Some Curious Consequences of the Minimal Length Uncertainty Relation

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Talk based on work presented in:

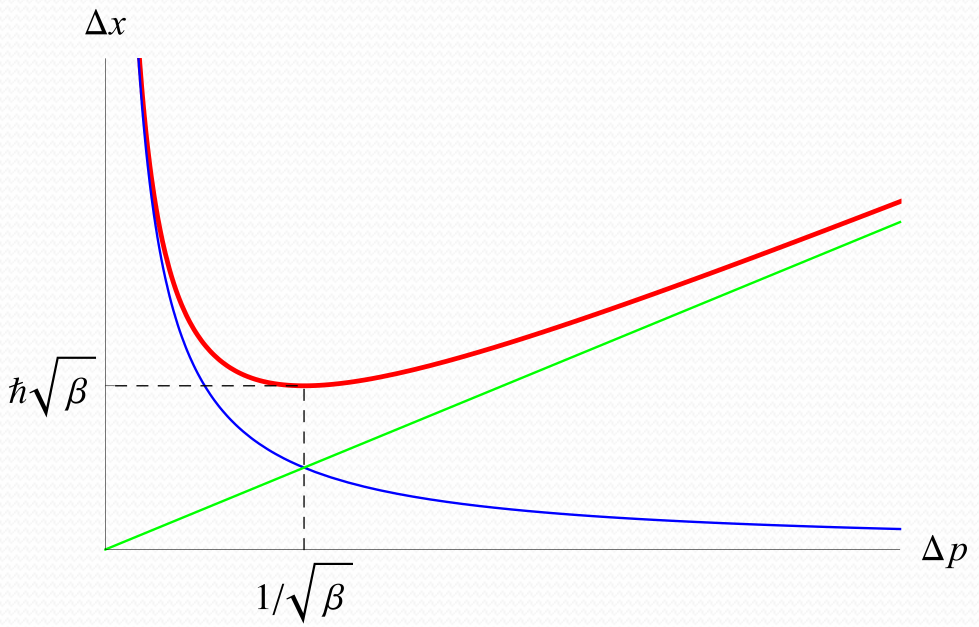
- Phys. Rev. D 84, 105029 (2011) [arXiv:1109.2680] with Zack Lewis,

and previous work with:

- Djordje Minic (Virginia Tech)
- Lay Nam Chang (Virginia Tech)
- Naotoshi Okamura (Yamanashi Univ.)
- Sandor Benczik (quit physics)
- Saif Rayyan (MIT, physics education)

The Minimal Length Uncertainty Relation

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + \beta \Delta p\right) \quad \Rightarrow \quad \Delta x \geq \Delta x_{\min} = \hbar\sqrt{\beta}$$



Suggested by Quantum Gravity. Observed in perturbative String Theory.

Deformed Commutation Relation

$$\frac{1}{i\hbar}\left[\hat{x},\hat{p}\right]=1+\beta\hat{p}^2 \qquad \Rightarrow \qquad \Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p}+\beta\Delta p\right)$$

The operators can be represented as:

$$\begin{cases} \hat{x} = i\hbar\left(1+\beta\hat{p}^2\right)\dfrac{d}{dp} \\ \hat{p} = p \end{cases}$$

and the inner product as

$$\langle f|g\rangle = \int_{-\infty}^{\infty}\frac{dp}{(1+\beta p^2)} f^*(p)\, g(p)$$

Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$$

$$\Rightarrow \quad \left[-\frac{\hbar^2 k}{2} \left\{ (1 + \beta p^2) \frac{d}{dp} \right\}^2 + \frac{p^2}{2m} \right] \psi(p) = E \psi(p)$$

