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Feynman Diagrams

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Richard Feynman

The **Nobel Prize in Physics** 1965 is awarded jointly to Sin-Itiro Tomonaga, Julian Schwinger and Richard P. Feynman for their fundamental work in **quantum electrodynamics**, with deep-ploughing consequences for the physics of elementary particles.

Calculating matrix elements for Perturbation Theory from first principles is awkward.

→ Need to do time-order sum of real particles whose production and decay does not conserve energy and momentum $p^\mu p_\mu = E^2 - |\vec{p}|^2 \neq m^2$

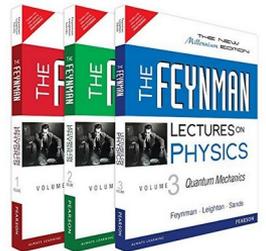
Complex mathematics was required to model extensive interactions with varying probabilities.

Feynman diagrams represent the maths of Perturbation Theory in a very simple way.

Use them to calculate matrix elements.

→ Exchanged particle is off-mass shell, however production or decay **conserves energy and momentum**.

R. Feynman recognized that interactions were representable by simple diagrams, which delineate the interplay of light. FD allows us to systematically represent **the probability amplitude** for a given process.



Basics

Vertices represent interactions (e.g., the emission or absorption of a boson by a particle). At each vertex, energy, momentum, and charge are conserved! Each vertex contributes a factor of **coupling constant**, g .

Propagators represent **virtual particles** moving between vertices – internal lines.

Spin ½	particle/antiparticle	
Spin 1	γ, W^\pm, Z	
	gluon	

Each propagator gives $1/(q^2 - m^2)$

EM interaction:
 $g = Q_e = Q \sqrt{\alpha}$

Weak int.:
 $g = g_w$

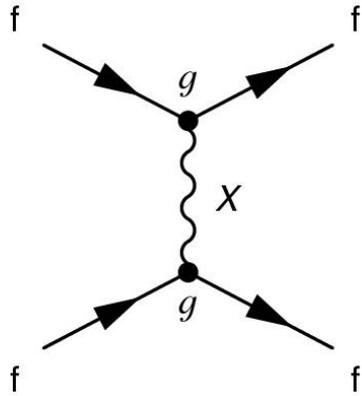
Strong int.:
 $g = \sqrt{\alpha_s}$

Outer lines represent **real particles** entering and leaving the process.

Spin ½	particle		Incoming
	antiparticle		Outgoing
			Incoming
			Outgoing
Spin 1	particle		Incoming
			Outgoing

Feynman rules are a set of rules that allow us to transform a diagram into a mathematical expression.

Vertices



The **transition matrix** in quantum mechanics contains a coupling factor g at each interaction vertex.

g is the strength of the interaction between gauge boson and a fermion.

Matrix element: $M \propto g^2$

Interaction probability: $|M|^2 = MM^* \propto g^4$

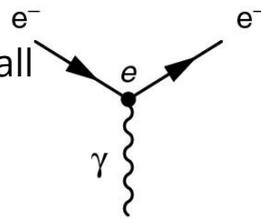
Dimensionless cons: $\alpha \propto g^2$

g is small \rightarrow the perturbation is small

QED:

$$g = e = \sqrt{4\pi\alpha} \sim 0.30$$

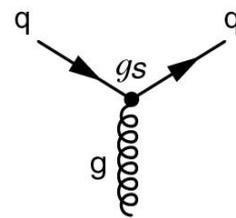
Electromagnetism



All charged particles
Never changes flavour

$$\alpha \approx 1/137$$

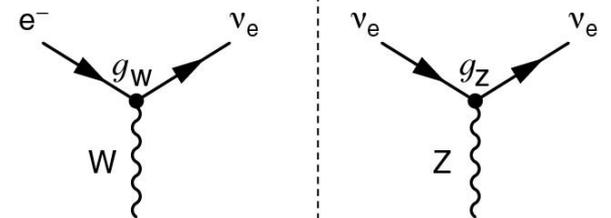
Strong interaction



Only quarks
Never changes flavour

$$\alpha_S \approx 1$$

Weak interaction

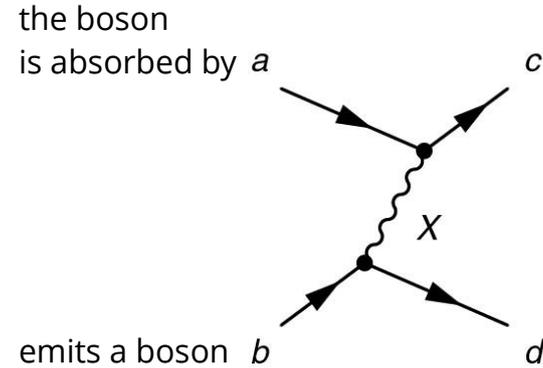
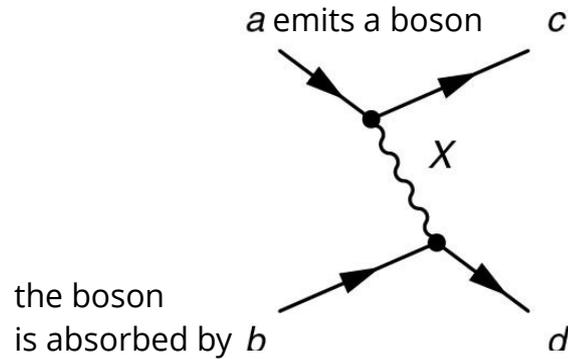


All fermions
Always changes flavour

All fermions
Never changes flavour

$$\alpha_{W/Z} \approx 1/30$$

Time-ordering



These processes depend on the choice of the system → are not invariant.

