The Four Dimensional Helicity Scheme Beyond One Loop

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Introduction

The Four Dimensional Helicity (FDH) scheme is widely used for computing QCD corrections at next-to-leading order in perturbation theory. It was invented for, and is particularly convenient for, use with the helicity method and unitarity techniques. Unfortunately, the FDH is not a proper regularization scheme. Its renormalization program fails to remove all of the ultraviolet poles.

The errors induced by this failure are at order ε at one loop and therefore do not affect next-to-leading-order calculations. At each successive order of perturbation theory, however, the errors occur at one lower power of ε. At two loops, the errors are at order ε^0 and at three loops they are at order ε^{-1}.

Introduction

Thus the FDH should be looked upon as a shortcut for obtaining scattering amplitudes in some other, renormalizable regularization scheme. Indeed, this is how it has always been used at one-loop; final results have been presented in the 't Hooft-Veltman (HV) scheme using the prescription of Kunszt, et al, to transform the FDH scheme result.

In the past, it was not clear whether this conversion was necessary or merely expedient, allowing one to match onto standard definitions of the running coupling, parton distributions, fragmentation functions, etc. It is now certain that one must convert the results of a calculation in the FDH scheme into results in a properly defined scheme.

Introduction

I have so far focused on the failure of the FDH scheme's renormalization program. For inclusive calculations, performed using the optical theorem, a prescription to patch up the renormalization of the FDH is sufficient. The real power of the FDH scheme, however, is that it facilitates the calculation of amplitudes, allowing access to the differential information they contain. To transform FDH scheme results into HV scheme results, one must not only renormalize the amplitudes, but also properly account for the infrared structure.

In this talk, I will present a new prescription for renormalizing FDH amplitudes and describe a procedure to transform the FDH scheme results into HV scheme results with the proper infrared structure and without spurious finite terms.

Plan of the talk

My plan for this talk is as follows:

1. A review of common regularization schemes.
2. A new definition of the FDH (including renormalization).
3. The infrared structure of the FDH.
4. Summary and Conclusions.

Dimensional Regularization

Dimensional Regularization is the basis for most regularization schemes in use today.

- Respects gauge invariance.
- Respects Lorentz invariance.
- Handles both UV and IR divergences.

The application of Dimensional Regularization to different kinds of problems has led to the development of a variety of regularization schemes which share the dimensional regularization of momentum integrals but differ in their handling of observed states and spin degrees of freedom.

Regularization Schemes

I will be discussing several different regularization schemes which commonly appear in the literature.

- The HV Scheme
- The CDR Scheme
- The FDH Scheme
- The DRED and DR Schemes

The first two are closely related and yield identical results in the calculation that I will be describing. The last two are also closely related as I will describe in the following slides.

The 't Hooft-Veltman Scheme

The original formulation of dimensional regularization (the HV scheme) specifies that external (observed) states are treated as four-dimensional, while internal states are to be treated as $D_m = 4 - 2\varepsilon$ dimensional. The D_m-dimensional vector space is *larger* than 4-dimensional space-time:

$$g^{\mu\nu} g_\nu^\alpha = g^{\mu\alpha}, \qquad g^{\mu\nu} \eta_\nu^\alpha = \eta^{\mu\alpha}, \qquad \eta^{\mu\nu} \eta_\nu^\alpha = \eta^{\mu\alpha},$$
$$g^{\mu\nu} g_{\mu\nu} = D_m, \qquad \eta^{\mu\nu} \eta_{\mu\nu} = 4.$$

In HV, internal gluons have $D_m - 2 = 2 - 2\varepsilon$ spin degrees of freedom. Internal fermions, however, still have exactly 2 spin degrees of freedom. Because it treats observed states as being four dimensional, the HV scheme is useful for presenting helicity amplitudes.

The Conventional Dimensional Regularization Scheme

In the CDR scheme, all states (observed or internal) are continued to $D_m = 4 - 2\varepsilon$ dimensions. This is in many ways simpler than the HV scheme, especially when dealing with infrared sensitive theories like QCD. In HV, if external states have an infrared overlap, they must be treated as internal (D_m-dimensional). In CDR, all states are already D_m-dimensional, so the overlap is automatically treated properly.