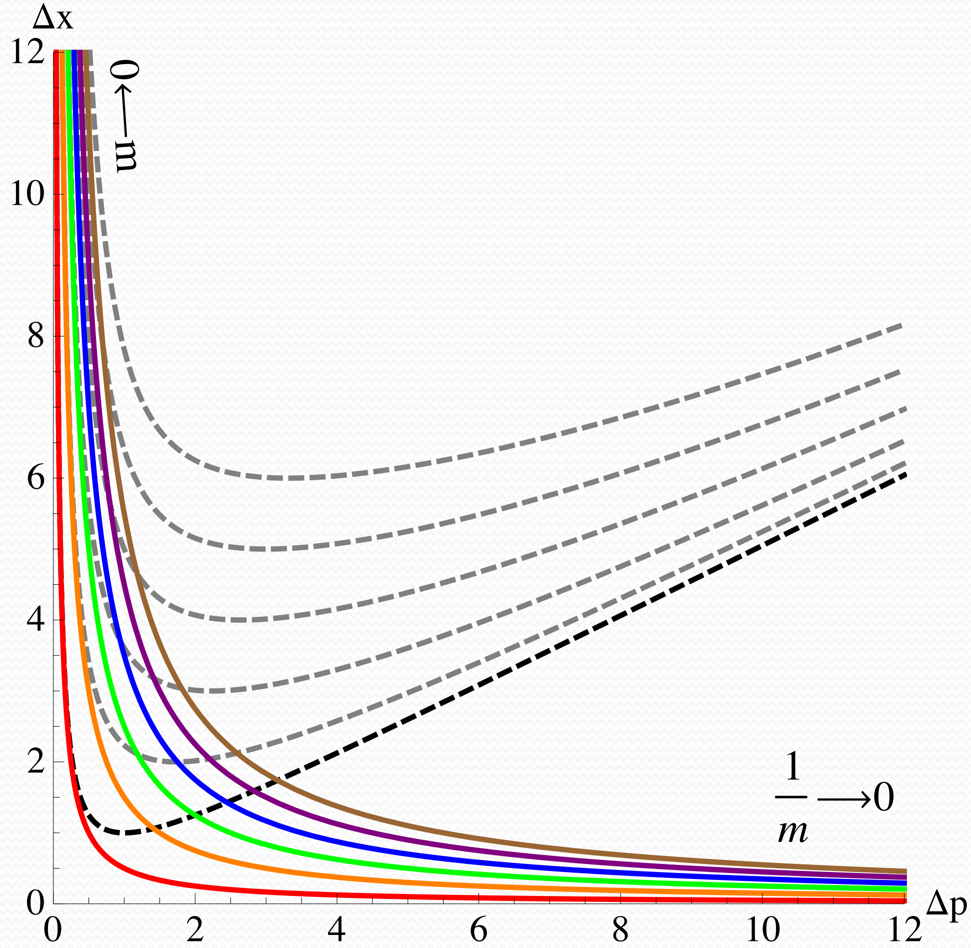
Can be solved exactly. Energy eigenvalues:

$$E_n = \frac{k}{2} \left[\left(n + \frac{1}{2} \right) \sqrt{(\Delta x_{\min})^4 + 4a^4} + \left(n^2 + n + \frac{1}{2} \right) (\Delta x_{\min})^2 \right]$$

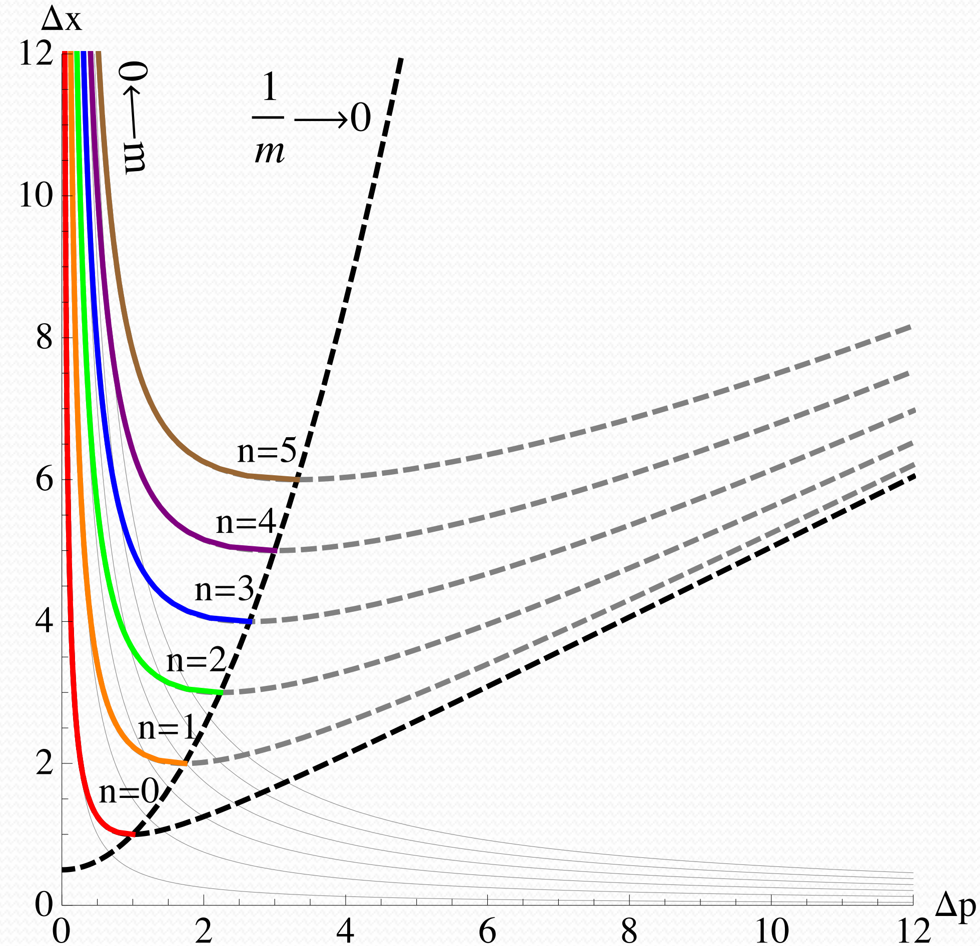
$$a = \sqrt[4]{\frac{\hbar^2}{km}} = \sqrt{\frac{\hbar}{m\omega}}$$