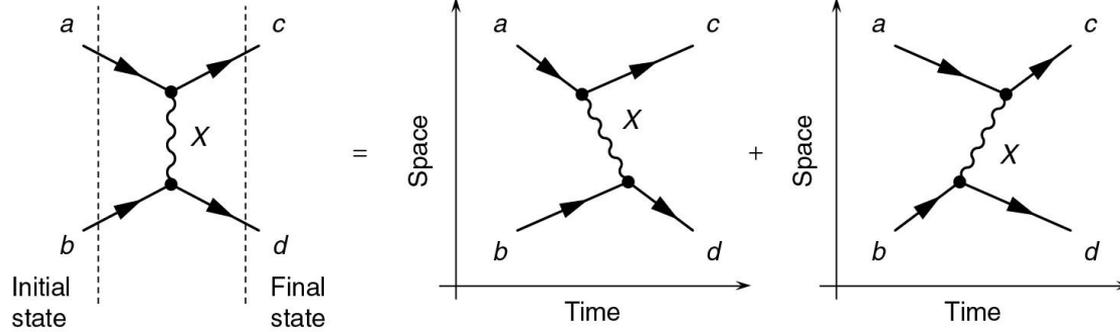
The exchanged particle is unobserved

- only the combined effect of possible sequences of events matters.

The result is **a transfer of momentum between *a* and *b*** and the creation of **force**.

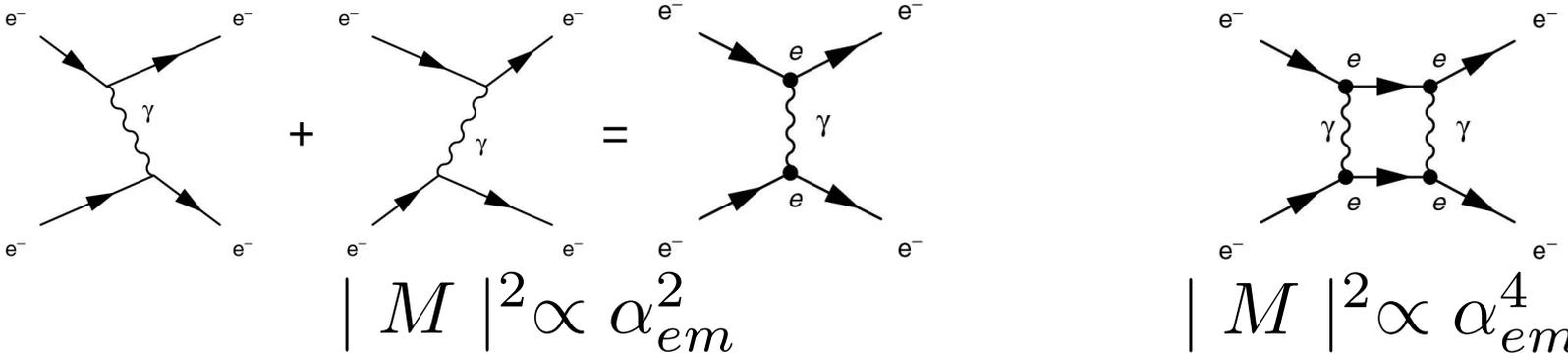
$$a + b \rightarrow c + d$$

sense, not order



A typical Feynman diagram is the sum of the amplitudes for two time sequences.

$$e^- + e^- \rightarrow e^- e^-$$



Every process has an infinite number of possible Feynman diagrams.

$a + b \rightarrow c + d$ - time-ordering

The greater the transition amplitude, the greater the prob. of the process.

Probability of processes: $\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$ **Fermi Golden Rule**

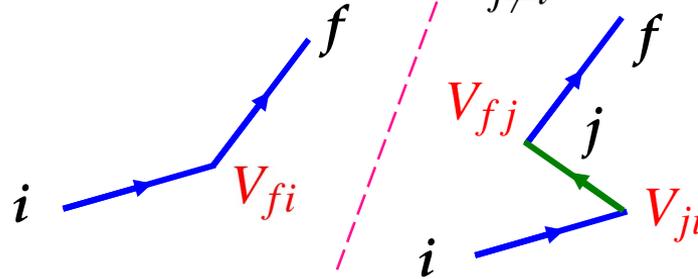
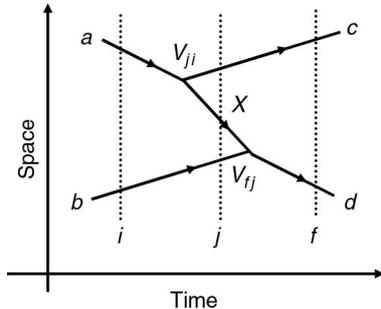
The more final states available, the greater the chance that one of them will be realized.

Transition matrix element:

$$T_{fi} = \langle f|V|i\rangle + \sum_{j \neq i} \frac{\langle f|V|j\rangle \langle j|V|i\rangle}{E_i - E_j} + \dots$$

FGR determines how quickly the system will "jump" to another state.

