Multiloop calculations in light cone perturbation theory

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Based on [2601.10374] and ongoing work
with Tuomas Lappi & Risto Paatelainen

Outline

1. Motivation: light cone perturbation theory at higher orders
2. Overview of loop calculations in light cone perturbation theory
3. Recent progress

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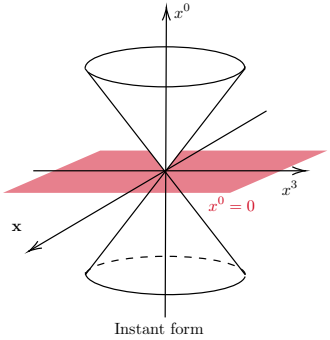
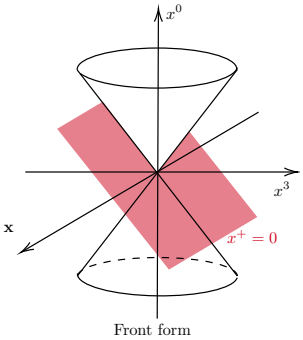
Quantum field theories on the light cone

- Light cone coordinates:

$$x^\mu = (x^+, x^-, \mathbf{x}),$$
$$x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right), \quad \mathbf{x} \equiv \left(x^1, x^2\right)$$

- Quantize along lightlike time x^+
  $\rightarrow$ "new" dynamics
- Light cone gauge: $A^+ = 0$
  $\rightarrow$ only two physical gluon polarizations, no ghosts





Light Cone Perturbation Theory (LCPT)

$=$ perturbation theory with field quantized along x^+

$\rightarrow$ Hamiltonian "old-school" time-ordered perturbation theory:

$$|\Psi\rangle_{\rm int} = |\Psi\rangle + \sum_{n_1} \frac{|n_1\rangle\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{1\Psi}} + \sum_{n_1,n_2} \frac{|n_2\rangle\langle n_2|H_{\rm int}|n_1\rangle\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{2\Psi}\mathrm{ED}_{1\Psi}} + \ldots$$

Everything is on-shell, but Lorentz covariance not explicit
(separate longitudinal and transverse dynamics, 3-momentum conservation at vertices)

Light cone wave functions

- Starting point: Fock space expansion:

$$|\Psi\rangle_{\rm int} = |\Psi\rangle + \sum_{n_1} |n_1\rangle \underbrace{\frac{\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{1\Psi}}}_{\psi^{\Psi\to n_1}} + \sum_{n_1,n_2} |n_2\rangle \underbrace{\frac{\langle n_2|H_{\rm int}|n_1\rangle\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{2\Psi}\mathrm{ED}_{1\Psi}}}_{\psi^{\Psi\to n_2}} + \ldots$$

- The coefficients of the Fock states $\psi^{\Psi\to n}$ are **Light Cone Wave Functions**: Probability amplitudes to find Fock state n in full physical state $|\Psi\rangle_{\rm int}$

Light cone wave functions

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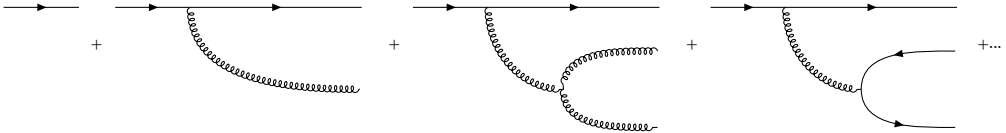
- The coefficients of the Fock states $\psi^{\Psi\to n}$ are **Light Cone Wave Functions**: Probability amplitudes to find Fock state n in full physical state $|\Psi\rangle_{\rm int}$

- E.g. expansion for a interacting quark:

$$|q\rangle_{\rm int} = \sqrt{Z_q}\Big[|q\rangle + \psi^{q\to qg}\otimes|qg\rangle + \psi^{q\to qgg}\otimes|qgg\rangle + \psi^{q\to qq\bar{q}}\otimes|qq\bar{q}\rangle + \ldots\Big]$$

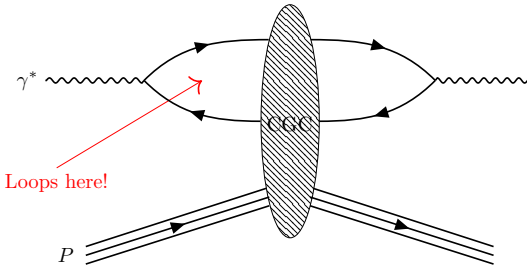


Why use LCPT?