No longer evenly spaced. n^2 - dependence is introduced.

Uncertainties of the Harmonic Oscillator: β=0



Uncertainties of the Harmonic Oscillator: β≠0



How can we get onto the Δx~Δp branch?

Harmonic Oscillator with negative mass:

$$\hat{H} = -\frac{\hat{p}^2}{2|m|} + \frac{1}{2}k\hat{x}^2$$

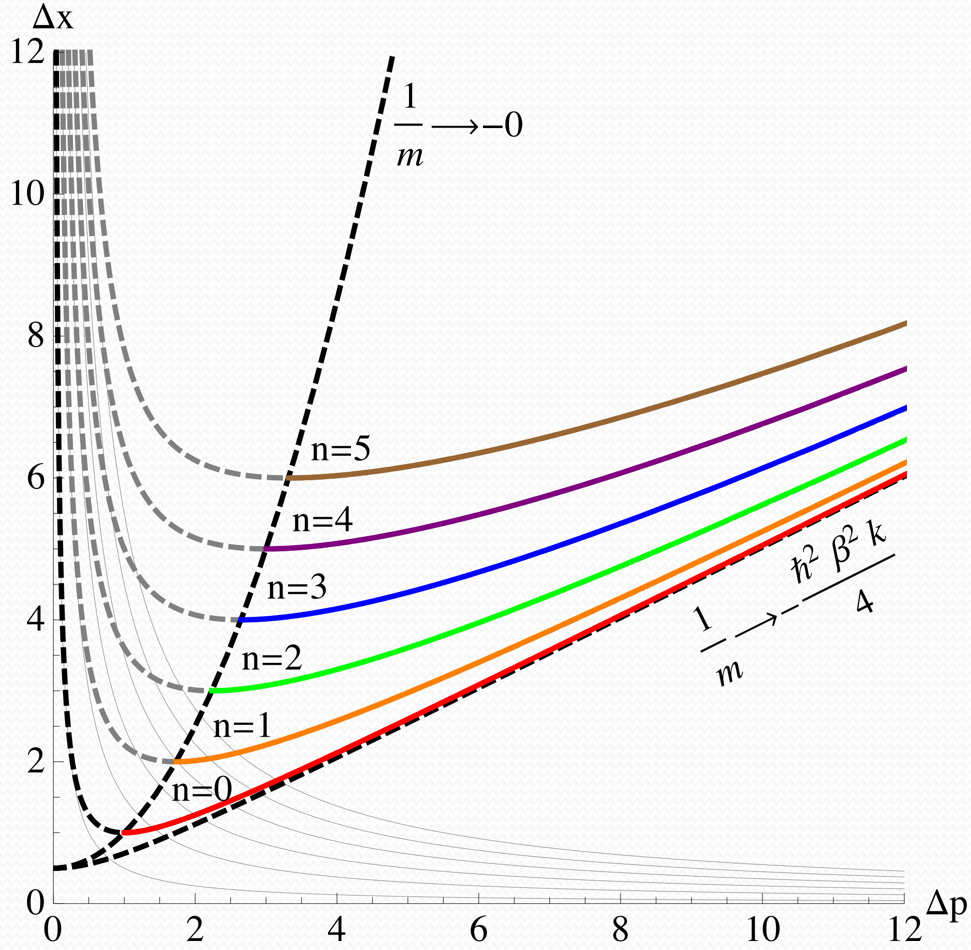
$$\Rightarrow \quad \left[-\frac{\hbar^2 k}{2}\left\{(1+\beta p^2)\frac{d}{dp}\right\}^2 - \frac{p^2}{2|m|}\right]\psi(p) = E\psi(p)$$