In Quantum Mechanics, the process of scattering in a static potential corresponds to the first-order term in the perturbation expansion $\langle f|V|i\rangle$



In Quantum Field Theory, interactions between particles are mediated by the exchange of other particles and there is no mysterious action at a distance.

Initial state $|i\rangle: a+b$

Final state $|f\rangle: c+d$

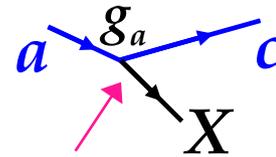
Intermediate state $|j\rangle: c+b+X$

Non-invariant matrix elements: V_{fj}
 V_{ji}

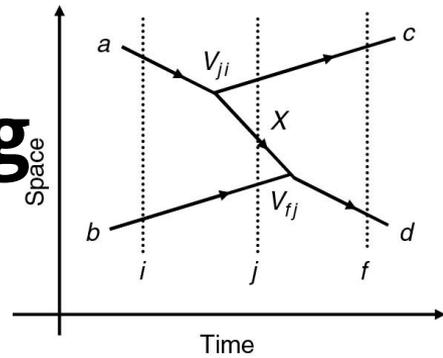
$$\begin{matrix} E_j \neq E_i \\ \Delta E \Delta t \sim 1 \end{matrix}$$

$$T_{fi} = \frac{\langle f|V|j\rangle \langle j|V|i\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle \langle c+x|V|a\rangle}{(E_a + E_b) - (E_c + E_x + E_b)}$$



$a + b \rightarrow c + d$ - time-ordering



The non-invariant matrix element is related to the Lorentz-invariant matrix element

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

$$V_{ji} = \langle c + x | V | a \rangle = \frac{M_{(a \rightarrow c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

The matrix element $M_{(a \rightarrow c+x)}$ is Lorentz invariant places strong constraints on its possible mathematical structure.

The simplest coupling \rightarrow scalar,

$$g_a = M_{(a \rightarrow c+x)}$$

$$V_{ji} = \langle c + x | V | a \rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

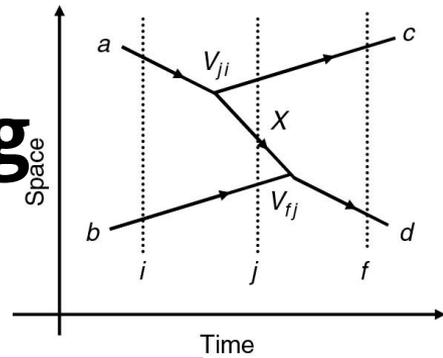
$$V_{fj} = \langle d | V | x + b \rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$$

Although individual pieces of the matrix element (such as fermion currents or gamma matrices) may carry Lorentz indices, **the total matrix element must be a Lorentz scalar**.

The indices are contracted through objects like **the metric tensor** $g_{\mu\nu}$ resulting in **a scalar quantity** suitable for computing probabilities and cross sections.

Any **non-scalar** intermediate form **is** only temporary and **not physically** meaningful until all indices are properly contracted.

$a+b \rightarrow c+d$ - time-ordering



$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a + E_b) - (E_c + E_x + E_d)}$$

$$= \frac{1}{2E_x} \cdot \frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$

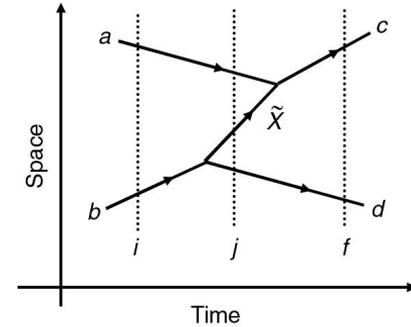
$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

The Lorentz-invariant matrix element
 For the 1st/2nd time-ordered diagram
 is related to the transition matrix element.
 Momentum is conserved
 at the interaction point,
 but $E_j \neq E_i$
 but the exchanged particle
 is "on-mass shell"

$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}$$

$$= \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$



$$E_x^2 = \vec{p}_x^2 + m^2$$

$a+b \rightarrow c+d$ - time-ordering

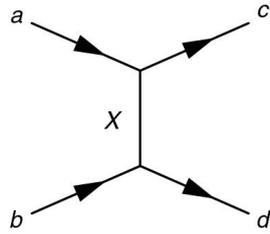
The different amplitudes for the same process need to be summed to obtain the total amplitude.

*

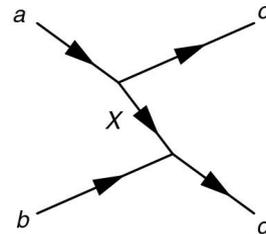
Momentum & energy conserved at interaction vertices

Exchanged particle "off mass shell"

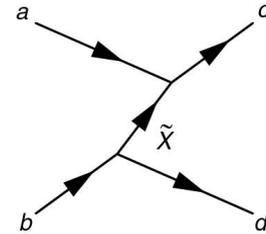
$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$



=



+



Momentum conserved at vertices

Energy **not** conserved at vertices

Exchanged particle "on mass shell"

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

$$M_{fi} = M_{fi}^{ab} + M_{fi}^{ba}$$

$$= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right)$$

$$= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right)$$

Energy conservation
($E_a + E_b = E_c + E_d$)

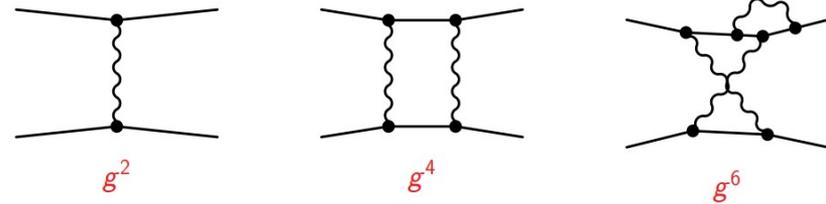
$$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2}$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}$$

Virtual Particles and Perturbative Orders

- **Virtual particle** differs from the real one. Real particles satisfy a relationship between E , p , m .