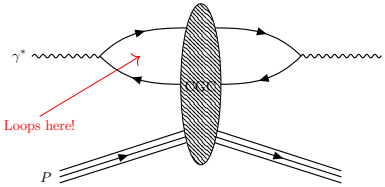
- Trivial vacuum and only physical d.o.fs
  $\rightarrow$ physical Fock space & partonic interpretation

- Works naturally with high-energy scattering
  $\mathcal{M} \sim \langle \text{out}|\hat{S} - 1|\text{in}\rangle$

$\Longrightarrow$ LCPT is the method of choice to perturbatively understand the dilute part of high-energy dilute-dense collisions



Multi-loop LCPT



- Currently, state-of-the-art in LCPT is one-loop level
- Increasing the precision to two loops would be important for small-x phenomenology
  $\rightarrow$ **precision physics for Electron-Ion Collider** (detection of gluon saturation)

Goal:

Completely new order in perturbative expansion in LCPT
$\rightarrow$ Eventually NNLO phenomenology in saturation regime

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Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

- Integration split into $(D-2)$-dimensional transverse integral and 1-dimensional longitudinal integral
- Numerator $\mathcal{N}$ looks quite different from ordinary perturbation theory
- Denominator structure is also different (Δ_i are functions of k^+, p^+)

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

1. **Numerator**: $\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p}) \sim [\bar{u}_h(p) \not{\epsilon}_\lambda(q) u_h(k)]^{\text{number of vertices}}$
   - Light cone spinors etc... typically hard to automatize
   - But can be written[1] in a way that can be fully automated with repeated use of spinor relations and Clifford algebra

[1] Hänninen, Lappi, Paatelainen [1711.08207]

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

2. **Reduction of tensor integrals to scalar ones**:

- e.g.:

$$\int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{k^i k^j}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]} = p^i p^j B_{21} + \delta^{ij} B_{22}$$

- Standard procedure, but now using $(D-2)$-dimensional rotational symmetry of transverse plane

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

3. **Reduction of scalar integrals to small set of master integrals**:
   - Integration by parts?

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

4. Evaluate scalar integrals

- Somewhat standard, but weird denominators due to Δ_i-weights
- Need do derive new integral formulae

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

5. **Expansion in ϵ and integration over longitudinal momentum**

   - One-dimensional integrals involving polynomials and (poly)logarithms
   - In some sense similar to Feynman parameter integrals
   - Expressions can get *very* long and contain divergences

Example loop expression & calculation steps

$$\mathcal{I} \sim \int \mathrm{d}k^+ \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}(k^+, \mathbf{k}; p^+, \mathbf{p})}{[\mathbf{k}^2 + \Delta_1 \mathbf{p}^2]\,[(\mathbf{k} + \Delta_2 \mathbf{p})^2 + \Delta_3 \mathbf{p}^2]}$$

Conclusion:

- Calculations similar to ordinary perturbation theory, but everything is a bit messier
- Systematization of calculation allow extension to higher orders
- Conventional multiloop tools work, but everything needs to be rederived for LCPT

Example integral structure at two loops

$$\mathcal{I} \sim \int \mathrm{d}k_1^+ \int \mathrm{d}k_2^+ \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} \frac{\mathcal{N}(k_1^+, \mathbf{k}_1; k_2^+, \mathbf{k}_2; p^+, \mathbf{p})}{\Big[(\mathbf{k}_1 + \Delta_1 \mathbf{p})^2 + \Delta_2 \mathbf{k}_2^2\Big]\Big[(\mathbf{k}_1 + \Delta_3 \mathbf{k}_2 + \Delta_4 \mathbf{p})^2 + \Delta_5 \mathbf{p}^2\Big]}$$

- Same steps, similar structure, but more complicated (especially denominator and longitudinal integrals)

Time ordering

- One more (big) complication: LCPT is time-ordered

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- Example: photon emission from electron in QED:

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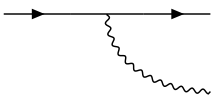
tree level:  1 diagram

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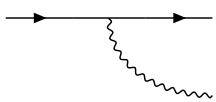
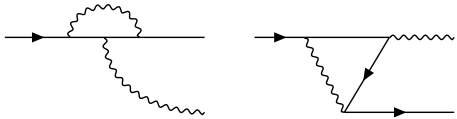