Energy eigenvalues :

$$E_n = \frac{k}{2}\left[\left(n+\frac{1}{2}\right)\sqrt{(\Delta x_{\min})^4 - 4a^4} + \left(n^2 + n + \frac{1}{2}\right)(\Delta x_{\min})^2\right]$$

$$a = \sqrt[4]{\frac{\hbar^2}{k|m|}}$$

Uncertainties of the Harmonic Oscillator: β≠0, m<0



Classical Limit:

$$\frac{1}{i\hbar}\left[\hat{x},\hat{p}\right]=\left(1+\beta\hat{p}^2\right) \quad \Rightarrow \quad \left\{x,p\right\}=\left(1+\beta p^2\right)$$

Classical Equations of Motion :

$$\dot{x}=\left\{x,H\right\}, \qquad \dot{p}=\left\{p,H\right\}$$

Liouville Theorem :

$$dx\wedge dp \quad \to \quad \frac{dx\wedge dp}{1+\beta p^2}$$

$\hbar\left(1+\beta p^2\right)$ can be considered a p - dependent effective $\hbar(p)$.

Classical Harmonic Oscillator:

Hamiltonian:

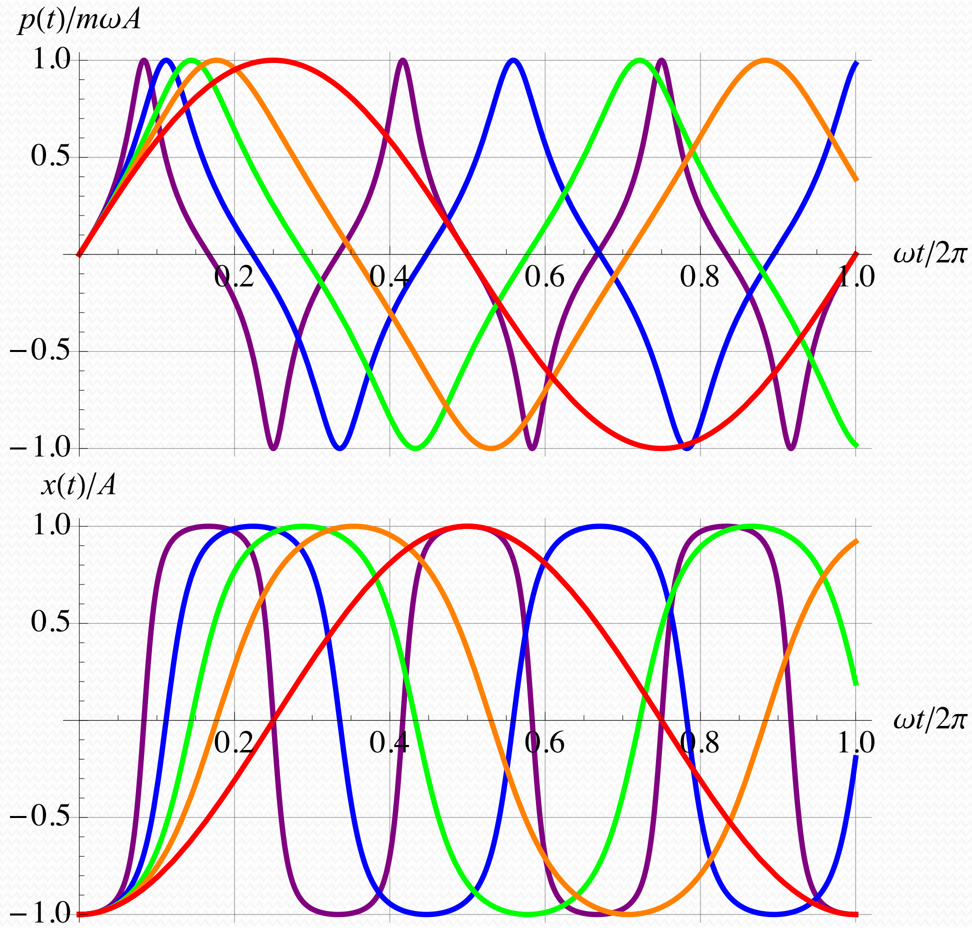
$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Classical Equations of Motion:

$$\begin{cases} \dot{x} = \{x, H\} = \dfrac{1}{m}\left(1 + \beta p^2\right)p \\ \dot{p} = \{p, H\} = -k\left(1 + \beta p^2\right)x \end{cases}$$