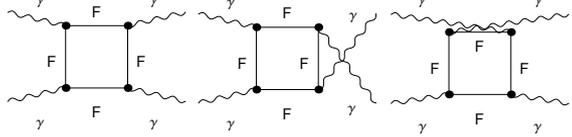
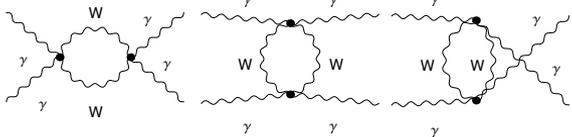
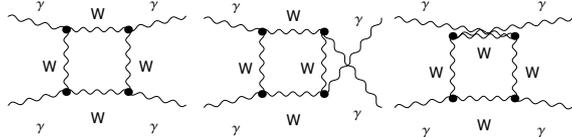
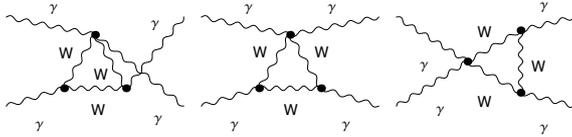
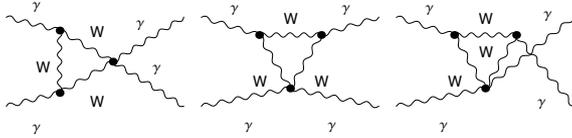
- They are as 'borrowed' particles that exist for a very short time, according to the Heisenberg uncertainty principle for $(\Delta E \Delta t \geq \hbar/2)$. One can 'borrow' energy (or mass) for a short time. This allows them to transmit interactions.



- To calculate the probability of a given process, it's not enough to draw only the simplest diagrams. We must consider all possible ways in which particles can interact, even very complex ones. More complicated diagrams (containing loops) \rightarrow higher-order perturbative diagrams. Each additional vertex in the diagram introduces an additional factor, which is/should be the coupling constant. **Higher-order diagrams** contribute less to the total amplitude. One should calculate the simplest diagrams first, and then add increasingly complex corrections until we achieve the required precision. The 1st term dominates, and subsequent terms are increasingly smaller corrections.
- Sometimes these higher-order diagrams lead to **the problem of infinity**. And this is where the concept of **renormalization** comes in. Renormalization is a mathematical procedure that allows us to subtract the infinite charge of the surrounding vacuum from the infinite charge of the bare electron, so as to obtain the finite value observed in experiments. The price of renormalization is that the charge and mass measurements depend on the four-momentum transfer q^2 .

$$\gamma \gamma \rightarrow \gamma \gamma$$

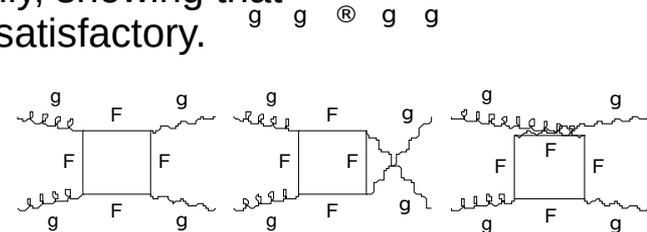
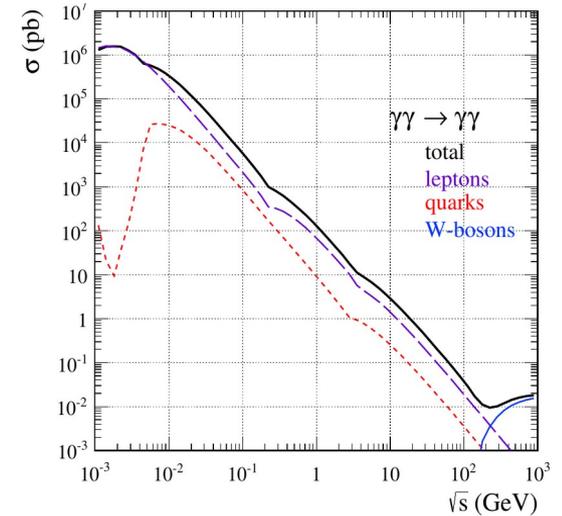
$$\gamma\gamma \rightarrow \gamma\gamma$$



A **Born diagram** is the simplest (lowest-order) Feynman diagram describing a given process in perturbative calculus, i.e., a loop-free diagram with the minimum number of vertices needed for the phenomenon to occur.

The box contributions to the $\gamma\gamma \rightarrow \gamma\gamma$ subprocess for on-shell fusing photons were calculated analytically by using the Mathematica - based FormCal Package.