The HV and CDR schemes are closely related. Their behaviors under the renormalization group (β-functions, anomalous dimensions) is identical. They differ only in the details of how they handle observed states.

The Four Dimensional Helicity Scheme

The FDH takes observed states to be four-dimensional and the D_m-dimensional space where momenta take values to be *larger* than 4-dimensional space-time, but also defines a *still larger* D_s-dimensional vector space where spin degrees of freedom take values. D_s is taken to be equal to 4 so that particles have the same number of spin degrees of freedom as they have in 4 dimensions.

$$
\begin{aligned}
g^{\mu\nu} g_{\mu\nu} &= D_s\,, & \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} &= D_m\,, & \eta^{\mu\nu}\eta_{\mu\nu} &= 4\,,\\
g^{\mu\nu}\hat{g}_{\nu}^{\rho} &= \hat{g}^{\mu\rho}\,, & g^{\mu\nu}\eta_{\nu}^{\rho} &= \eta^{\mu\rho}\,, & \hat{g}^{\mu\nu}\eta_{\nu}^{\rho} &= \eta^{\mu\rho}\,,\\
g^{\mu\nu}\delta_{\nu}^{\rho} &= \delta^{\mu\rho}\,, & \hat{g}^{\mu\nu}\delta_{\nu}^{\rho} &= 0\,, & \eta^{\mu\nu}\delta_{\nu}^{\rho} &= 0\,.
\end{aligned}
$$

It is also useful to define the complement space $D_x = D_s \setminus D_m$ which has dimension $D_x = 2\varepsilon$.

The Dimensional Reduction Scheme

The DRED scheme was invented for regularizing supersymmetric theories. One starts from 4-dimensional space-time and compactifies to a *smaller* vector space of dimension $D_m = 4 - 2\varepsilon$ in which momenta take values.

$$g^{\mu\nu} g_\nu^\alpha = g^{\mu\alpha}, \qquad g^{\mu\nu} \eta_\nu^\alpha = g^{\mu\alpha}, \qquad \eta^{\mu\nu} \eta_\nu^\alpha = \eta^{\mu\alpha}.$$

Particles in the spectrum retain their 2 spin degrees of freedom from 4 dimensions. This preserves supersymmetry.

BUT: The Ward Identity only applies to the vector subspace in which momenta are defined!

In non-SUSY theories, the "evanescent" (2ε-dimensional) gluons are independent degrees of freedom from the D_m-dimensional gluons. The fields and their couplings renormalize independently!

The DR scheme

Defining the DRED scheme in terms of a compactification has semantic appeal, but presents practical difficulties. There is no consistent way to describe chirality, a 4-dimensional property, if momenta take values in a vector space of fewer than 4-dimensions. If $D_m > 4$, as in CDR, chirality can be defined on the 4-dimensional subspace of D_m, with evanescent gluons living in a still-larger vector space D_s, just like in the FDH.

In the absence of chiral operators (and ignoring the semantics of power counting) there are no computational differences between the DR and DRED schemes. In particular, the Ward identity only holds on D_m, so the evanescent gluons must still be treated as independent degrees of freedom from the D_m-dimensional gluons. I will refer to this as the DR scheme.

The failure of the FDH

The FDH and DR schemes bear almost the same relation to one another as the HV and CDR schemes. The difference is in the handling of the evanescent (D_x-dimensional) spin degrees of freedom. The FDH renormalization program of treating them just like gluons is inconsistent and leads to incorrect results beyond one loop.

Last year, I presented results for the calculation of the total cross section for $e^+e^- \to$ hadrons at NLO, NNLO and N^3LO. I found that the FDH results were correct (to order ε^0) at NLO, finite but incorrect at NNLO, and contained unsubtracted poles in ε at N^3LO. With the DRED scheme, however, and its handling of the evanescent states, I was able to reproduce the known results originally computed in the CDR scheme.