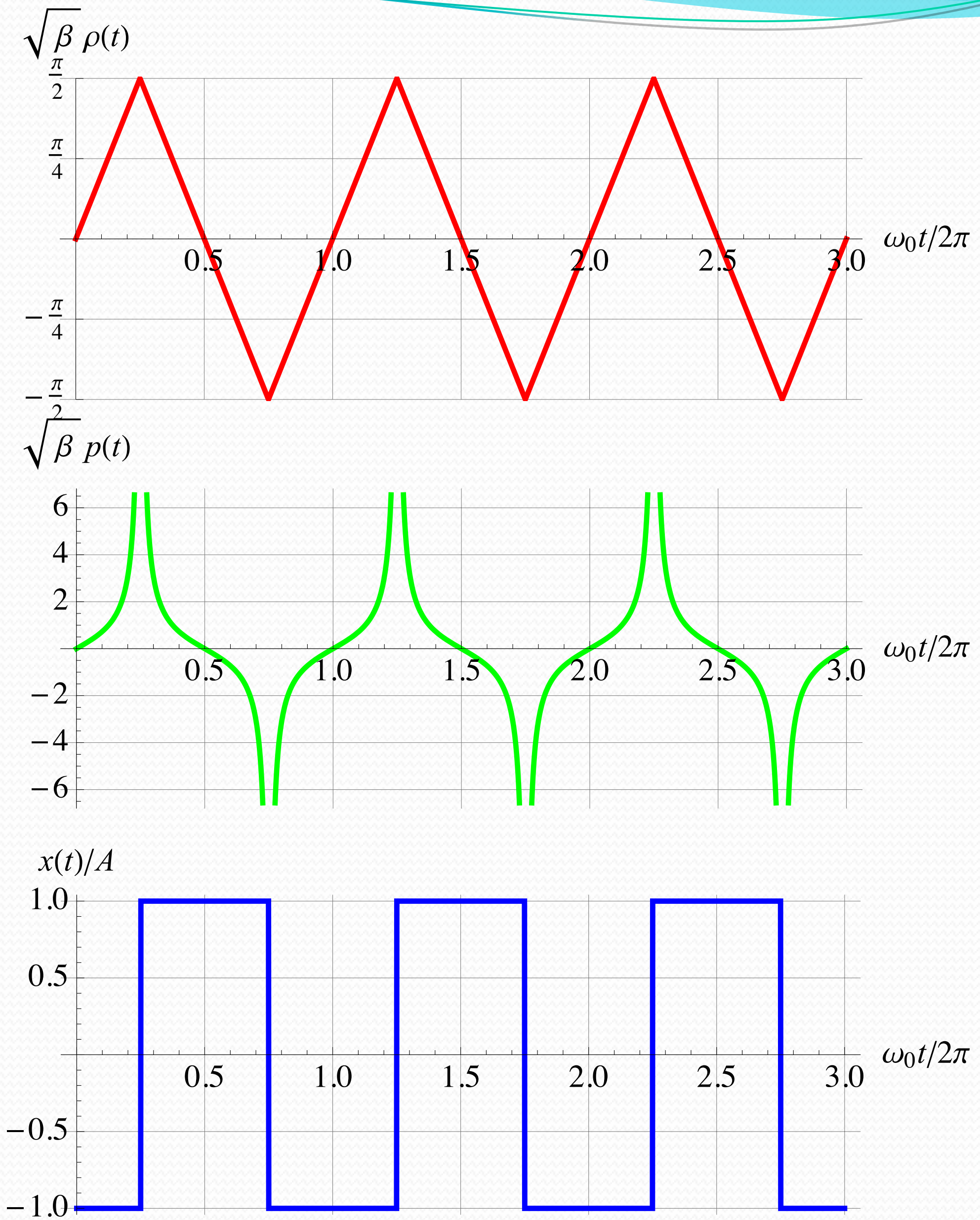
Time-dependence of x and p are different, but the trajectories in phase space are the same as the $\beta = 0$ case since the Hamiltonian is the same.

m>0 case:



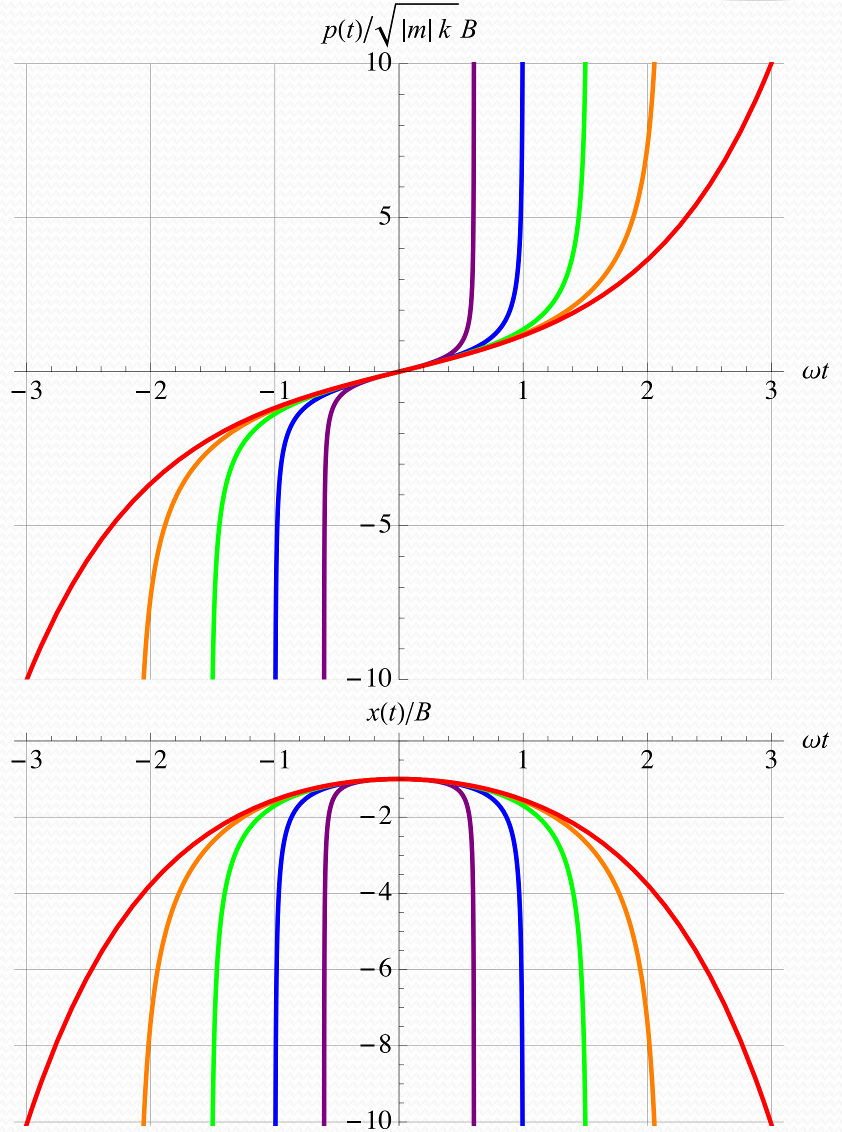
$p(t)/m\omega A$
1.0
0.5
−0.5
−1.0
0.2
0.4
0.6
0.8
1.0
$\omega t/2\pi$
$x(t)/A$
1.0
0.5
−0.5
−1.0
0.2
0.4
0.6
0.8
1.0
$\omega t/2\pi$

m→∞ limit:

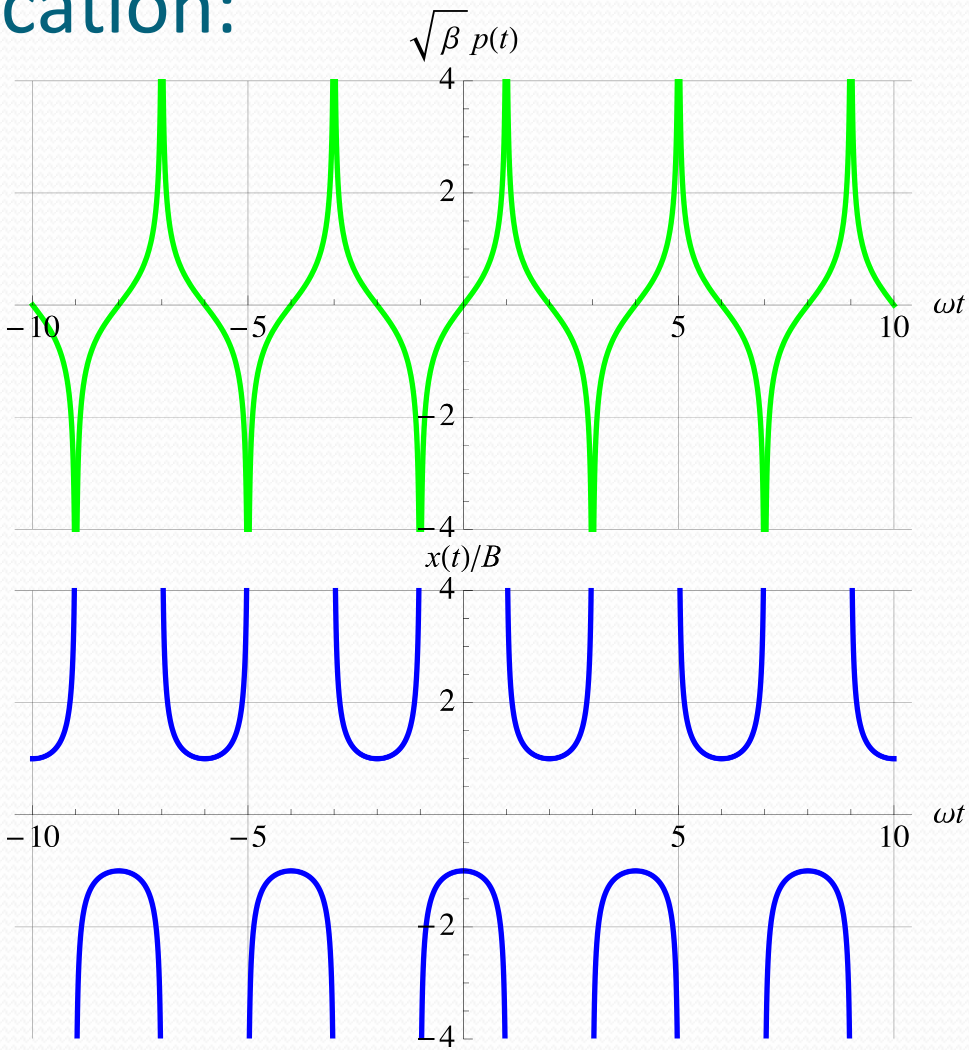


$\sqrt{\beta}\,\rho(t)$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
$\omega_0 t/2\pi$
0.5
1.0
1.5
2.0
2.5
3.0
$\sqrt{\beta}\,p(t)$
6
4
2
−2
−4
−6
$\omega_0 t/2\pi$
$x(t)/A$
1.0
0.5
−0.5
−1.0
$\omega_0 t/2\pi$

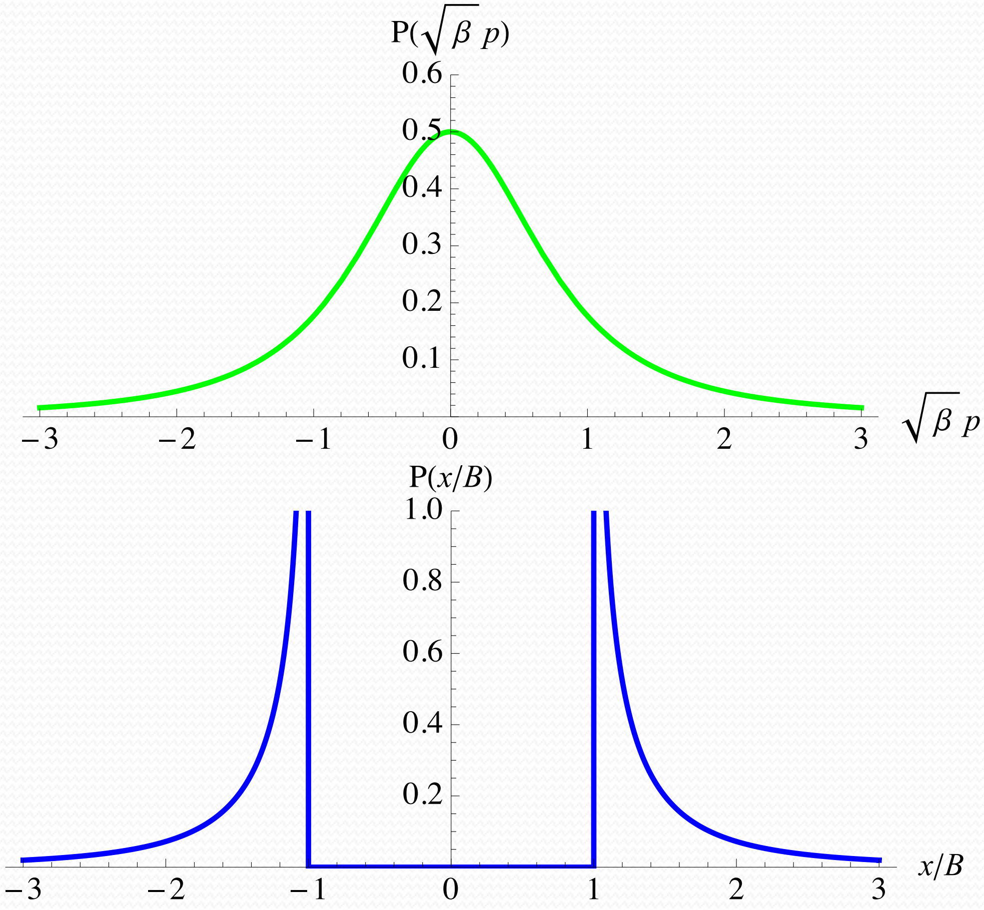
m<0 case:



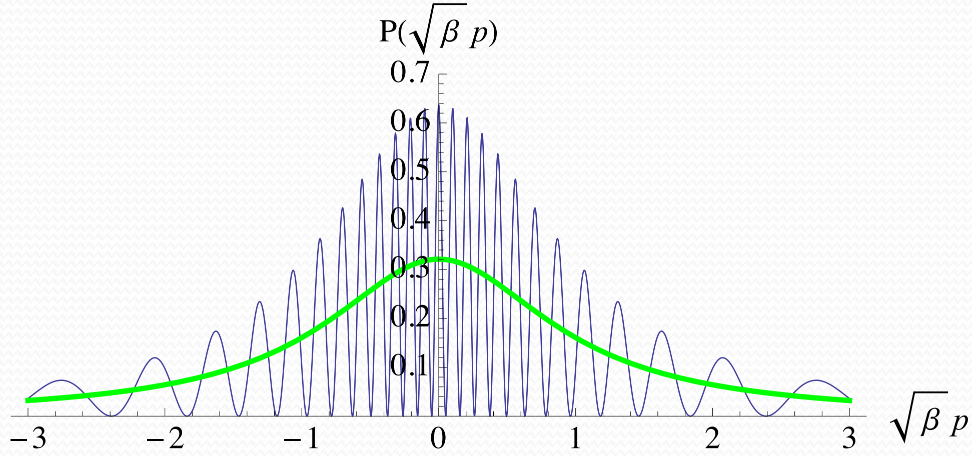
Compactification:

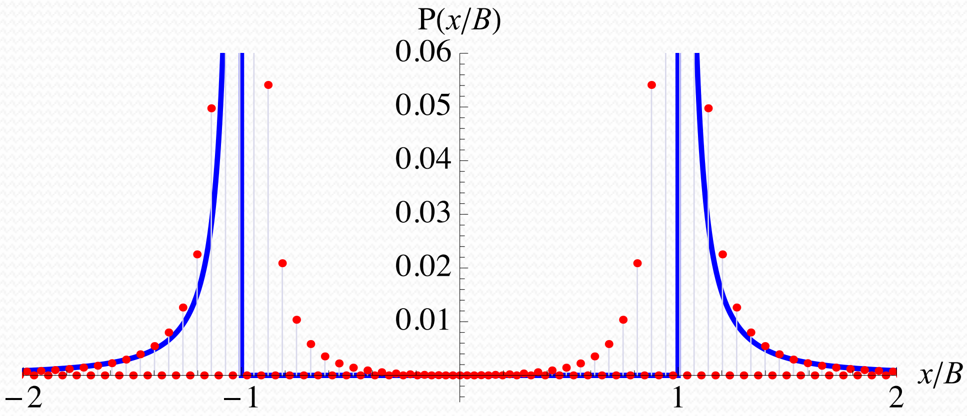


Classical Probabilities:



Comparison with Quantum Probabilities:





Conclusions:

- The minimal length uncertainty relation allows discreate energy “bound” states for “inverted” potentials.
- In the classical limit, these “bound” states can be understood to be due to the finite time the particle spends near the phase-space origin.
- Particles move at arbitrary large velocites. Do the non-relativistic negative mass states correspond to relativistic imaginary mass tachyons?