The QCD and QED **corrections** (the two-loop Feynman diagrams) to the one-loop fermionic contributions are **quite small** numerically, showing that the LO computations are satisfactory.



$$a + b \rightarrow c + d$$

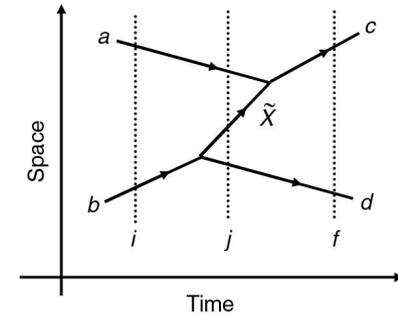
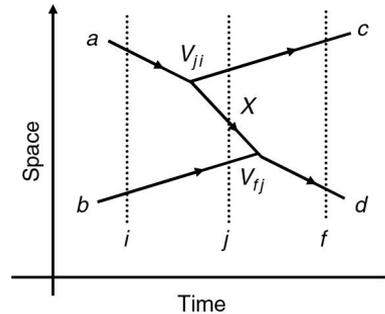
both time-ordered diagrams \rightarrow Feynman Diagram

Momentum is conserved at each interaction point, $\mathbf{p}_X = (\mathbf{p}_a - \mathbf{p}_c)$

$$\mathbf{p}_{\tilde{X}} = (\mathbf{p}_b - \mathbf{p}_d) = -(\mathbf{p}_a - \mathbf{p}_c)$$

The energy of the exchanged particle is related to its momentum

$$E_x^2 = \vec{p}_x^2 + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$$



$$\begin{aligned} M_{fi} &= \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} \\ &= \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2} \end{aligned}$$

The 4-momentum of the exchanged virtual particle: $q = p_a - p_c$

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

The matrix element

=

The fundamental strength of the interaction at the two vertices

x

the propagator

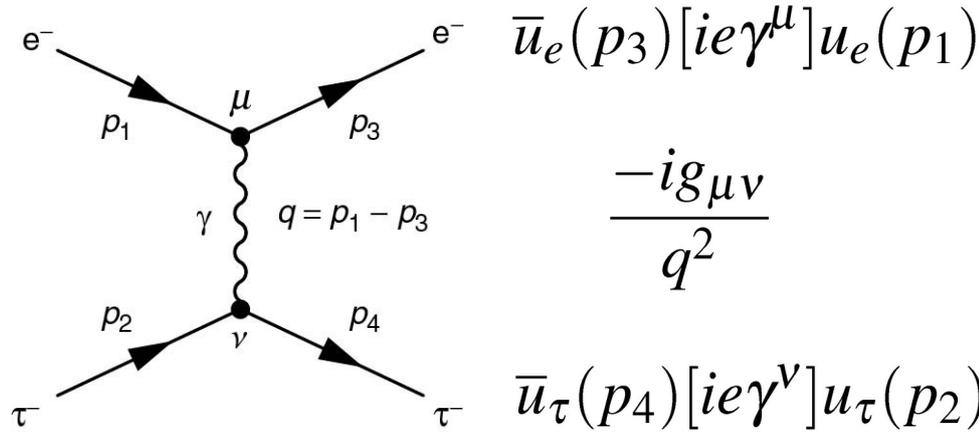
Feynman rules for QED

initial-state particle:	$u(p)$		} Spin 1/2	} The Dirac spinors for each external line	
final-state particle:	$\bar{u}(p)$				
initial-state antiparticle:	$\bar{v}(p)$				
final-state antiparticle:	$v(p)$				
initial-state photon:	$\varepsilon_\mu(p)$		} Spin 1		
final-state photon:	$\varepsilon_\mu^*(p)$				
photon propagator:	$-\frac{ig_{\mu\nu}}{q^2}$		} Spin 1/2		} A propagator factor for each internal line
fermion propagator:	$-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$				
QED vertex:	$-iQe\gamma^\mu$		} Spin 1/2	} A vertex factor for each vertex	

Matrix element $-iM = \text{product of all factors}$

QED scattering process

elastic scattering



$$\bar{u}_e(p_3) [ie\gamma^\mu] u_e(p_1)$$

electron current

$$\frac{-ig_{\mu\nu}}{q^2}$$

photon propagator

$g_{\mu\nu}$ - the sum over the γ^* polarization states

$$\bar{u}_\tau(p_4) [ie\gamma^\nu] u_\tau(p_2)$$

tau-lepton current

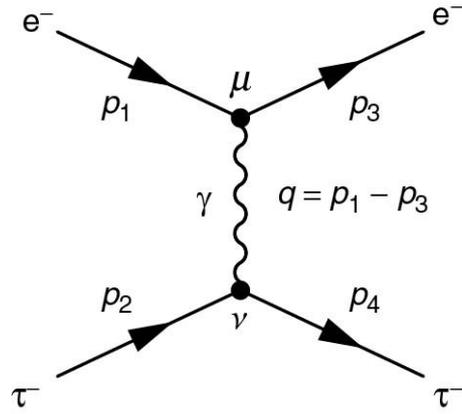
an adjoint spinor for the final-state electron a spinor for the initial-state electron

