\mu^2)}{\sigma^P_{\rm Born}(N)} = {\rm Tr}\left[\mathbf{H}^P(m^2,\mu^2)\ \exp\left\{\int_0^1 dx \frac{x^{N-1}-1}{1-x}\right.\right.$$
$$\left.\left.\times\left(\int_{\mu_F^2}^{4m^2(1-x)^2}\frac{dq^2}{q^2}\,2\,A_P\left(\alpha_s\left[q^2\right]\right)\mathbf{1}+D^P_{Q\overline{Q}}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right)\right\}\right]$$

Jet functions (from Drell-Yan/Higgs)

And the anomalous dimension is:

$$D^P_{Q\overline{Q}} = \frac{\alpha_s(\mu^2)}{\pi}(-C_A)\begin{pmatrix}0&0\\0&1\end{pmatrix}+\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left\{D^{(2)}_{\rm P}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\left(-C_A\frac{K}{2}-\frac{\zeta_3-1}{2}C_A^2-C_A\frac{\beta_0}{2}\right)\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$

Fixed by the small-mass limit of the massive formfactor!

Czakon, Mitov, Sterman '09
Beneke, Falgari, Schwinn '09