QCD at small x and saturation

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Contents

1 Practicalities **3**

2 Plan for classes **3**

3 Literature **3**

4 Preliminary stuff **4**
4.1 Partons . . . 4
4.2 Rapidity . . . 5
4.3 Light cone variables . . . 5

5 Eikonal scattering **6**
5.1 Eikonal approximation . . . 6
5.2 Eikonal scattering off target of glue . . . 9

6 One gluon exchange **10**
6.1 One gluon exchange, high energy limit . . . 11
6.2 Eikonal vertex . . . 12
6.3 One gluon exchange with the eikonal vertex . . . 14

7 DIS in dipole picture **14**
7.1 DIS introduction . . . 14
7.2 Partons . . . 18
7.3 Virtual photon scattering . . . 19
7.4 Dipole picture . . . 21

8 The photon light cone wave function **24**
8.1 Light Cone Perturbation Theory . . . 24
8.2 Virtual photon light cone wave function . . . 28

9 BK equation **32**
9.1 Soft gluon radiation . . . 32
9.2 Idea of a renormalization group equation . . . 33
9.3 The Balitsky-Kovchegov equation: a short derivation . . . 35

10 Gluon saturation & CGC **40**
10.1 Spacetime structure in 2 gauges . . . 40
10.2 McLerran-Venugopalan model . . . 44
10.3 Dipole cross section . . . 45
10.4 Weizsäcker-Williams distribution . . . 48
10.5 Glasma . . . 49

11 Hybrid formalism for pA collisions **52**
11.1 Forward rapidity in proton-nucleus collisions 52
11.2 Quark-nucleus scattering 53
11.3 Going to higher orders 57

12 Diffractive DIS at NLO **58**
12.1 Physical situation: diffractive DIS 58
12.2 LO calculation 59
12.3 NLO corrections 60

1 Practicalities

Schedule

- We have 5 classes
- Lectures: blackboard
- Material: this lecture note
- During or after the lecture: small exercise problems

2 Plan for classes

Plan for classes

1. Eikonal scattering Secs 5, 6
2. Light cone quantization and dipole picture of DIS 7 8
3. Soft gluons, Balitsky-Kovchegov equation 9
4. Spacetime structure of the CGC field, the MV model 10
5. Dilute-dense cross sections à la Bjorken, Kogut, Soper: "Hybrid formalism" of particle production in proton-nucleus collisions, 11. Example of a NLO calculation: diffractive DIS 12

3 Literature

A large part of these lectures are based on courses lectured at the University of Jyväskylä: "High energy scattering in QCD" lectured in the spring of 2011 , http://users.jyu.fi/~tulappi/fysh560kl11/ (original handwritten notes for the blackboard lectures are partly in Finnish.) and later a part of the course "QCD" lectured in the spring of 2014, http://users.jyu.fi/~tulappi/fysh555kl14/ (slides and handwritten lecture notes in English), the course "High energy scattering in QCD" in the spring 2023 https://users.jyu.fi/~tulappi/heqcd23/. Some earlier versions of a subset have also been delivered at various summer schools, including lectures on "Early stages of Heavy-Ion Collisions" at the 2018 ECT* Doctoral Training Program https://indico.ectstar.eu/event/14/, lectures at the Galileo Galilei Institute in 2023 and at Cape Town in 2013 and 2023.

Literature

Mostly these lectures are based on the following books:

[1] Y. V. Kovchegov and E. Levin, "Quantum chromodynamics at high energy," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33** (2012).

[2] V. Barone and E. Predazzi, "High-Energy Particle Diffraction," Springer 2002

Potentially relevant are recent review articles:

[3] F. Gelis, "Initial state and thermalization in the Color Glass Condensate framework," Int. J. Mod. Phys. E **24** (2015) no.10, 1530008, [arXiv:1508.07974 [hep-ph]]

[4] J. L. Albacete and C. Marquet, "Gluon saturation and initial conditions for relativistic heavy ion collisions," Prog. Part. Nucl. Phys. **76** (2014) 1 [arXiv:1401.4866 [hep-ph]]

[5] S. Schlichting and D. Teaney, "The First fm/c of Heavy-Ion Collisions," Ann. Rev. Nucl. Part. Sci. **69** (2019), 447-476, [arXiv:1908.02113 [nucl-th]].

[6] J. Berges, M. P. Heller, A. Mazeliauskas and R. Venugopalan, "QCD thermalization: Ab initio approaches and interdisciplinary connections," Rev. Mod. Phys. **93** (2021) no.3, 035003 [arXiv:2005.12299 [hep-th]].

and other slightly less new reviews and others:

[7] E. Iancu and R. Venugopalan, "The Color glass condensate and high-energy scattering in QCD," In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 249-3363 [hep-ph/0303204]

[8] F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, "The Color Glass Condensate," Ann. Rev. Nucl. Part. Sci. **60** (2010) 463 [arXiv:1002.0333 [hep-ph]]

[9] T. Lappi, "Small x physics and RHIC data," Int. J. Mod. Phys. E **20** (2011) no.1, 1 [arXiv:1003.1852 [hep-ph]]

[10] H. Weigert, "Evolution at small x(bj): The Color glass condensate," Prog. Part. Nucl. Phys. **55** (2005) 461 [hep-ph/0501087]

[11] J. Jalilian-Marian and Y. V. Kovchegov, "Saturation physics and deuteron-Gold collisions at RHIC," Prog. Part. Nucl. Phys. **56** (2006) 104 [hep-ph/0505052]

[12] F. Gelis, T. Lappi and R. Venugopalan, "High energy scattering in Quantum Chromodynamics," Int. J. Mod. Phys. E **16** (2007) 2595 [arXiv:0708.0047 [hep-ph]]

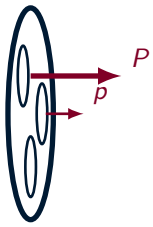
[13] S. J. Brodsky, H. C. Pauli and S. S. Pinsky, "Quantum Chromodynamics and Other Field Theories on the Light Cone," Phys. Rept. **301** (1998) 299 [arXiv:hep-ph/9705477].

[14] C. Marquet, "Chromodynamique quantique à haute énergie, théorie et phénoménologie appliqué aux collisions de hadrons," PhD thesis, in French (!) http://tel.archives-ouvertes.fr/tel-00096416/fr/

[15] H. Hänninen, "Deep inelastic scattering in the dipole picture at next-to-leading order," PhD thesis, [arXiv:2112.08818 [hep-ph]].

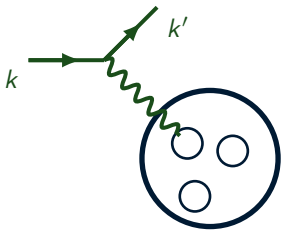
4 Preliminary stuff

4.1 Partons



J.D. Bjorken: At high energy hadrons consist of pointlike constituents: "partons".

- Longit. momentum fraction x: $p = xP$
- Transverse scale p_T or Q



Measured in *D*eep *I*nelastic *S*cattering

$$Q^2 = -(k-k')^2 = -q^2$$
$$x = \frac{Q^2}{2P\cdot(k-k')}$$

Bj scaling:
Q^2-dependence is just like for a point-like particle.

Here the "scale" Q can roughly be thought of as the inverse transverse size of the parton. In textbook calculations in the parton model, the partons are treated as collinear, i.e. they only have longitudinal but not transverse momentum, $k_T = 0$. The resolution scale Q is determined by the physical scattering process that one studies, for example for DIS it is given by the virtuality of the process. In fact it is better to really think of the parton, when measured in such a process as having $k_T \sim Q$. The feature that the parton really has $k_T \sim Q$, but nevertheless one can calculate as if it was collinear, as long as the parton distribution that the parton comes from depends on Q, is the magic of collinear factorization! In this course, we will go beyond collinear factorization: when discussing the CGC and the physics of gluon saturation it is essential to actually think and calculate of small-x gluons as having an intrinsic transverse momentum.

