LHC to Terascale Physics
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Meaningful characterisation of perturbative theoretical uncertainties

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(MC and N. Houdeau, arXiv:1105.5152)

Disclaimer

I am going to point out an issue, stimulate discussion, and propose a possible angle of attack.

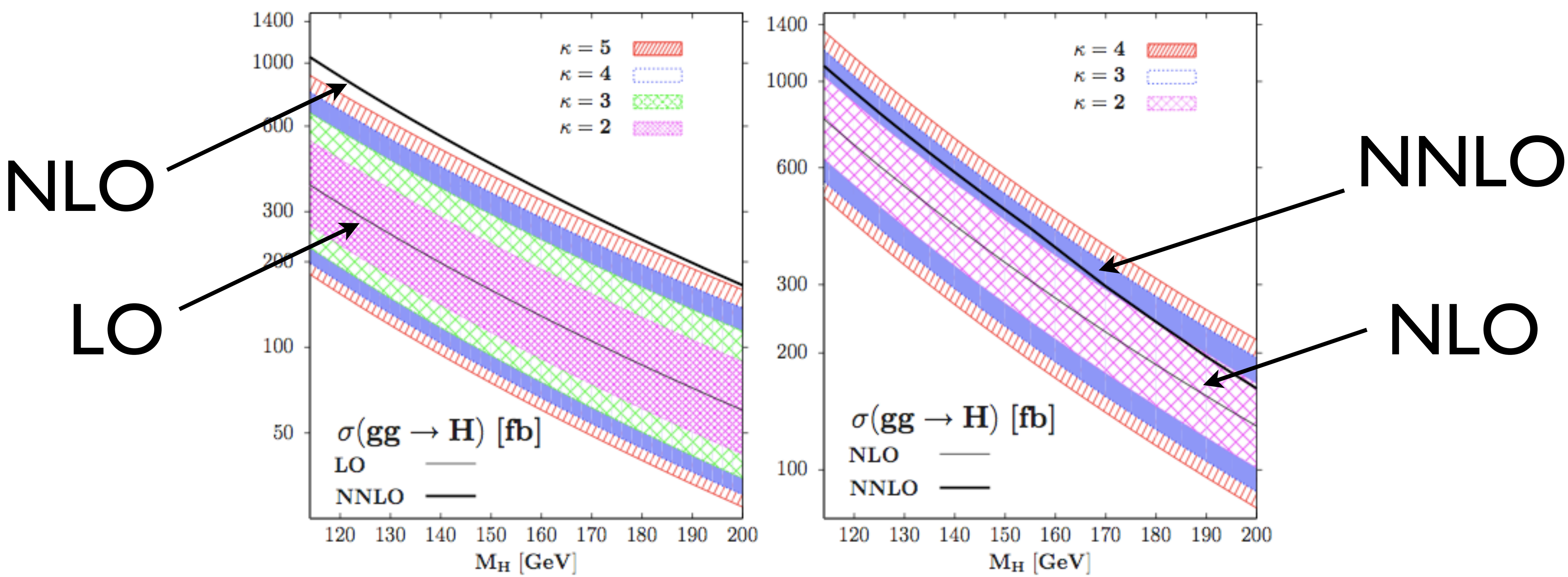
I am not (yet) going to offer a (full) solution to a concrete problem.

Conventional uncertainty estimate

Standard recipe:

1) choose a range
2) vary scales (possibly independently/with constraints) in said range
3) draw band

Baglio, Djouadi, arXiv:1003.4266



Choosing the "correct" range is a priori not obvious

What 'is' the band?

Even if something told you what the 'proper' range is, **what does the band mean?** How confident are we in our way of estimating theoretical uncertainties?

Would you feel ready to bet on the result of a scale variations uncertainty estimate? How much?