1-loop: + … 10 diagrams



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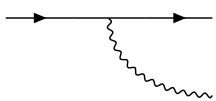

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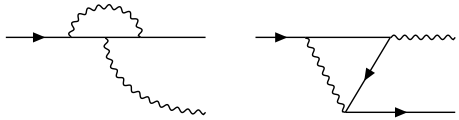
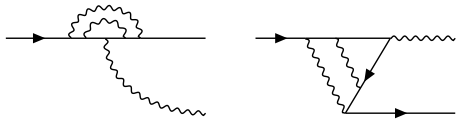


2-loop: + … 426 diagrams



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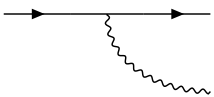
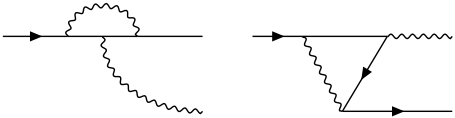
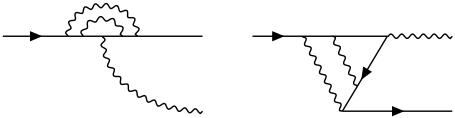
tree level:  1 diagram $\mathcal{O}(1)$ PhD student

1-loop:  + … 10 diagrams $\mathcal{O}(1)$ PhD student

2-loop:  + … 426 diagrams $\mathcal{O}(100)$ PhD students

Automation

Full automation with computer programs required!

(and no such automation pipeline exists for LCPT...)

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1. Motivation: light cone perturbation theory at higher orders
2. Overview of loop calculations in light cone perturbation theory
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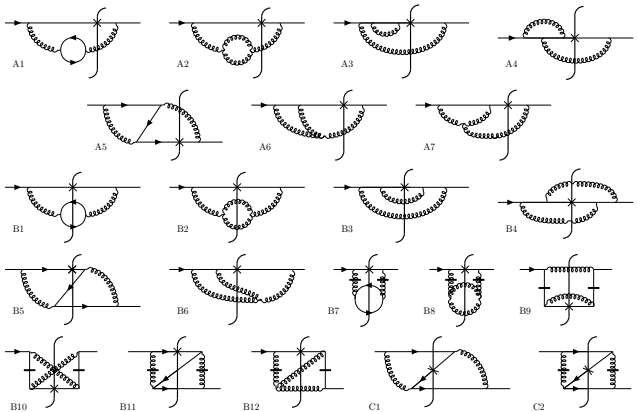
Two-loop DGLAP splitting functions from LCPT

- We computed[a] the first ever two-loop calculation in LCPT by computing two-loop splitting functions
- Simple (but important) two-loop process (1-to-1)



[a] [T. Lappi, R. Paatelainen, M.S., 2601.10374]

Two-loop DGLAP splitting functions from LCPT

Results:

- Matches known result $\Longrightarrow$ **LCPT feasible at two loops!**
- Developed systematic calculation procedures
- Gained understanding of PDF evolution in LCPT language
  - Very clear physical picture: PDF = $\langle h|a^\dagger a|h\rangle$
    (expectation value of parton number operator in hadron state)

Automation code

- I am currently developing a program for fully automated LCPT calculations:
  - Generation of time-ordered diagrams
  - Generation of wave function expression from diagrams
  - Evaluation of numerator and loop integrals
  - Results in a (maybe) analytical expression for the wave function that can then be connected with e.g. CGC scattering

Automation code

- Due to fundamental differences in LCPT and covariant theory, I'm writing everything from scratch

- Many fun puzzles:
  - Structure and notation for loop integrals?
  - Simplification with Galilean invariance
  - etc...

- The code is written with Symbolica[2], a modern computer algebra system

[2] https://symbolica.io/

Automation code

First goal:

Compute (first-ever) two-loop light cone wave functions for a simple QED process to check that everything works (well underway)

Ultimate goal:

General purpose LCPT automation code to be used for one-loop and two-loop small-x phenomenology

Conclusions

LCPT needed for dilute-dense scattering at high energy

Try to push LCPT to two loops for precision phenomenology in small-x QCD

More diagrams, more algebra...

...but we now understand how to do calculations systematically and efficiently

First two-loop calculations in LCPT at hand

Fully automated LCPT program being developed

Thank you!