- Textbooks: partons collinear
- Now: partons have k_T

When Bjorken introduced the term "parton," quarks and gluons were not yet known. Instead, Bjorken used the new word to describe some, so far unknown, elementary, pointlike constituents of the proton. Only later it became understood that partons are quarks and gluons. Today, one might consider the term "parton" to be a synonym of "quark and gluon". But usually the term "parton" is used in a slightly more limited sense, meaning roughly "quarks and gluons as constituents of a proton or nucleus, especially one accelerated to near the speed of light." If one is discussing e.g. the weak interaction, supersymmetry or the quark model for hadrons (i.e. with "constituent quarks"), one does not usually use the term "parton."

We shall be interested in these lectures at particles traveling practically at the speed of light, at very high energy. In this case it is convenient to think of the scattering problem in a coordinate system well adapted to things traveling at the speed of light in the z- or 3-direction. For this purpose, we introduce the kinematical variable of *rapidity* and *light cone coordinates*

4.2 Rapidity

Successive boosts in one direction with velocity v become easy if we define the *rapidity* ξ corresponding to the boost as $v = \tanh \xi$. Now for successive boosts $\xi = \xi_1 + \xi_2$. (exercise)

Rapidity

Momentum space rapidity y; $v^z = \tanh y$

$$E = \sqrt{m^2 + p_T^2} \cosh y = m_T \cosh y \qquad \mathbf{p}_T = (p^1, p^2) \tag{1}$$

$$p^3 = \sqrt{m^2 + p_T^2} \sinh y = m_T \sinh y \qquad p_T^2 = \mathbf{p}_T^2 \tag{2}$$

When boosting with a boost ξ this behaves additively. (exercise)

The pseudorapidity η is defined by taking $m = 0$ in Eq. (2):

$$p^3 = p_T \sinh \eta \tag{3}$$

+ Pseudorapidity just depends on the scattering angle (different components of the 3-momentum), so it easy to measure $\Longrightarrow$ it is the preferred quantity for experimentalists.
- Pseudorapidity is the same as rapidity y for massless particles.
- However, pseudorapidity does not Lorentz-transform easily for massive particles, it depends on the rest frame in a nontrivial way.

4.3 Light cone variables

Light cone variables

For any 4-vector v^μ:

$$v^\pm = \frac{1}{\sqrt{2}}\left(v^0 \pm v^3\right) \tag{4}$$

Scalar product

$$\begin{aligned} x \cdot y &= x^0 y^0 - x^3 y^3 - \mathbf{x}\cdot\mathbf{y} = x^+ y^- + x^- y^+ - \mathbf{x}\cdot\mathbf{y} && (5)\\ v^2 &= (v^0)^2 - (v^3)^2 - \mathbf{v}^2 = 2v^+v^- - \mathbf{v}^2 && (6) \end{aligned}$$

Thus metric is

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu} \tag{7}$$

Raising and lowering indices is not just a sign:

$$v^+ = v_- \qquad v^- = v_+ \tag{8}$$

High energy particle traveling in $+$ direction

$$p^+ \approx \sqrt{2}E \quad \gg \quad p^- = \frac{m^2 + \mathbf{p}^2}{2p^+} \tag{9}$$

Lorentz-invariant phase space

$$\frac{\mathrm{d}^4 p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2) = \frac{\mathrm{d}p^+\,\mathrm{d}p^-\,\mathrm{d}^2\mathbf{p}}{(2\pi)^3}\delta(2p^+p^- - \mathbf{p}^2 - m^2) = \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}p^+}{2p^+} = \frac{\mathrm{d}^2\mathbf{p}}{2(2\pi)^3}\,\mathrm{d}y \tag{10}$$

Rapidity and light cone variables

$$\begin{aligned} k^+ &= \frac{1}{\sqrt{2}}\sqrt{m^2 + k_T^2}\,e^{y} && (11)\\ k^- &= \frac{1}{\sqrt{2}}\sqrt{m^2 + k_T^2}\,e^{-y} && (12) \end{aligned}$$

Later in the course: Light cone quantization

Using light cone coordinates is just a change of variables and does not affect the physics in any way. However, the coordinate system is particularly useful in the context of light cone quantization [13, 1]. In ordinary ("instant form") quantization one quantizes the theory by imposing equal time ($t = t'$) commutation relations between Heisenberg time-dependent operators, and studies the evolution of the system in time t generated by a Hamiltonian H, which is the 0-component of a Lorentz 4-vector. In light-cone (or light front) quantization one treats instead the coordinate x^+ as the time coordinate (light cone time), and imposes canonical commutation relations at equal x^+. The evolution of the Heisenberg picture operators is then generated by the light cone Hamiltonian $\hat{P}^-$, which is the $-$-component of a Lorentz 4-vector.

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5 Eikonal scattering

5.1 Eikonal approximation

Scattering at high energy becomes much simpler with the *eikonal approximation*. Many textbooks, (e.g. Barone & Predazzi (Sec 2.3) start by describing the eikonal approximation for nonrelativistic scattering off a classical potential). A more formal field theoretic derivation for gauge theory (e.g. [12]) follows the important work of Bjorken et al [16] by starting from the operator definition of the scattering amplitude and looking at its properties when the incoming state is boosted to a very high energy. We will here follow an

approach that is between these in the sense that we look just at solutions of wave equation, but inspect a situation that is

- relativistic and
- involves a vector potential (for the Abelian theory at first)

Klein-Gordon eq. in external vector potential:

$$\left[(i\partial_\mu + eA_\mu(x))(i\partial^\mu + eA^\mu(x)) - m^2\right]\phi(x) = 0 \tag{13}$$

We are now interested in the scattering problem where the boundary condition is that of a high energy particle approaching the target (represented by the gauge potential A_μ). Thus the boundary condition for the equation is a plane wave $e^{-ik\cdot x}$ at $x^+ \to -\infty$. The assumption that we are interested in high energy means that k^+ is much larger than any other momentum scale in the problem, in particular any gradients of the potential A_μ. It is natural to make this assumption about the momentum scales explicit and to separate out the rapidly oscillating behavior of the wave function from the smoother scales by using an ansatz where it is explicitly factorized. We can also assume that we are in Lorenz gauge $\partial_\mu A^\mu = 0$.

$$\phi(x) \to e^{-ik\cdot x}, \quad x^+ \to -\infty \tag{14}$$

Ansatz

$$\phi(x) = e^{-ik\cdot x}\varphi(x) \tag{15}$$

$$k^+ \gg \frac{\partial_\mu\varphi(x)}{\varphi(x)} \tag{16}$$

$$\partial_\mu A^\mu = 0 \tag{17}$$

Inserting the ansatz (15) into the original equation (13) leads to

$$e^{-ik\cdot x}\Big[\overbrace{k^2 - m^2}^{=0} + \underline{2ek^\mu A_\mu + 2ik^\mu\partial_\mu} - \partial_\mu\partial^\mu + 2ieA^\mu\partial_\mu + e^2A_\mu A^\mu\Big]\varphi(x) = 0 \tag{18}$$

Now comes the crux of the eikonal approximation. We are assuming that k^+ is a large variable, thus at leading order we only need to keep the terms proportional to k^+ in equation (18). These only appear in the terms that are underlined in the equation. Thus, and recalling that now the initial condition is $\varphi(x) \to 1, \quad x^+ \to -\infty$, and factoring out the $2ik^\mu$ we end up with

$$\left[\partial_+ - ieA^-(x)\right]\varphi(x) = 0 \implies \varphi(x) = \exp\left\{ie\int_{-\infty}^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x})\right\} \tag{19}$$

Note that at this point only the A^- component of the target gauge potential matters, due to the large k^+ momentum of the probe. In the far future after the target, $x^+ \to \infty$, the outgoing wave is written in terms of the incoming one in terms of the

Eikonal phase

$$\chi(\mathbf{x}) = e\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}) \tag{20}$$

$$\phi(x)|_{x^+\to\infty} \approx e^{-ik\cdot x}\left[1 - \left(1 - e^{i\chi(\mathbf{x})}\right)\right] \tag{21}$$