If you can't (or don't wish to) answer, then we don't have a proper estimate, and one that can be meaningfully and safely combined with other sources of uncertainty

Proposal

I am going to propose a method to obtain not only an uncertainty band, but also establish a **degree of belief** for the true sum[1] of the series to be within the band

[1] Yes, a proper definition of this may be tricky, but it's quite immaterial here

Understanding scale variations

Theoretical prediction

$$\sigma_{QCD,k} = \sum_{n=1}^{k} c_n(Q,\mu)\alpha_s^n(\mu)$$

Remainder

$$\Delta_k \equiv \sum_{n=k+1}^{\infty} c_n \alpha_s^n$$

Example:

$$\frac{\sigma(e^+e^- \to \text{hadrons}, Q)}{\sigma(e^+e^- \to \mu^+\mu^-, Q)} = R_{EW}(Q)(1 + \sigma_{QCD}(Q))$$

Known exactly up to k=3

'Theoretical uncertainty'

$$\sigma_k^{\pm} = \sigma_k \pm \frac{\delta_k}{2} \quad \text{where} \quad \delta_k \equiv |\sigma_k(Q, \mu = 2Q) - \sigma_k(Q, \mu = Q/2)|$$

Why do we say that δ_k represents the theoretical uncertainty?

Understanding scale variations

Approximate δ_k as $\delta_k \simeq \left| \frac{d\sigma_k}{d \ln \mu^2} \right|_{\mu=Q} [\ln(2Q)^2 - \ln(Q/2)^2] \simeq 3k\beta_0\alpha_s^{k+1}|c_k|$

This is the last known coefficient, multiplied by a further power of α_s

The true uncertainty, $O(\Delta_k)$, starts at α_s^{k+1} c_{k+1}

Logically equating $|\Delta_k|$ and δ_k means assuming that $|c_k| \approx |c_{k+1}|$

Bayesian model

Even if they can somehow estimate Δ_k, the bands given by scale variations have no statistical meaning: what is our **degree of belief** that they contain the right result?

We want to construct a credibility distribution for Δ_k from which to calculate the degree of belief of a given interval:

$$f(\Delta_k | c_0, \ldots c_k)$$

For this, we use explicitly the implicit assumption that allows the scale variations method to work, i.e. that $|c_k| \approx |c_{k+1}|$

Bayesian model

We suppose that all coefficients of the series are bounded by an (unkown) maximum value

$$f(c_n|\bar{c}) = \frac{1}{2\bar{c}} \begin{cases} 1 & \text{if } |c_n| \le \bar{c} \\ 0 & \text{if } |c_n| > \bar{c} \end{cases} \qquad ①$$

whose orders of magnitude are a priori equally probable

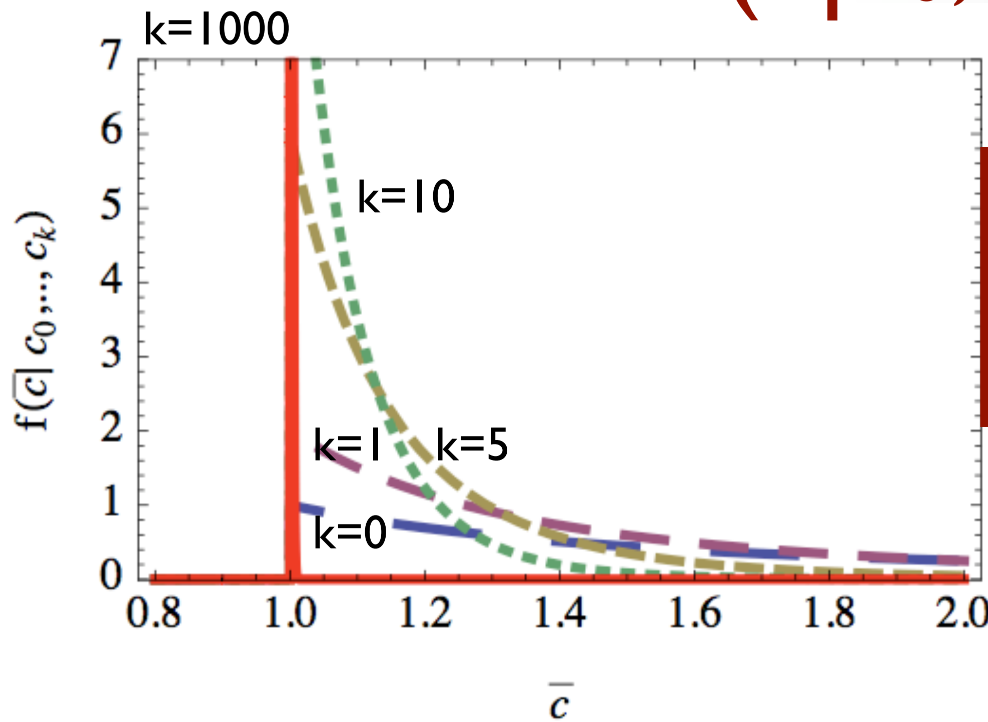
$$f_\epsilon(\ln \bar{c}) = \frac{1}{2|\ln \epsilon|} \chi_{|\ln \bar{c}| \le |\ln \epsilon|} \qquad ②$$