Dimensional Reconstruction

I have shown that the FDH fails beyond one loop because its renormalization program does not correctly handle the “extra” spin degrees of freedom. Boughezal, et al. have subsequently described a way to patch up the renormalization program. Their method, called Dimensional Reconstruction, involves calculating lower-loop amplitudes with one or more scalar multiplets, allowing one to solve for the correct renormalizations.

While dimensional reconstruction allows one to correctly renormalize FDH amplitudes, there are still problems that need to be addressed.

Remaining Problems

1. Dimensional Reconstruction treats the FDH as a non-renormalizable theory. Each process at each order needs a new renormalization constant. It seems that there there should be a more universal prescription.
2. The power of the FDH is that it gives access to full amplitudes and the differential information they contain. To make use of amplitudes, renormalization is not enough; we need to understand the infrared structure of the FDH amplitude.

A solution to both problems can be found by closely tying the FDH to a renormalizable scheme.

Observations about FDH and DR calculations

In formulating a functional definition of the FDH, I am looking for a scheme that works well with unitarity and helicity methods. However, several facts become apparent when performing Feynman diagram calculations:

1. By breaking up the metric tensor of the FDH scheme, $g^{\mu\nu} = \hat{g}^{\mu\nu} + \delta^{\mu\nu}$ one finds that the FDH and DR scheme calculations are term-by-term identical, except for the identification of the couplings in the FDH.
2. The DR calculation contains *all* of the diagrams from the CDR scheme calculation, plus a (large) set of diagrams where one or more evanescent fields are exchanged.

Defining the FDH

The first observation yields a precise definition of the FDH: It is identical to the DR scheme ($D_s > D_m > 4$) with the following conditions:

1. Observed states are four dimensional.
   - This is equivalent to the distinction between HV and CDR,
   - and eliminates evanescent external states.
2. The evanescent couplings are taken to be equal to the QCD coupling.
   - $\mathscr{M}_{FDH} = \mathscr{M}_{DR}|_{\{\alpha_e,\eta_1,\ldots\}\to\alpha_s}$
   - This is the step that breaks the renormalization program.

The FDH is to be used only to compute bare quantities. I will derive quantities within the DRED scheme that will allow us to transform bare FDH amplitudes into renormalized HV scheme amplitudes.

Finite terms in DR

The second observation explains why it is sufficient to merely renormalize the DR scheme and understand its infrared structure.

When evaluating the “extra” diagrams that do not occur in a CDR calculation, one always encounters an evanescent spin-sum, meaning that the evanescent diagrams are suppressed by a factor of $D_x = 2\varepsilon$. Thus, for an evanescent diagram to make a finite (or singular) contribution to the amplitude, the factor of D_x must be balanced by ultraviolet and/or infrared poles.

Renormalizing the FDH

Renormalization is accomplished by computing the counterterms in the DR scheme, correctly treating the evanescent couplings as distinct from the QCD coupling, and then, at the end, setting the evanescent couplings equal to α_s.

$$\mathscr{M}_{\mathrm{CT},FDH}(\alpha_s) = \mathscr{M}_{\mathrm{CT},DR}(\alpha_s, \alpha_e, \ldots)|_{\alpha_e,\cdots\to\alpha_s}$$

Adding these counterterms to the FDH amplitude will correctly remove the ultraviolet poles and those apparently finite terms that arise from UV poles.

The correct infrared and finite terms can be determined by a similar procedure, but first I will review the IR structure of QCD amplitudes in the CDR scheme.

The Infrared Structure in CDR

The infrared structure of QCD amplitudes is completely universal and can be predicted entirely in terms of the identities and momenta of the external states.