At this point we will just state, without trying to go through a formal derivation, that one minus the exponential of the eikonal phase is analogous to the opacity/transparency of the optical problem, and that the elastic cross section is given by

$$1 - e^{i\chi(\mathbf{x})} \iff \Gamma(\mathbf{x}) \implies \frac{\mathrm{d}\sigma_{\text{el.}}}{\mathrm{d}^2\mathbf{q}} = \left|\frac{-i}{2\pi}\int \mathrm{d}^2\mathbf{x}e^{-i\mathbf{q}\cdot\mathbf{x}}\left(1 - e^{i\chi(\mathbf{x})}\right)\right|^2 \tag{22}$$

We can also try to just directly calculate the cross section starting from the outgoing wavefunction (21) in terms of the definition of the cross section. In the outgoing wavefunction (21) we clearly see which wave is the incoming wave and which one the scattered one:

$$\phi(x)|_{x^+\to\infty} = \phi_{\text{in}} + \phi_{\text{scat}} \tag{23}$$

with

$$\phi_{\text{in}} = e^{-ik\cdot x} \tag{24}$$

$$\phi_{\text{scat}} = e^{-ik\cdot x}\left(1 - e^{i\chi(\mathbf{x})}\right) \tag{25}$$

This derivation follows the way a cross section is calculated from a scattering amplitude in nonrelativistic quantum mechanics. A cross section is defined as

$$\mathrm{d}\sigma = \frac{\mathrm{d}(\text{ number of particle in phase space})/\text{time}}{\text{flux in}} \tag{26}$$

Now the square of our wavefunction in coordinate space is related to the probability density of particles. Let us calculate the flux:

$$\text{flux} = \frac{\text{number}}{\text{time}\times\text{area}} = \frac{\text{number}}{\text{volume}}\frac{\text{length}}{\text{time}}, = |\phi_{\text{in}}|^2 \times \frac{k}{E(k)} = |e^{-ik\cdot x}|^2 \times \frac{k}{E(k)} = \frac{k}{E(k)} \tag{27}$$

where the length (= volume/area) is some distance in the direction of the incoming particles which, divided by the unit of time that it takes the particles to travel this distance, gives us the velocity. The velocity of a relativistic particle is its momentum divided by energy. Now, for the outgoing particles, we do not want to measure the density as a function of transverse coordinate, but rather transverse momentum. Thus we need to Fourier-transform the wavefunction to transverse momentum, to get the number of outgoing particles. Define the mixed space wavefunction as

$$\phi(\mathbf{q}, z, t) = \int \mathrm{d}^2\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{x}, z, t) \tag{28}$$

Now the square of this quantity gives the number of particles per unit length and unit transverse momentum, with a $(2\pi)^2$ that always goes with the Fourier transform[1]

$$|\phi(\mathbf{q}, z, t)|^2 = (2\pi)^2 \frac{\mathrm{d}N}{\mathrm{d}z} \tag{29}$$

Again we need to change this from a number of particles per unit length to a number of particles per unit time. This involves the velocity of the particle. Here, we need to again evoke the fact that we are working in the high energy limit, where the scattering angle is very small. Thus, even if the particle does acquire some transverse momentum $\mathbf{q}$ in the scattering, we consider that the longitudinal momentum k is still much larger, $k \gg |\mathbf{q}|$, and the velocity (in the z-direction) is given by $k/E(k)$. Then we get the elastic cross section as

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\text{el.}}}{\mathrm{d}^2\mathbf{q}} = \frac{\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{q}\,\mathrm{d}z}\times\frac{k}{E(k)}}{\text{flux}} = \frac{1}{(2\pi)^2}|\phi_{\text{scat}}(\mathbf{q}, z, t)|^2 = \frac{1}{(2\pi)^2}\left|\int \mathrm{d}^2\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} e^{ik\cdot x}\left(1 - e^{i\chi(\mathbf{x})}\right)\right|^2 \\ = \left|\frac{-i}{2\pi}\int \mathrm{d}^2\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}}\left(1 - e^{i\chi(\mathbf{x})}\right)\right|^2,\end{aligned} \tag{30}$$

using the fact that $\mathbf{k} = 0$ and thus $e^{ik\cdot x}$ just cancels in the absolute value. The final $-i$ of course inside the integral just gives one in the absolute value. We scale it out so that we recover the conventional complex phase of the scattering amplitude.

[1] You can check this by checking that the total number of particles is

$$N = \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}z|\phi(\mathbf{x}, z, t)|^2 = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,\mathrm{d}z|\phi(\mathbf{q}, z, t)|^2.$$

5.1.1 Summary

This was a sketchy derivation. Some points of interest:

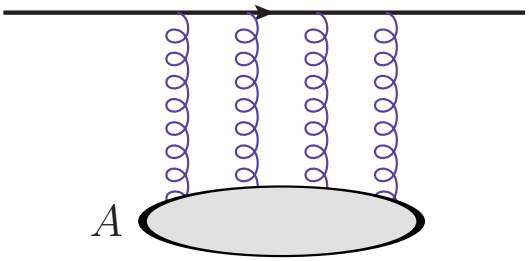
- We haven't specified anything about x^-: we will come back to this later
- The outgoing wave does not appear to expand spherically as was the case for our optical problem. This is mainly an artefact of the coordinate system; and a result of the eikonal approximation that assumes small angle scattering.
- If we do expand to leading order in the eikonal phase χ, we see that Γ is imaginary, i.e. the scattering amplitude real. But when expanded to second order, the scattering amplitude develops an imaginary phase. The eikonal approximation builds in a relation between the real and complex parts of the scattering amplitude, i.e. between elastic and inelastic (absorptive) scattering, that is consistent with unitarity of the scattering S-matrix.

5.2 Eikonal scattering off target of glue

So far we have only been discussing Abelian theory. Let us now move to QCD in the high energy limit. We will, however, still stay within the eikonal picture of scattering. In particular, we will assume that QCD scattering is dominated by scattering off the gluon field of the target. This is in fact not just an assumption, but a consequence of the spin of the gluons: the coupling strength of a high energy particle to the gluon field is proportional to the momentum of the particle. The coupling to the fermion field of the target, on the other hand, is proportional to the square root of the high energy, because the fermion field has spin half.

We also know from experiment, and can show in perturbative QCD (BFKL evolution), that the number of gluons in the proton or nucleus grows at higher energy (i.e. smaller x). We may therefore assume that there are many of these gluons, in fact so many that eventually at some point the gluon field becomes nonperturbatively strong.

Color Glass Condensate (CGC)
Target has many gluons: $A_\mu \gg 1/g \implies$ described as classical gluon field



A careful reader will note that the dimensions do not match: the correct dimensions for $A_\mu(x)$ will have to come from some (transverse) momentum that a typical gluon has, together with the dimensionless coupling g.

To see how a colored particle interacts with such a target color field, we can follow the same procedure as previously, by solving the Dirac equation of a colored fermion in this field. The work essentially the same as previously in the abelian theory. The only difference with respect to eq. (19) is that now the field is a matrix in color space, and does not commute with the field at a different coordinate x^+. Thus, when multiplying or exponentiating matrices, one naturally ends up with a path ordered exponential.