$\gamma^{0,\nu}$ - matrices are the original Dirac representation

$$-iM = \underbrace{[\bar{u}_e(p_3) ie\gamma^\mu]}_{\text{a factor for the interaction vertex}} \underbrace{u_e(p_1)}_{\text{a spinor for the initial-state electron}} \frac{-ig_{\mu\nu}}{q^2} \underbrace{[\bar{u}_\tau(p_4) ie\gamma^\nu]}_{\text{a factor for the interaction vertex}} \underbrace{u_\tau(p_2)}_{\text{a spinor for the initial-state electron}}$$

a factor for the interaction vertex

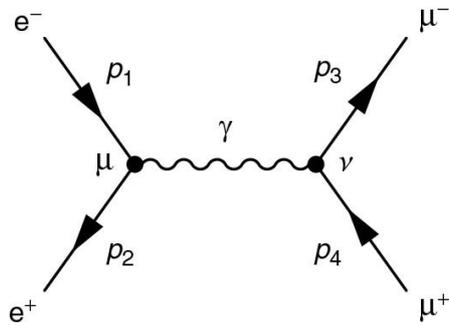
QED processes



t - channel scattering process $e^- \tau^- \rightarrow e^- \tau^-$

$$-iM = [\bar{u}_e(p_3) ie\gamma^\mu u_e(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4) ie\gamma^\nu u_\tau(p_2)]$$

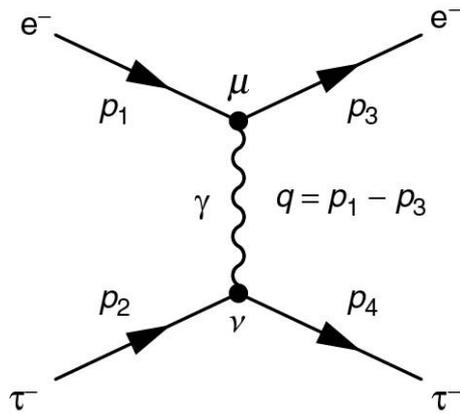
Fermions have their own spinors,
but when they exit, we need a adjoint spinor, $\bar{\psi} = \psi^\dagger \gamma^0$
to the left.



s - channel annihilation process $e^- e^+ \rightarrow \mu^- \mu^-$

$$-iM = [\bar{v}(p_2) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3) ie\gamma^\nu v(p_4)]$$

$e^- \tau^- \rightarrow e^- \tau^-$



$$-iM = [\bar{u}_e(p_3) i e \gamma^\mu u_e(p_1)] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4) i e \gamma^\nu u_\tau(p_2)]$$

$$-iM = (-ie) \bar{u}(p_3) \gamma^\mu u(p_1) \cdot \frac{-i g_{\mu\nu}}{t} \cdot (-ie) \bar{u}(p_4) \gamma^\nu u(p_2)$$

$$(-ie)^2 \cdot \left(\frac{-i g_{\mu\nu}}{t} \right) = -e^2 \cdot \left(\frac{-i g_{\mu\nu}}{t} \right) = \frac{ie^2}{t} g_{\mu\nu}$$

metric tensor property: $g_{\mu\nu} J_e^\mu J_\tau^\nu = J_e^\mu J_\mu^\tau$

$$\mathcal{M} = \frac{e^2}{t} [\bar{u}(p_3) \gamma^\mu u(p_1)] [\bar{u}(p_4) \gamma_\mu u(p_2)]$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{t^2} \sum_{\text{spins}} |[\bar{u}(p_3) \gamma^\mu u(p_1)] [\bar{u}(p_4) \gamma_\mu u(p_2)]|^2$$

spin sum identities: $\sum_s u(p) \bar{u}(p) = \not{p} + m$

$$\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{t^2} \cdot \text{Tr} [(p_3 + m_e) \gamma^\mu (p_1 + m_e) \gamma^\nu] \cdot \text{Tr} [(p_4 + m_\tau) \gamma_\mu (p_2 + m_\tau) \gamma_\nu]$$

For massless limit:
$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{t^2} \cdot \text{Tr} [p_3 \gamma^\mu p_1 \gamma^\nu] \cdot \text{Tr} [p_4 \gamma_\mu p_2 \gamma_\nu]$$

Identity for gamma traces:
$$\text{Tr}[a \gamma^\mu b \gamma^\nu] = 4 (a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu} a \cdot b)$$

$$\text{Tr} [p_3 \gamma^\mu p_1 \gamma^\nu] = 4 (p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - g^{\mu\nu} p_3 \cdot p_1)$$

$$\text{Tr} [p_4 \gamma_\mu p_2 \gamma_\nu] = 4 (p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - g_{\mu\nu} p_4 \cdot p_2)$$

$$|M|^2 = \frac{16e^4}{t^2} (p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - g^{\mu\nu} p_3 \cdot p_1) (p_{4\mu} p_{2\nu} + p_{4\nu} p_{2\mu} - g_{\mu\nu} p_4 \cdot p_2)$$

Solving tensor products yields 9 scalar 4-vector products

$$(p_3^\mu p_1^\nu)(p_{4\mu} p_{2\nu}) = (p_3 \cdot p_4)(p_1 \cdot p_2)$$

$$(p_3^\nu p_1^\mu)(p_{4\mu} p_{2\nu}) = (p_3 \cdot p_2)(p_1 \cdot p_4)$$

$$(-g^{\mu\nu} p_1 \cdot p_3)(p_{4\mu} p_{2\nu}) = -(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$(p_3^\mu p_1^\nu)(p_{4\nu} p_{2\mu}) = (p_3 \cdot p_2)(p_1 \cdot p_4)$$

$$(p_3^\nu p_1^\mu)(p_{4\nu} p_{2\mu}) = (p_3 \cdot p_4)(p_1 \cdot p_2)$$

$$(-g^{\mu\nu} p_1 \cdot p_3)(p_{4\nu} p_{2\mu}) = -(p_1 \cdot p_3)(p_2 \cdot p_4)$$