and that they are independent with the exception of this common bound

$$f(\{c_i, i \in I\}|\bar{c}) = \prod_{i \in I} f(c_i|\bar{c}) \qquad ③$$

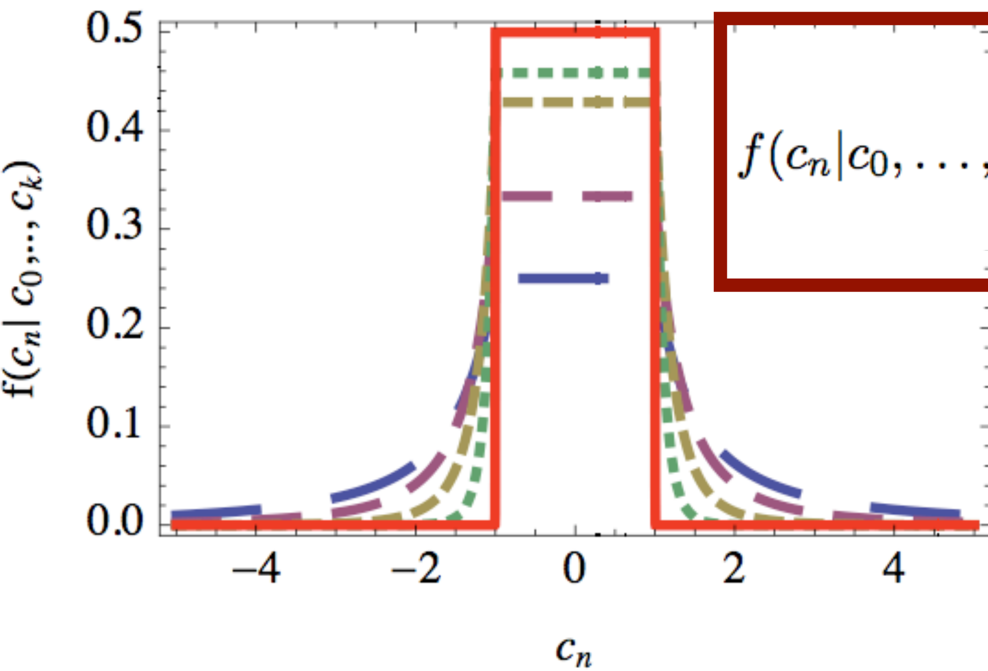
The results will be found in the $\varepsilon \to 0$ limit