Quark in color field, Dirac:

$$(i\not{\partial} - g\not{A})\psi(x) = 0 \tag{31}$$

Ansatz $\psi(x) = V(x)e^{-ip\cdot x}u(p)$ + eikonal approx.

$$\partial_+ V(x^+, x^-, \mathbf{x}) = -igA^-(x^+, x^-, \mathbf{x})V(x^+, x^-, \mathbf{x}) \tag{32}$$

Inserting the ansatz, with $p^\mu = (p^+, 0, \mathbf{0})$ into the Dirac equation we get

$$e^{-ip\cdot x}(\not{p} + i\not{\partial} - g\not{A})V(x)u(p) = 0 \tag{33}$$

We now use the massless Dirac equation $\not{p}u(p) = 0$. From this equation it follows, since $p^\mu = (p^+, 0, \mathbf{0})$, that $\gamma^- u(p) = 0$. We an now multiply the remaining equation from the left by γ^- to get

$$\gamma^-(i\not{\partial} - g\not{A})V(x)u(p) = 0 \tag{34}$$

Since

$$\gamma^-\not{a} = \gamma^-\left(a^+\gamma^- + a^-\gamma^+ - \mathbf{a}\cdot\boldsymbol{\gamma}\right) = -\left(a^-\gamma^+ - \mathbf{a}\cdot\boldsymbol{\gamma}\right)\gamma^- + a^-, \tag{35}$$

this implies

$$\left(i\partial_+ - gA^-\right)V = 0. \tag{36}$$

Wilson line = path-ordered exponential

$$V(x^+, x^-, \mathbf{x}) = \mathbb{P}\exp\left\{-ig\int^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x})\right\} \tag{37}$$

$$\mathbb{P}\left[A(y^+)A(x^+)\right] \equiv \theta(y^+ - x^+)A(y^+)A(x^+) + \theta(x^+ - y^+)A(x^+)A(y^+) \tag{38}$$

These path ordered exponentials are the same quantities that one uses to formulate lattice field theory (the link matrices). The difference is that now, we are following a light-like path, whereas in lattice field theory the links are in the spatial or imaginary time (also $\sim$ spatial) direction. This is because we are interested in the propagation of a very energetic particle moving at the speed of light in stead of a static equilibrium system.

Recap

- We used an eikonal approximation: assume that longitudinal momentum is larger than other momenta involved
- Our result is not perturbative, i.e. it sums a series of different powers of the external field A_μ
- Only one component of the vector field A^- matters. This is related to the high energy approximation: the current of charged particles with a large momentum k^+ has a J^+-component that is larger than all others, and this component of the current only couples to the A^--component of the gauge field. Incidentally, this feature is independent of the spin of the probe, and is thus the same also if we replace our incoming charged scalar particles by fermions or charged vector bosons. On the other hand, this feature does very much depend on the fact that we are scattering off a spin-1 field (a vector field), it is this feature that causes the strength of the interaction to be proportional to the momentum of the incoming particle. Theh graviton field would couplt to the momentum squared, whereas a scalar field does not couple to the momentum. The same feature can easily be seen by calculating some tree level Feynman diagrams for such scattering [exercise]. In terms of Feynman diagrams one can state the eikonal approximation in terms of the "eikonal vertex". The fact that only A^- matters indicates that there is a gauge dependence in our result. It should be possible to do a gauge transformation that changes A^-, but leaves the cross section invariant. Thus there are gauges where obtaining the correct result requires taking into account other components of the gauge field, which in some gauge might be large enough to compensate the lack of an explicit enhancement by the large component k^+.

Digression: Probe and target light cone gauges

If we wanted to count the photons of the target in a rigorous well-defined quantum mechanical way, we would want to go to the light-cone gauge of the leftmoving target, $A^- = 0$. Then our interaction would look very different from what we have now. Thus, even the perturbative limit (power series expansion in A_μ) of our result is a perturbation series in the interaction with a Coulomb field of the target, not really with a specific number of actual radiation quanta (photons) in it.

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6 One gluon exchange

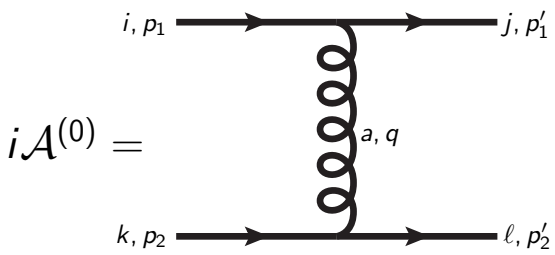
Exercise: Spin-0 exchange

Calculate (to lowest order, one tree diagram) the cross section for fermion-fermion (e.g. $e^-\mu^- \to e^-\mu^-$) scattering in the Yukawa interaction, i.e. via a scalar particle. The Feynman rule for the fermion-scalar particle vertex is just g (the coupling), with spinors for the external legs, and the scalar particle propagator is just i/q^2. What is the high energy limit?

We will first try to compute the high energy behavior of scattering amplitudes in QCD from normal Feynman diagrams. For simplicity, we consider first $qq \to qq$ scattering. But I will try to convince you that the result we get is in fact much more general

6.1 One gluon exchange, high energy limit

As a warm-up exercise let's calculate the scattering amplitude for the proess $qq \to qq$ via one gluon exchange. We first define the kinematical variables.



$$
\begin{aligned}
p_1 &= (p_1^+, 0, \mathbf{0}) && (39)\\
p_2 &= (0, p_2^-, \mathbf{0}) && (40)\\
p_1' &= (p_1^+ - q^+, -q^-, -\mathbf{q}) && (41)\\
p_2' &= (q^+, p_2^- + q^-, \mathbf{q}) && (42)\\
s &= (p_1 + p_2)^2 = 2p_1^+ p_2^- && (43)\\
t &= q^2 && (44)
\end{aligned}
$$

H. E. limit $\boxed{s \gg t}$

$$0 = (p_1')^2 = -2q^- p_1^+ - t \quad \Longrightarrow q^- = -\frac{t}{2p_1^+} = -\frac{t}{s}p_2^- \tag{45}$$

$$0 = (p_2')^2 = 2q^+ p_2^- + t \quad \Longrightarrow q^+ = -\frac{t}{2p_2^-} = -\frac{t}{s}p_1^+ \tag{46}$$

$$\Longrightarrow p_1^+ \gg q^+ \quad p_2^- \gg q^- \quad , \quad t \approx \mathbf{q}^2. \tag{47}$$

This has some very typical features for scattering kinematics at high energy:

- We are considering momenta as ordered: the plus-going quark has a larger plus-momentum than the exhanged gluon.
- This kinematics is equivalent to the limit $s \gg t$
- This limit leads to the momentum transfer t being dominated by transverse momentum. Thus one can also think of the H.E. limit as the limit of *transverse momenta* $\ll$ *longitudinal momenta*.
- If one thinks in terms of scattering angles this means that the scattering angle is very small. Thus the high energy limit is often also called the *forward* limit.

Now we want to calculate the square of the amplitude, and average over incoming and sum over outgoing colors and spins.

The color algebra can be done as follows

$$\mathcal{A}^{(0)} = g^2 t^a_{ji} t^a_{\ell k} \bar{u}(p_1')\gamma^\mu u(p_1)\bar{u}(p_2')\gamma_\mu u(p_2)\frac{1}{q^2} \tag{48}$$

$$\overline{\left|\mathcal{A}^{(0)}\right|^2} \sim \overbrace{\frac{1}{N_c{}^2}}^{\text{avg}} \underbrace{t^a_{ji} t^a_{\ell k}}_{\mathcal{A}} \underbrace{t^b_{ij} t^b_{l\ell}}_{\mathcal{A}^*} = \frac{1}{N_c{}^2}\,\mathrm{tr}\left(t^a t^b\right)\mathrm{tr}\left(t^a t^b\right) = \frac{1}{N_c{}^2}\,\frac{1}{2}\delta^{ab}\,\frac{1}{2}\delta^{ab} = \frac{N_c{}^2 - 1}{4N_c{}^2} \tag{49}$$

Note that the generators t^a_{ij} are Hermitian, ie. the complex conjugate of the transpose is the matrix itself, or equivalently complex conjugation is the same as taking the transpose. Thus if the amplitude $\mathcal{A}$ has t^a_{ji}, the conjugate amplitude $\mathcal{A}^*$ has $(t^a_{ji})^* = t^a_{ij}$.