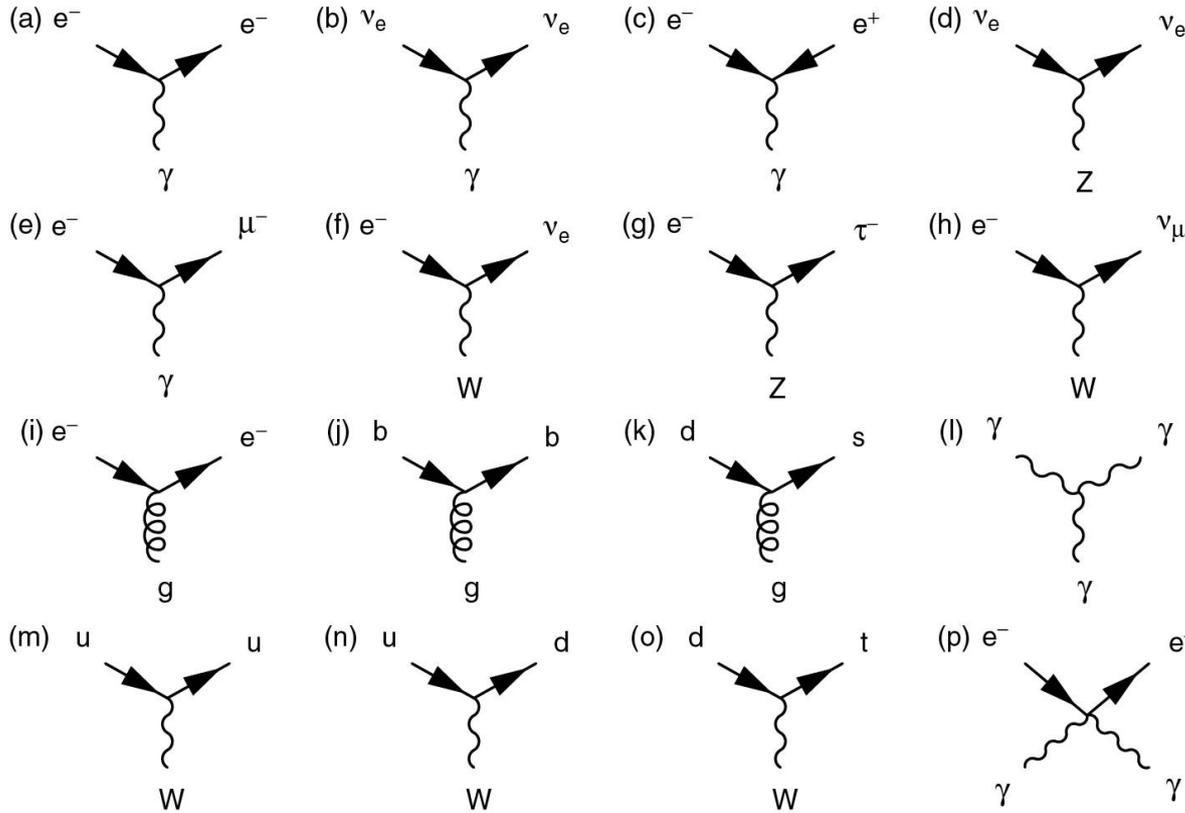
$$(p_3^\mu p_1^\nu)(-g_{\mu\nu} p_2 \cdot p_4) = -(p_3 \cdot p_1)(p_2 \cdot p_4)$$

$$(p_3^\nu p_1^\mu)(-g_{\mu\nu} p_2 \cdot p_4) = -(p_3 \cdot p_1)(p_2 \cdot p_4)$$

$$(-g^{\mu\nu} p_1 \cdot p_3)(-g_{\mu\nu} p_2 \cdot p_4) = +4(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$|\mathcal{M}|^2 = \frac{16e^4}{t^2} [2(p_3 \cdot p_4)(p_1 \cdot p_2) + 2(p_3 \cdot p_2)(p_1 \cdot p_4)]$$

Exercises



Feynman diagrams are constructed out of the Standard Model vertices shown in Figure.

Only the weak charged-current (W^\pm) interaction can change the flavour of the particle at the interaction vertex.

Each of the 16 diagrams represents a valid SM vertex.

TIP: if you see a ν or ν^- , it must be a weak interaction

In a nutshell

- Feynman diagrams provided **a visual tool** for making otherwise impossible calculations practical.
- Antiparticles are drawn in Feynman diagrams with arrows pointing in the “backwards in time” direction.
- In the Standard Model, particles and antiparticles can be created or annihilated only in pairs. Arrows on the fermionic lines indicate the flow of fermionic number (charge).
- Each **vertex** describes the probability of gauge boson emission. This probability is proportional to the gauge boson's propagator.
- The **propagator** determines how virtual the particle is (off-mass shell). The greater the virtuality, the lower the chance of producing such a particle. The most probable photon emission is a low-virtual photon ($q^2 = 0$ – **quasi-real photons**).
- If all are particles (or all are antiparticles), only scattering diagrams involved. If particles and antiparticles, may be able to have scattering (t- and u-) and/or annihilation diagrams (s-channel).

Important rules to remember

Solid lines correspond to fermions (quarks, leptons)

Wavy lines correspond to bosons (in QED, these are photons)

Arrows on the lines indicate the direction of evolution in time – time increases from L to R

An arrow pointing to an electron moving backward in time corresponds to a positron moving forward in time

Fermion and boson lines meet at **vertices** (vertex)

At each vertex, energy momentum, angular momentum, charge, and baryon number are conserved

- Conservation of electric charge:

The sum of charges entering a vertex must equal the sum of charges leaving.