$f(\bar{c}|c_0,\ldots,c_k)$ and $f(c_n|c_0,\ldots,c_k)$



$$f(\bar{c}|c_0,\ldots,c_k) = (k+1)\frac{\bar{c}_{(k)}^{k+1}}{\bar{c}^{k+2}}\chi_{\bar{c}\geq\bar{c}_{(k)}}$$

$$\bar{c}_{(k)} = \max(|c_0|,\ldots,|c_k|)$$



$$f(c_n|c_0,\ldots,c_k) = \left(\frac{k+1}{k+2}\right)\frac{1}{2\bar{c}_{(k)}}\begin{cases} 1 & \text{if } |c_n|\leq\bar{c}_{(k)} \\ \frac{1}{(|c_n|/\bar{c}_{(k)})^{k+2}} & \text{if } |c_n|>\bar{c}_{(k)} \end{cases}$$

Knowing more perturbative coefficents improves our estimates of $\bar{c}$ and of the unknown coefficients

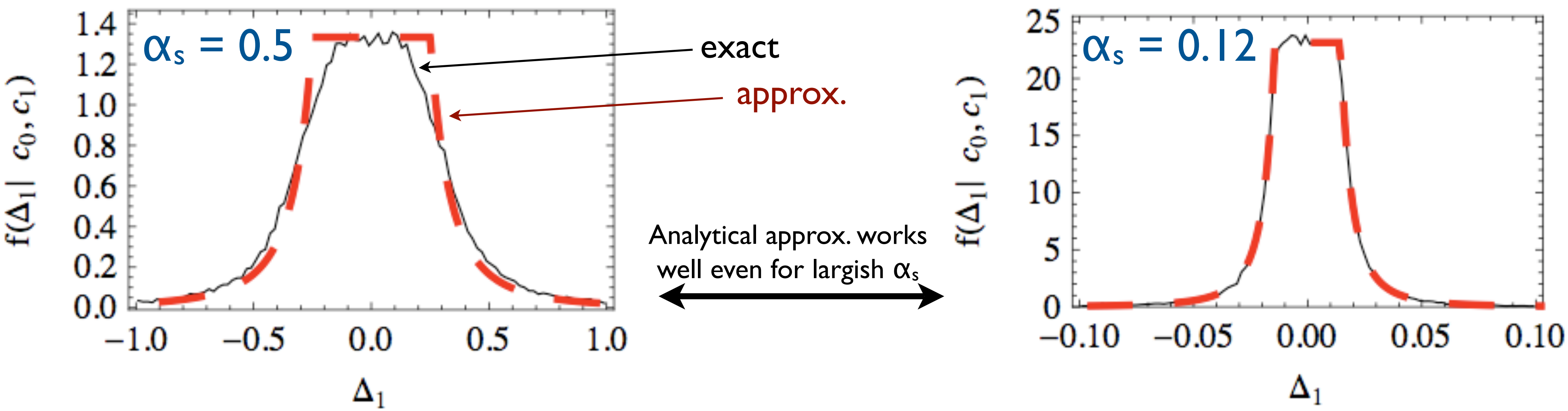
$f(\Delta_k|c_0,\ldots,c_k)$

$$f(\Delta_k|c_0,\ldots,c_k) = \int \left[\delta(\Delta_k - \sum_{n=k+1}^{\infty} \alpha_s^n c_n)\right] f(c_{k+1}, c_{k+2}, \ldots |c_0,\ldots,c_k)\, dc_{k+1} dc_{k+2} \ldots$$

Making the approximation $\Delta_k \simeq \alpha_s^{k+1} c_{k+1}$

i.e. assuming that the coupling is reasonably small, one finds

$$f(\Delta_k|c_0,\ldots,c_k) \simeq \left(\frac{k+1}{k+2}\right) \frac{1}{2\alpha_s^{k+1}\bar{c}_{(k)}} \begin{cases} 1 & \text{if } |\Delta_k| \le \alpha_s^{k+1}\bar{c}_{(k)} \\ \frac{1}{(|\Delta_k|/(\alpha_s^{k+1}\bar{c}_{(k)}))^{k+2}} & \text{if } |\Delta_k| > \alpha_s^{k+1}\bar{c}_{(k)} \end{cases}$$