Now let us do the spin algebra as you would do it in the particle physics course, assuming massless quarks.

$$\begin{aligned}
\overline{\left|\mathcal{A}^{(0)}\right|^2} &\sim \overbrace{\frac{1}{2^2}}^{\text{avg}} \operatorname{tr} \not{p}'_1\gamma^\mu \not{p}_1\gamma^\nu \operatorname{tr} \not{p}'_2\gamma_\mu \not{p}_2\gamma_\nu && (50)\\
&= \frac{1}{4}\, 4(p_1'^\mu p_1^\nu - g^{\mu\nu} p_1\cdot p_1' + p_1'^\nu p_1^\mu)\; 4(p'_{2\mu}p_{2\nu} - g_{\mu\nu}p_2\cdot p'_2 + p'_{2\nu}p_{2\mu}) && (51)\\
&= s^2 - t^2 + u^2 - t^2 + 4t^2 - t^2 + u^2 - t^2 + s^2 = 2s^2 + 2u^2 && (52)
\end{aligned}$$

$$\begin{aligned}
s &= (p_1+p_2)^2 = 2p_1\cdot p_2 = (p_1'+p_2')^2 = 2p_1'\cdot p_2' && (53)\\
t &= (p_1 - p_1') = -2p_1\cdot p_1' = (p_2 - p_2') = -2p_2\cdot p_2' && (54)\\
t &= (p_1 - p_2') = -2p_1\cdot p_2' = (p_2 - p_1') = -2p_2\cdot p_1' && (55)
\end{aligned}$$

$$\overline{\left|\mathcal{A}^{(0)}\right|^2} = g^4 \frac{N_c{}^2 - 1}{4N_c}\frac{2s^2+2u^2}{t^2} \qquad (56)$$

Very fine and good, this is the way one does amplitude computations in a normal particle physics or field theory course. With some practice this way of calculating becomes routine. But we want to do the same thing a little bit differently:

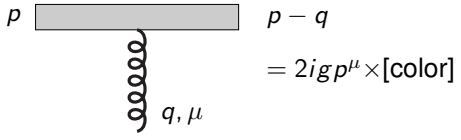
- We want to use the H.E. kinematical approximation to make our life simpler
- We want to look a the amplitude only, not have to square it.
- We want to understand what happens to individual spin states, now we just summed over the spin.

6.2 Eikonal vertex

We are interested in the interactions of spin-1 gauge bosons (gluons, also photons) with different kinds of matter. Very generically, a spin-1 particle is represented by a vector field A_μ, which couples in the Lagrangian to some 4-current carrying the appropriate charge: $J^\mu A_\mu$. For an elementary vertex particle→particle+gauge boson in perturbation theory, the vector A_μ becomes the polarization vector of the gauge boson, i.e. the Lorentz-index in the gauge boson propagator. This polarization index then couples to some 4-vector, and the question is: which one?

We have basically two options. The 4-vector that the gauge boson couples to can be made up from 4-vectors of particle momenta, and from spins. Without proof, I claim that at in the high energy limit (when the momentum of the matter particle is much larger than that of the gauge boson), the spin does not matter. Thus, apart from the color factor, the interaction of a gauge boson with a high energy particle is always the same.

At high energy: spin does not matter



Let us derive the eikonal vertex for some known interactions.

- Interaction with a scalar field. The standard vertex (with momenta as above) is

$$ig(p^\mu + (p-q)^\mu) \approx 2igp^\mu, \quad p \gg q \qquad (57)$$

- Interaction with a fermion: the vertex is

$$ig\bar{u}_{\lambda'}(p-q)\gamma^\mu u_\lambda(p) \qquad (58)$$

Here λ', λ are the quark spins, and we are not writing out the color factor). How does one treat this? One can always use the explicit forms, but a more straightforward option is to use the "Gordon identity"

$$ig\bar{u}_{\lambda'}(p-q)\gamma^\mu u_\lambda(p) = ig\bar{u}_{\lambda'}(p-q)\left[(p+(p-q))^\mu + i\sigma^{\mu\nu}(p-(p-q))_\nu\right]u_\lambda(p). \tag{59}$$

Now we see that explicitly the spin-dependent part ($\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is the generator of Lorentz-tranformations, including rotations, and thus related to the spin) is proportional to the small momentum q. Neglecting terms proportional to q and using the spinor normalization

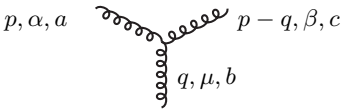
$$\bar{u}_{\lambda'}(p)u_\lambda(p) = \delta_{\lambda'\lambda} \tag{60}$$

we arrive at the eikonal vertex. At this point the reader might wonder why we say that the vertex has the spinors $\bar{u}$ and u which, in the usual Feynman rules, are only attached to external lines. To see why also internal quark lines entering a vertex can be thought of as contributing a similar factor, one can write the quark propagator as

$$\frac{i}{\not{p} - m + i\varepsilon} = i\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = i\sum_\sigma \frac{u_\sigma(p)\bar{u}_\sigma(p)}{p^2 - m^2 + i\varepsilon}. \tag{61}$$

The first u is then associated with the vertex at the beginning of the propagator (following the fermion line), and the $\bar{u}$ with the one at the end.

- Gluon: we start from the elementary vertex (note that now p is incoming, $p-q$ and q outgoing momenta)



$$gf^{abc}\left[g^{\alpha\mu}(p+q)^\beta + g^{\mu\beta}(-q+(p-q))^\alpha + g^{\beta\alpha}(-(p-q)-p)^\mu\right] \tag{62}$$

We first neglect the small momentum components $q \ll p$, and this becomes

$$gf^{abc}\left[g^{\alpha\mu}p^\beta + g^{\mu\beta}p^\alpha - 2g^{\beta\alpha}p^\mu\right] \tag{63}$$

Now we must, in a way that is similar to the insertion of the spinors needed for the fermionic vertex, discuss the polarization states of the gluons. If the incoming (p) and outgoing ($p-q$) hard gluons are external, on-shell particles, the vertex must be contracted with their polarization vectors $\varepsilon_\alpha(p)$ and $\varepsilon^*_\beta(p-q) \approx \varepsilon^*_\beta(p)$. These polarization vectors are transverse, i.e. $k^\mu\varepsilon_\mu(k) = 0$. Thus, in this case at least, the terms $\sim p^\alpha$ and $\sim p^\beta$ do not contribute. Also if these gluons are internal lines, we can do the calculation assuming that they are transverse (i.e. taking the propagators in Landau gauge); we know that the result for a scattering amplitude is gauge invariant. Finally, relating the structure constant to the generator of the adjoint representation of SU(3) $(T^c)_{ba} = if_{abc}$ we can write the vertex as

$$2igp^\mu g^{\beta\alpha}(T^c)_{ba} \tag{64}$$

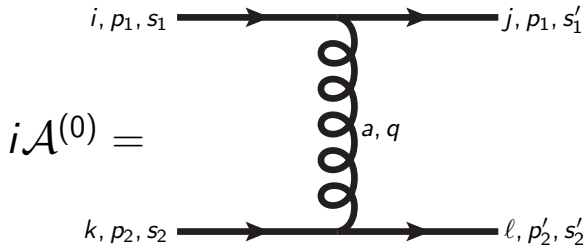
(signs $\sim \ldots$). The way to read this is that the eikonal 3-gluon vertex has the universal eikonal coupling of a color charged particle to gluons, $2igp^\mu$. This is multiplied by a color factor $(T^c)_{ba}$ (reading the color indices starting from the end of the diagram) corresponding to the adjoint representation color charge. There is also a polarization delta function $g^{\beta\alpha}$ which ensures that the polarization of the fast particle does not change, just like the spin of a quark does not change in the eikonal interaction.

Notes:

- This eikonal vertex should only be used in covariant gauge, where the polarization vector of the gauge particle $\varepsilon^\mu(q)$ has all 4 components. Later in this course we will also work in light cone gauge, where one chooses the gauge in such a way that $\varepsilon^\mu(q)p_\mu = 0$, i.e. so that the component of the polarization vector that couples to the eikonal vertex vanished by a gauge choice. In this case the interaction cannot be taken to be given by the same eikonal vertex, and one needs to take into account the other components.

- The eikonal interaction conserves the spin/helicity/polarization of the "matter" particle.
- Note that the two approximations: neglecting the momentum of the gauge particle from the point of view of the matter particle $p^+ \approx p^+ - q^+$ and the interaction not depending on the spin of the matter particle together mean that we are neglecting the *quantum* nature of the radiation. The eikonal approximation or eikonal vertex corresponds to the classical interaction with charged matter with a gauge field: acceleration or deceleration of the matter due to the Lorentz force, and potential classical radiation due to this acceleration or deceleration. Thus physics in the leading high energy limit can, of one does it carefully, be represented by classical (color) charges interacting with classical gauge fields.