In the language of Sterman and Tejeda-Yeomans, an amplitude factorizes into three functions: the Jet function, the Soft function and the Hard Scattering Function.

$$\left|\mathcal{M}_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right\rangle = \mathcal{J}_{\mathbf{f}}\left(\alpha_s(\mu^2), \epsilon\right) \mathbf{S}_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \left|H_{\mathbf{f}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)\right\rangle .$$

The Jet Function

The jet function $\mathcal{J}_{\mathbf{f}}$ describes the collinear evolution of the amplitude's external legs and is found to be the product of individual jet functions $\mathcal{J}_{f_i}$, one for each external leg, which are naturally defined in terms of their Sudakov form factors and the anomalous dimensions that appear there,

Because the infrared structure is universal, the anomalous dimensions of the jet functions can be extracted from the direct calculation of QCD amplitudes rather than the eikonal amplitudes that define the Sudakov form factors.

$$\ln \mathcal{J}_i\left(\alpha_s(\mu^2),\varepsilon\right) = -\left(\frac{\alpha_s}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\gamma_{K\,i}^{(0,1)} + \frac{1}{4\,\varepsilon}\mathcal{G}_i^{(0,1)}(\varepsilon)\right]$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\left\{\frac{\beta_{QCD}^{(0,2)}}{8}\frac{1}{\varepsilon^2}\left[\frac{3}{4\,\varepsilon}\gamma_{K\,i}^{(0,1)} + \mathcal{G}_i^{(0,1)}(\varepsilon)\right] - \frac{1}{8}\left[\frac{\gamma_{K\,i}^{(0,2)}}{4\,\varepsilon^2} + \frac{\mathcal{G}_i^{(0,2)}(\varepsilon)}{\varepsilon}\right]\right\} + \dots$$

The Soft Function

The soft function, $\mathbf{S_f}$, describes soft exchanges between the external legs. It is a matrix in color space because soft gluon exchanges can rearrange the color flow that took place in the hard scattering.

Like the jet function, the soft function can be defined in terms of eikonal amplitudes and is determined entirely by the soft anomalous dimension matrix $\mathbf{\Gamma}_{S_f} \propto \frac{1}{2}\sum_{i\in\mathbf{f}}\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j \ln\left(\frac{\mu^2}{-s_{ij}}\right)$. As in the case of the jet function, the coefficients of the soft anomalous dimension matrix are universal and can be extracted from Feynman diagram calculations of particular scattering amplitudes.

$$\mathbf{S_f}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = 1 + \frac{1}{2\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)\mathbf{\Gamma}^{(1)}_{S_f} - \frac{\beta_0}{4\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(1)}_{S_f} + \frac{1}{8\,\varepsilon^2}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(1)}_{S_f}\times\mathbf{\Gamma}^{(1)}_{S_f} + \frac{1}{4\,\varepsilon}\left(\frac{\alpha_s}{\pi}\right)^2\mathbf{\Gamma}^{(2)}_{S_f} + \ldots$$

The Hard-Scattering Function

The hard-scattering function, $|H_{\mathbf{f}}\rangle$ describes the short-distance scattering process and is a vector in the color-space representation. As with any factorization, there is considerable freedom to move terms about from one function to the others. I adopt the convention of Sterman et al. that the logarithms of the jet and soft functions contain only infrared poles, while all infrared finite terms are absorbed into the hard scattering function.

$$
\begin{aligned}
\left|\mathcal{M}^{(0)}\right\rangle &= \left|H^{(0)}\right\rangle \\
\left|\mathcal{M}^{(1)}\right\rangle &= \left[\left(\mathcal{J}^{(1)} + \mathbf{S}^{(1)}\right)\left|H^{(0)}\right\rangle + \left|H^{(1)}\right\rangle\right] \\
\left|\mathcal{M}^{(2)}\right\rangle &= \left[\left(\mathcal{J}^{(2)} + \mathcal{J}^{(1)}\mathbf{S}^{(1)} + \mathbf{S}^{(2)}\right)\left|H^{(0)}\right\rangle \right. \\
&\qquad \left. + \left(\mathcal{J}^{(1)} + \mathbf{S}^{(1)}\right)\left|H^{(1)}\right\rangle + \left|H^{(2)}\right\rangle\right]
\end{aligned}
$$