Intervals

From the credibility distribution of Δ_k one can calculate by integration the **degree of belief of any given interval**, or a ***p*%-credible interval**

$$\mathbb{C}(\Delta_k \in [\Delta_k^-, \Delta_k^+]|c_0,\ldots,c_k) = \int_{\Delta_k^-}^{\Delta_k^+} f(\Delta_k|c_0,\ldots,c_k)\, \mathrm{d}\Delta_k$$

$$p\% = \int_{-d_k^{(p)}}^{d_k^{(p)}} f(\Delta_k|c_0,\ldots,c_k)\mathrm{d}\Delta_k$$

Using analytical approx. for Δ_k

$$\mathbb{C}(\Delta_k \in [-\frac{\delta_k}{2},\frac{\delta_k}{2}]|c_l,\ldots,c_k) = \begin{cases} 1 - \frac{1}{n_c+1}\left[\frac{2}{3k\beta_0}\frac{\bar{c}_{(k)}}{|c_k|}\right]^{n_c} & \text{if } \frac{\delta_k}{2} \geq \alpha_s^{k+1}\bar{c}_{(k)} \Leftrightarrow |c_k| \geq \frac{2}{3k\beta_0}\bar{c}_{(k)} \\ \frac{n_c}{n_c+1}\frac{3k\beta_0}{2}\frac{|c_k|}{\bar{c}_{(k)}} & \text{if } \frac{\delta_k}{2} < \alpha_s^{k+1}\bar{c}_{(k)} \Leftrightarrow |c_k| < \frac{2}{3k\beta_0}\bar{c}_{(k)} \end{cases}$$

$$d_k^{(p)} = \begin{cases} \alpha_s^{k+1}\bar{c}_{(k)}\frac{k+2}{k+1}p\% & \text{if } p\% \leq \frac{k+1}{k+2} \\ \alpha_s^{k+1}\bar{c}_{(k)}\left[(k+2)(1-p\%)\right]^{-1/(k+1)} & \text{if } p\% > \frac{k+1}{k+2} \end{cases}$$