6.3 One gluon exchange with the eikonal vertex



Now let us calculate the scattering amplitude again using the eikonal vertex. We write out the spins of the quarks s_1, s_2, s_1', s_2' explicitly and use the eikonal vertex for the gluon coupling. Now the amplitude is

$$\mathcal{A}^{(0)} \approx -i \left[\delta_{s_1,s_1'} 2ip_1^\mu t^a_{ji}\right] \frac{-ig_{\mu\nu}}{q^2} \left[\delta_{s_2,s_2'} 2ip_2^\nu t^a_{\ell k}\right] = g^2 t^a_{ji} t^a_{\ell k} \frac{2s}{t} \delta_{s_1,s_1'} \delta_{s_2,s_2'}, \tag{65}$$

remembering that $p_1 \cdot p_2 = s/2$. (I think that even the sign is right. In the Abelian limit the potential is repulsive, and the scattering amplitude should be negative.) We can square this and sum or average over the spins and colors to get the squared amplitude as

$$\overline{\left|\mathcal{A}^{(0)}\right|^2} = \left(\frac{2sg^2}{t}\right)^2 \left[\frac{1}{2} \sum_{s_1=\pm, s_1'=\pm} (\delta_{s_1,s_1'})^2\right] \left[\frac{1}{2} \sum_{s_2=\pm, s_2'=\pm} (\delta_{s_2,s_2'})^2\right] \frac{{N_c}^2 - 1}{4{N_c}^2} = g^4 \frac{{N_c}^2 - 1}{4{N_c}^2} \frac{4s^2}{t^2}, \tag{66}$$

where the color sum is done just as before. We notice that this is exactly the same as the full expression (56) in the H.E: limit $s \approx -u \gg |t|$, obtained with a lot less effort.

Exercise: Spin-2 exchange

Take the Feynman rules for the graviton propagator (160) and the graviton-spin-0 particle vertex (162) from arXiv:1702.00319. Calculate the elastic scattering cross section $d\sigma/dt$ between two unidentical massless spin-0 particles via the exchange of a graviton (just one diagram, similar to the previous problem). What is the dimension of the coupling κ so that the cross section has the right dimensions? What is the high energy behavior $s \approx -u \gg -t > 0$?

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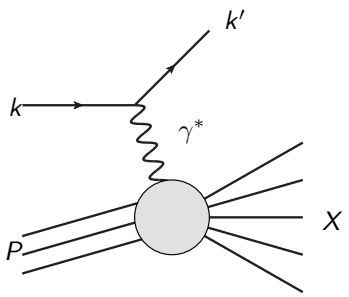
7 DIS in dipole picture

7.1 DIS introduction

Before understanding what happens when two nuclei collide at high energy, we have to know what the nuclei are made of. We know that they consist of quarks and gluons, but how many of these are there? We know that the nuclei are nonperturbative objects, so we cannot understand them from first principles at weak coupling alone. We can, however, *measure* the content of the hadron experimentally, by studying it with a simple probe: an electron. This experimental situation is known as deep inelastic scattering.

DIS kinematics



$$s = (k+P)^2 \tag{67}$$
$$q = k - k' \quad q^2 \equiv -Q^2 \tag{68}$$
$$W^2 = (P+q)^2 \tag{69}$$
$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{W^2+Q^2-m_N^2} \tag{70}$$

High energy limit

$$x \to 0 \quad W^2 \to \infty \tag{71}$$

In addition to the variables above, one also conventionally defines the following Lorentz-invariant variables

$$y = \frac{2P\cdot q}{2P\cdot k} = \frac{W^2+Q^2-m_N^2}{s-m_N^2} \tag{72}$$
$$\nu = P\cdot q/m_N = \frac{W^2+Q^2-m_N^2}{2m_N} \tag{73}$$

Before moving on let us discuss a little the physical interpretatation of the kinematical variables.

- The total e^-p center of mass energy s is of course important for an experimentalist. For the QCD part of the scattering process it is, completely irrelevant; its only practical consequence is that kinematically it sets the upper limit for the quantity we are really interested in, which is
- W^2, the c.m.s. energy of the γ^*p-system. This is the collision energy that is relevant for the QCD part of the scattering, and the high W^2 is what defines the high energy limit that we are interested in. The collision energy W^2 is practically the same as
- ν, which is the energy of the γ^*, i.e. the energy lost by the electron, in the target rest frame. This is important historically when most experiments are done at fixed targets, but W^2 is nicer for a theorist since it is Lorentz-invariant.
- The "inelasticity" y is really related to the electron-photon vertex kinematics. It determines how much of the electron energy is taken by the photon, i.e. the relation between the uninteresting variable s and the interesting one W^2, and is thus uninteresting for us here.[2]
- Q^2 should be thought of as the spatial resolution scale, it tells us what is the wavelength of the space-like virtual photon and therefore the resolution scale at which we can probe the contents of the target. Unless we have heavy quarks, Q^2 has to be large enough to justify weak coupling. As we will see later, it is useful to think of Q^2 as a transverse momentum scale, and W^2 as a longitudinal momentum, or energy scale.
- Finally there is x; which for the purposes of these lectures is mostly just a way of parametrizing W^2. However, it is conventionally thought of as a momentum fraction of the struck parton, and we will first take a short detour to see why this is the case.

[2] Of course, if you are building a detector for DIS experiments at the Electron-Ion Collider, y is the most important variable for you: it determines what direction the electron flies in, and you really have to measure very precisely this electron.

Let us look at the diagram in a bit more detail, and introduce the concept of the *leptonic tensor*, the *hadronic tensor* and the structure functions F_1 and F_2. In general the cross section is

$$\equiv \frac{e^4 g^{\mu\mu'} g^{\nu\nu'}}{q^4} L_{\mu\nu}(2\pi) W_{\mu'\nu'}$$

$$d\sigma = \frac{1}{2(s-m^2)} \overbrace{\frac{1}{2}\sum_{s_e,s_{e'}} \frac{1}{2}\sum_{s_p,s_X}\sum_X \int d\Pi_X \left|\mathcal{M}\right|^2}^{} \frac{d^3\bar{\mathbf{k}}'}{2E'(2\pi)^3} \tag{74}$$

Here we have even kept the full proton mass dependence, just for fun (m is the proton mass). The electron is massless $m_e = 0$). The spins of the incoming electron and proton s_e and s_p are averaged over, and we sum over the spin of the outgoing electron $s_{e'}$ and all the spins of all the outgoing particles in the final state X. We all summing over all possible final states X of the proton system X, and integrating over their phase spaces $d\Pi_X$. There is a separate integration over the phase space of the outgoing electron, whose momentum is $\bar{\mathbf{k}}'$ and energy $E' = |\bar{\mathbf{k}}'|$. We have included the overall momentum conservation delta function in the phase space of the system X, with the idea that it should be thought of as restricting the total four-momentum of the system X to $P^\mu + q^\mu = P^\mu + k^\mu - k'^\mu$. Now let us split out of the scattering amplitude $\mathcal{M}$ all the ingredients that we know. The amplitude will be, in Feynman gauge,

$$\mathcal{M} = e^2 \bar{u}_{s_{e'}}(k')\gamma^\mu u_{s_e}(k) \frac{-ig_{\mu\mu'}}{q^2} \langle X, P+q| J^{\mu'}(0) |P\rangle , \tag{75}$$

where all the unknown QCD physics of the target is in the matrix element of the electric current operator $J^{\mu'}$ in the target state.

Digression: Why current operator?