- Conservation of momentum and energy:

At each vertex, energy and momentum **must be conserved**.

- Conservation of lepton and baryon numbers:

In many quantum processes, the number of leptons (e.g., electrons and neutrinos) and the number of baryons (e.g., protons and neutrons) remain constant.

Application

Feynman diagrams are an indispensable tool in many fields of physics, and their applications are incredibly broad.

- **High-Energy (Particle) Physics:**

Predicting how often specific processes occur.

Data Analysis: They allow us to distinguish the signal from the bckg and understand what happens in collisions.

- **Condensed Matter Physics:**

Feynman diagrams are also used to describe the behavior of electrons in solids. They help us understand phenomena such as conduction, superconductivity, and magnetism. Although the energy scale is different, the fundamental principles of quantum interactions remain the same.

- **Cosmology and Astrophysics:**

Feynman diagrams are crucial for modeling the processes that led to the formation of the elements, dark matter, and the structure of the universe.

Drawing Tools

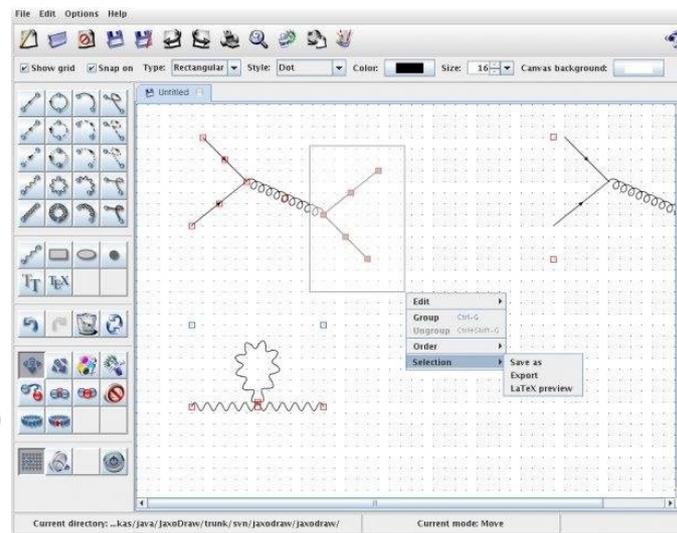
JaxoDraw

graphical interface
Generates LaTeX (PSTricks)
or exports to images.

Fast for simple diagrams.

Website:

<http://jaxodraw.sourceforge.net/>



Calculation Tools

FeynCalc (for Mathematica)

Supports:

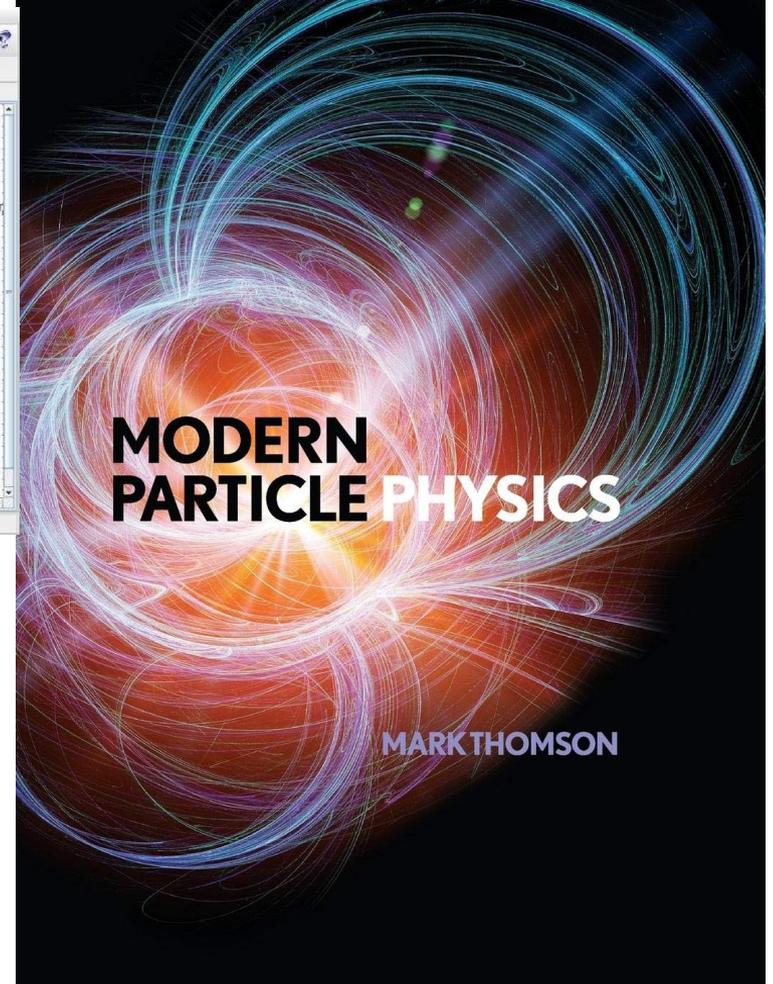
spinors,

gamma matrix algebra,

index tracking,

Feynman rules,

automatic calculation of amplitudes and their squares.



All drawings were borrowed from the book.

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