$\bar{c}_{(k)} = \max(|c_0|,\ldots,|c_k|)$

n_c = number of known coefficients

DoB of the conventional interval

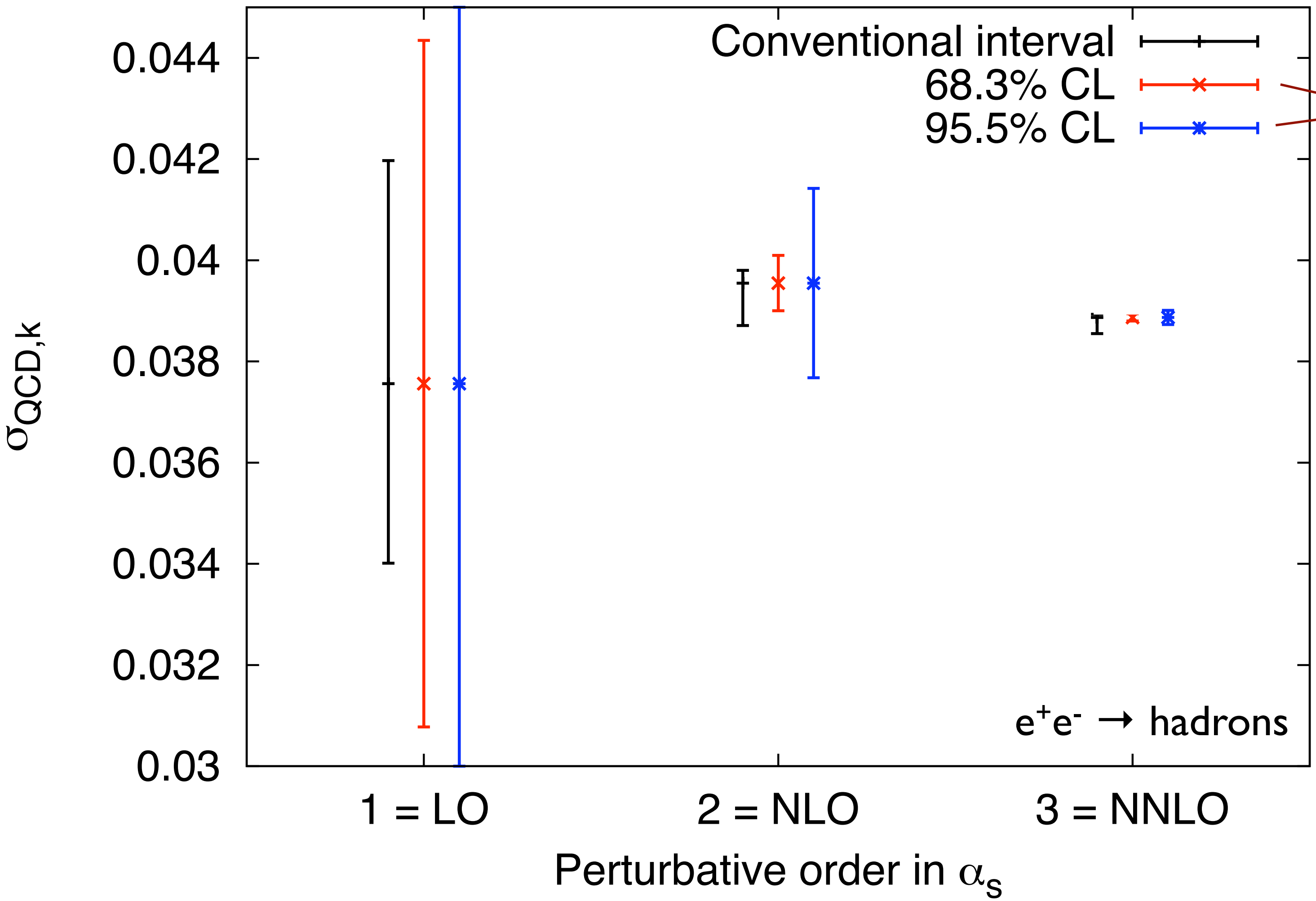
Degree of belief of the interval given by scale variations for $e^+e^- \rightarrow$ hadrons at 90 GeV

Perturbative order	Degree of Belief (approx)	Degree of Belief (exact)
k=1 (LO)	46%	45.8%
k=2 (NLO)	54.8%	58.4%
k=3 (NNLO)	98.8%	77.2%

This discrepancy is due to the asymmetry of the scale variations interval at this order. One should always use the exact expression for accurate results.

p%-credible intervals

Independently from scale variations, one can calculate the **interval [σ_k-$d_k^{(p)}$, σ_k+$d_k^{(p)}$] that has a *p*% degree of belief**



p%-credible intervals in Bayesian model

Conclusions (1)

- The Bayesian model allows one to calculate the degree of belief of given intervals of the remainder of a perturbative series
- The aim is **not** to 'add knowledge' to the perturbative calculation, but rather to **formalise** one way of estimating its uncertainty

The Bayesian model replaces a 2-step process

(1) Choose a procedure/range for scale variations

(2) Decide what the band you obtain means

with a 1-step one

(1) Decide on a prior for your coefficients

The DoB of the band is then automatic

Conclusions (2)

- The priors we have used are no more arbitrary than the conventional method of scale variations. The added value is that the resulting intervals have a proper interpretation in terms of degree of belief, in principle allowing a 'coherent bet'

- Outlook: more extensive exploration of priors, extension to hadronic processes, comparisons with known perturbative calculations, efficient numerical implementation