It might not be obvious why the electric current operator expectation value is the one that appears. One way to think about it is in the field theory language: the interaction term between photons and charged particles in the Lagrangian is $e\int d^4x A_\mu(x) J^\mu(x)$, where we have scaled out the coupling, so that the current J^μ is really a number current and not a charge current. For example for an elementary charged fermion field, the current operator is $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, with fermion field operators $\bar{\psi}(x), \psi(x)$. When doing perturbation theory, the Lagrangian is in the exponent, and at each order one power of the interaction term drops down: we therefore calculate expectation values of $e\int d^4x A_\mu(x) J^\mu(x)$ between the incoming and outgoing states. The photon field A_μ becomes a part of the photon propagator (which is $\langle A_\mu A_\nu\rangle$) and the integral over coordinates x becomes a four-momentum conserving delta function. What is left is the QCD part: the expectation value of the current-current correlator in the hadronic state.

Now we split this into a few parts. First is the *Leptonic tensor*

$$L_{\mu\nu} = \frac{1}{2}\sum_{s_e,s_{e'}} \bar{u}_{s_{e'}}(k')\gamma_\mu u_{s_e}(k) \left[\bar{u}_{s_{e'}}(k')\gamma_\nu u_{s_e}(k)\right]^\dagger = 2(k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k\cdot k') \tag{76}$$

Remember that we are not squaring the lepton tensor with the same index in the amplitude and the conjugate amplitude: instead each index (μ for the amplitude and ν for the conjugate is separately contracted with the respective indices in the hadron, via the photon propagator).

The structure of the hadron is encoded in the *Hadronic tensor*

$$W_{\mu'\nu'} \tag{77}$$

Gauge invariance

$$q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = q^{\mu'} W_{\mu'\nu'} = q^{\nu'} W_{\mu'\nu'} = 0 \tag{78}$$

Electromagnetic gauge invariance is encoded in the Ward identities, which lead to the leptonic and hadronic tensors being transverse to the photon momentum. This follows from the fact that both $L_{\mu\nu}$ and $W_{\mu\nu}$ are expectation values of electromagnetic current operators, and the current is conserved: $\partial_\mu J^\mu(x) = 0$. If we are not measuring spin (of any of the particles), the hadronic tensor can only depend on two Lorentz

four-vectors: q^μ and P^μ. Out of these two vectors, it is possible to construct only two independent symmetric ($W_{\mu\nu} = W_{\nu\mu}$) Lorentz structures. The coefficients of these Lorentz tensors are known as the structure functions F_1 and F_2.

$$W_{\mu\nu} = 2\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{2}{P\cdot q}\left[\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\right] F_2(x, Q^2) \tag{79}$$

The way one should read this is relation is that

- F_2 is the structure corresponding to the current (remember $W_{\mu\nu} \sim \langle J_\mu J_\nu \rangle$) in the direction of the proton, but removing the part that is parallel to the photon q_μ.
- F_1 is the rest, so in some appropriately Lorentz and gauge invariant sense both the part parallel to P_μ and the part orthogonal to it.

Let us now simplify our cross section (74) with the help of the structure functions. We also will write it as differentially in Lorentz-invariant variables (for this also there are several different options). We start by contracting the leptonic and hadronic tensors. Since we have expressions for all the scalar products in terms of the Lorentz-invariants, we can, after some algebra, easily see that

$$L_{\mu\nu} W^{\mu\nu} = 4Q^2 F_1(x, Q^2) + \frac{4Q^2}{xy^2}\left[1 - y - \frac{x^2 m^2 y^2}{Q^2}\right] F_2(x, Q^2) \tag{80}$$

Let us then treat the phase space integration for the final state electron in the cross section (74). These manipulations are easiest to do in the target (proton) rest frame, but the final result will be Lorentz-invariant. We first write the electron phase space in polar coordinates: $\mathrm{d}^3\overline{\mathbf{k}}' = (E')^2\,\mathrm{d}\Omega\,\mathrm{d}E'$, with $E' = |\overline{\mathbf{k}}'|$ for a massless electron. We also have $s - m^2 = 2mE$ with $E = |\overline{\mathbf{k}}|$ for the incoming electron. Our cross section is now

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\,\mathrm{d}\Omega} = \frac{\alpha_{\text{e.m.}}^2}{2mQ^4}\frac{E'}{E} L_{\mu\nu} W^{\mu\nu}. \tag{81}$$

This is the useful form if you actually want to measure this, assuming your detector is in the target rest frame. But we can do some changes in the integration variables. First integrate over the azimuthal angle of the electron (assuming the cross section does not depend on it, which it will not if we are averaging over spins and averaging over the azimuthal angle of the scattered electron, as we will discuss in a moment), which leads to $\mathrm{d}\Omega = 2\pi\,\mathrm{d}\cos\theta$. We then replace $\cos\theta$ with Q^2 (you remeber already from nonrelativistic scattering theory that the momentum transfer is closely tied in with the scattering angle). This is done with the help of

$$Q^2 = 2EE'(1 - \cos\theta) \quad\Longrightarrow\quad \mathrm{d}\sigma = \frac{2\pi\alpha_{\text{e.m.}}^2}{Q^4} L_{\mu\nu} W^{\mu\nu} \frac{\mathrm{d}Q^2\,\mathrm{d}E'}{4mE^2} \tag{82}$$

We then also want to get rid of the scattered electron energy E'. We can exchange this to the Lorentz-invariant variable x with the help of the relation (now for fixed Q^2, because Q^2 is at this point our other integration variable:

$$E' = \frac{1}{2m}\left(s - \frac{Q^2}{x} - Q^2\right) \quad\Longrightarrow\quad \mathrm{d}E' = \frac{Q^2}{2mx^2}\,\mathrm{d}x \quad\Longrightarrow\quad \mathrm{d}\sigma = \frac{2\pi\alpha_{\text{e.m.}}^2}{Q^4} L_{\mu\nu} W^{\mu\nu} \frac{Q^2\,\mathrm{d}Q^2\,\mathrm{d}x}{8m^2E^2x^2} \tag{83}$$

Using $Q^2 = xy(s - m^2) = 2mExy$ this leads to another commonly seen form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{2\pi\alpha_{\text{e.m.}}^2}{Q^4} L_{\mu\nu} W^{\mu\nu} \frac{y}{4mEx}, \tag{84}$$

where using (80)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{4\pi\alpha_{\text{e.m.}}}{xQ^4}\left[\left(1 - y - \frac{x^2m^2y^2}{Q^2}\right) F_2(x, Q^2) + xy^2 F_1(x, Q^2)\right]. \tag{85}$$

We can still go further and change the variable from Q^2 to y using $Q^2 = xy(s - m^2) = 2mExy$ and thus $\mathrm{d}Q^2 = 2xmE\,\mathrm{d}y$, remembering that now x is fixed. This leads to

$$\mathrm{d}\sigma = \frac{2\pi\alpha_{\text{e.m.}}^2}{Q^4} L_{\mu\nu} W^{\mu\nu} \frac{y\,\mathrm{d}y\,\mathrm{d}x}{2}, \tag{86}$$

and to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}y} = \frac{4\pi\alpha_{\text{e.m.}}^2(s - m^2)}{Q^4}\left[\left(1 - y - \frac{x^2m^2y^2}{Q^2}\right) F_2(x, Q^2) + xy^2 F_1(x, Q^2)\right], \tag{87}$$

where for example in [17] this is written with $s - m^2 = 2mE$.

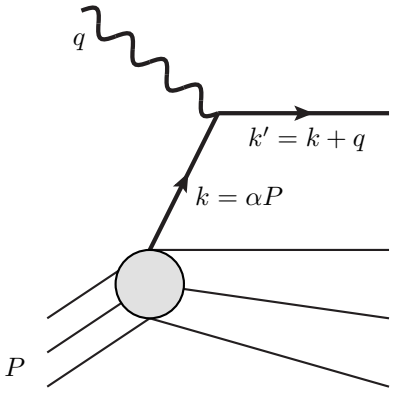
7.2 Partons

Usually the textbook description of DIS is done in the infinite momentum frame, where the hadron is moving fast. One then makes the *parton model* assumption that the hadron consistss of partons, quarks and gluons, whose momenta are parallel to that of the hadron (since the hadron momentum is very high, the transverse momenta of the partons with respect to the large longitudinal momenta are small). The physical picture of a DIS process is then that the virtual photon strikes one of these partons (at LO it has to be a quark, since photons couple to gluons only through a higher order process with intermediate quarks).