IR structure of the DR scheme

The infrared structure in the DR scheme is very similar to that of the CDR scheme. The anomalous dimensions get extra terms from the evanescent interactions and there are new terms that depend on the evanescent coupling. For instance, the jet function becomes:

$$\ln \mathscr{J}_{i,DR}\left(\alpha_s(\mu^2),\alpha_e(\mu^2),\varepsilon\right) = -\left(\frac{\alpha_s}{\pi}\right)\left[\frac{1}{8\,\varepsilon^2}\overline{\gamma}^{(1)}_{K\,i} + \frac{1}{4\,\varepsilon}\overline{\mathscr{G}}^{(1)}_{i}(\varepsilon)\right]$$

$$+\left(\frac{\alpha_s}{\pi}\right)^2\left\{\frac{\overline{\beta}^{0}_{QCD}}{8}\frac{1}{\varepsilon^2}\left[\frac{3}{4\,\varepsilon}\overline{\gamma}^{(1)}_{K\,i} + \overline{\mathscr{G}}^{(1)}_{i}(\varepsilon)\right] - \frac{1}{8}\left[\frac{\overline{\gamma}^{(2)}_{K\,i}}{4\,\varepsilon^2} + \frac{\overline{\mathscr{G}}^{(2)}_{i}(\varepsilon)}{\varepsilon}\right]\right\}$$

$$+\left(\frac{\alpha_e}{\pi}\right)\frac{\overline{\mathscr{G}}^{(0,1)}_{i,e}(\varepsilon)}{4\,\varepsilon} - \left(\frac{\alpha_s}{\pi}\right)\left(\frac{\alpha_e}{\pi}\right)\frac{1}{8}\left[\frac{\overline{\beta}^{(1,1)}_{e}\,\overline{\mathscr{G}}^{(0,1)}_{i,e}(\varepsilon)}{\varepsilon^2} - \frac{\overline{\mathscr{G}}^{(1,1)}_{i,e}(\varepsilon)}{\varepsilon}\right]$$

$$+\left(\frac{\alpha_e}{\pi}\right)^2\frac{1}{8}\left[\frac{\overline{\beta}^{(0,2)}_{e}\,\overline{\mathscr{G}}^{(0,1)}_{i,e}(\varepsilon)}{\varepsilon^2} - \frac{\overline{\mathscr{G}}^{(0,2)}_{i,e}(\varepsilon)}{\varepsilon}\right] + \ldots$$

Converting FDH amplitudes to HV amplitudes

The two-loop hard-scattering function in the HV scheme can be extracted from the two-loop FDH scheme amplitude by adding the DR scheme counterterms, subtracting the DR scheme infrared structure and then identifying the evanescent couplings with α_s.

$$\left|H^{(2)}_{HV}\right\rangle = \left|\mathcal{M}^{(2)}_{FDH}\right\rangle + \left|\mathcal{M}^{(2)}_{DR}\right\rangle^{CT}\Big|_{\{\alpha_e,\eta_1,\ldots\}\to\alpha_s}$$
$$- \Big[\left(\mathcal{I}^{(1)}_{DR} + \mathbf{S}^{(1)}_{DR}\right)\left|H^{(1)}_{DR}\right\rangle$$
$$+ \left(\mathcal{I}^{(2)}_{DR} + \mathcal{I}^{(1)}_{DR}\mathbf{S}^{(1)}_{DR} + \mathbf{S}^{(2)}_{DR}\right)\left|H^{(0)}_{DR}\right\rangle\Big]_{\{\alpha_e,\eta_1,\ldots\}\to\alpha_s}.$$

Although cast in the language of factorization, this is the two-loop variation of the shift prescribed by Kunszt, Signer and Trócsányi for one-loop amplitudes.

Summary

- The four dimensional helicity scheme is a good match for helicity/unitarity based calculations but is not renormalizable.
- Results calculated in the FDH must be transformed to a renormalizable schme (preferably HV).
- The transformation must account for renormalization and for the infrared structure of both the FDH and HV schemes.
- I have defined the FDH in terms of the DR scheme and have computed the anomalous dimensions that describe the ultarviolet and infrared structure of the DR scheme.
- The connection between the FDH and DR, combined with knowledge of the anomalous dimensions describing the universal structure of DR and HV amplitudes allows me to transform bare FDH amplitudes into renormalized HV scheme amplitudes.