Let us now assume that

1. the 4-momentum of the quark before it hits the photon is parallel to the hadron and carries a fraction α of the momentum of the hadron P,
2. the quark after it is struck by the photon is an asymptotic final state particle and thus on shell (this is certainly always the case in inclusive cross sections, because there one is summing over all possible final states, and this can be done using any complete set of states, including of course the free particle states)
3. we can neglect the mass of the hadron (we always do in these lectures whenever we can).

This allows us to show that actually the momentum fraction in the infinite momentum frame α is *with these assumptions* is actually the same as the kinematical variable x:



$$0 = (q + \alpha P) = -Q^2 + 2\alpha P \cdot q + \overbrace{\alpha^2 m_N^2}^{\approx 0} \quad \Longrightarrow \quad \alpha = \frac{Q^2}{2P \cdot q} = x \tag{88}$$

$x =$ momentum fraction

The identification of the kinematical variable x (or Bjorken x, i.e. x_{Bj}), which is a well defined Lorentz-invariant constructed from the 4-momenta of the incoming and outgoing electron and the incoming proton, with the momentum fraction of the quark that is struck is so common, that the two concepts are often used completely interchangeably. It is good to remember, however, that x and the momentum fraction are conceptually separate things and identifying them requires a certain physical picture, that of the quarks as completely collinear to the hadron moving at infinite velocity.

One can with relative ease show that if the scattering happens off spin-$1/2$ quarks only, and these quark behave like free particles like the electrons, one has a relation between the structure functions Free spin-$\frac{1}{2}$ quarks

$$F_2 = 2xF_1 \tag{89}$$

This is easy to show by checking that if this condition is true, the decomposition (79) reduces to the same form as the leptonic tensor. If the Callan-Gross relation is satisfied, quarks are free, and there are no gluons. Consequently in QCD the violation

$$F_L = F_2 - 2xF_1 \sim \alpha_s x g(x, Q^2) \tag{90}$$

only arises at higher order in perturbation theory, and is directly sensitive to gluons. We will see why this is called the "longitudinal structure function" shortly.

7.3 Virtual photon scattering

We can use the gauge invariance relation (78) to replace in the photon propagator

$$-g_{\mu\mu'} \to -\left(g_{\mu\mu'} - \frac{q_\mu q_{\mu'}}{q^2}\right). \tag{91}$$

The propagator with this replacement is called the Landau gauge propagator. Now $q^2 < 0$, i.e. the virtual photon is spacelike and off shell. Thus we can go to a frame where it does not have an energy component: this is the Breit frame. In this frame it is also very easy to write down sensible longitudinal and transverse polarization vectors that are orthogonal to the momentum transfer.

Breit frame

$$q^\mu = (0, 0, 0, Q) \quad (= (q^0, q^1, q^2, q^3)). \tag{92}$$

$$\varepsilon_1^\mu = (0, 1, 0, 0) \quad \varepsilon_2^\mu = (0, 0, 1, 0) \quad \varepsilon_L^\mu = (1, 0, 0, 0) \tag{93}$$

The first two are (linear) transverse polarizations, and the third one a longitudinal one. The usual transverse polarization states with a specific angular momentum are linear combinations of our polarization states $\varepsilon_1^\mu, \varepsilon_2^\mu$. They satisfy

$$q \cdot \varepsilon_\lambda(q) = 0. \tag{94}$$

With the help of these polarization vectors we can write the polarization index structure of the photon propagator as:

$$-g_{\mu\mu'} \to -\left(g_{\mu\mu'} - \frac{q_\mu q_{\mu'}}{q^2}\right) = -\varepsilon_L^{*\mu}\varepsilon_L^{\mu'} + \varepsilon_1^{*\mu}\varepsilon_1^{\mu'} + \varepsilon_2^{*\mu}\varepsilon_2^{\mu'} = \sum_{\lambda=0}^{2}(\eta_\lambda)\varepsilon_\lambda^{*\mu}\varepsilon_\lambda^{\mu'}, \tag{95}$$

with $\eta_{L=0} = -1, \eta_1 = \eta_2 = 1$.

Note the sign of the longitudinal polarization vector squared in this expression. Because the virtual photon is spacelike, it is impossible to have a Lorentz-covariant form for the the transverse projection operator Eq. (91) in terms of a sum of squares (which are positive) of polarization vectors with equal signs. In the frame (92) the projection operator (91) is $diag(-1, 1, 1, 0)$; this is not a positive definite tensor that could be a sum of squared polarization vectors with positive coefficients. Nevertheless, it is useful to think of the propagator as being a sum of polarization states. Now we can write our DIS cross section (in the original Lorentz-covariant form) as

$$\mathrm{d}\sigma = \frac{1}{2(s-m^2)} \sum_{\lambda,\lambda'=0}^{2} (\eta_\lambda \eta_{\lambda'}) \varepsilon_\lambda^{*\mu} \varepsilon_\lambda^{\mu'} \varepsilon_{\lambda'}{}^{\nu} \varepsilon_{\lambda'}^{*\nu'} L_{\mu\nu}(2\pi) W_{\mu'\nu'} \frac{\mathrm{d}^3\overline{\mathbf{k}}'}{2E'(2\pi)^3}. \tag{96}$$

The electron momentum only appears in the leptonic tensor $L_{\mu\nu}$ Let us now integrate over the angle of the electron, for fixed values of the Lorentz-invariants s, x, y, Q^2. It is convenient to do this in the Breit frame, where we can choose the momenta as:

$$k = (E_B, \kappa\cos\varphi, \kappa\sin\varphi, Q/2) \tag{97}$$

$$k' = (E_B, \kappa\cos\varphi, \kappa\sin\varphi, -Q/2) \tag{98}$$

$$q = (0, 0, 0, Q) \tag{99}$$

$$P = (\sqrt{m^2 + Q^2/(4x^2)}, 0, 0, -Q/(2x)) \tag{100}$$

The value of the incoming electron energy E_B can be fixed from $s = (k+P)^2 = Q^2/(xy) + m^2$ and the value of κ from $k^2 = 0$. In this frame it is easy to take the dot products of the photon polarization vectors with the leptonic tensor, integrate them over the outgoing electron angle φ and see that

$$\varepsilon_\lambda^{*\mu} \varepsilon_{\lambda'}{}^{\nu} L_{\mu\nu} \propto \delta_{\lambda,\lambda'}. \tag{101}$$

Although we did this calculation in a specific frame, our expression for the cross section is Lorentz-invariant, and thus this result is completely general. This leads to an important result that very commonly used:

$\int \mathrm{d}\varphi_{e'}\ (Q^2, x = \text{cst.}) \implies \lambda' = \lambda$

In other words, if we integrate over the angle of the electron (and don't measure its spin), the photon polarizations do not interfere with each other. If we do not measure or track (i.e. we integrate over) the scattered electron angle, the polarization of the virtual photon in the amplitude and the conjugate amplitude is the same. This means that we can write the electron-proton cross section as an ("incoherent") sum of cross sections that are mediated by different polarization states, with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y\,\mathrm{d}x} = \frac{2\pi\alpha_{\text{e.m.}}^2}{Q^4}\frac{y}{2}\sum_{\lambda=0}^{2}[\varepsilon_\lambda^{*\mu}\varepsilon_\lambda{}^{\nu}L_{\mu\nu}]\overbrace{\left[\varepsilon_\lambda^{\mu'}\varepsilon_\lambda^{*\nu'}(2\pi)W_{\mu'\nu'}\right]}^{\sim\sigma^{\gamma_\lambda^*+p}} \tag{102}$$

Note that now there is only one sum over polarizations, at the cross section level, not any more in the amplitude and conjugate separately. With a little bit more time we would evaluate these integrals and calculate $[\varepsilon_\lambda^{*\mu}\varepsilon_\lambda{}^{\nu}L_{\mu\nu}]$.

We can think of this as a photon flux from the electron, multiplied by a virtual photon-target cross section. Which parts of the expression one actually factorizes into the cross section and which part into the photon flux. My convention, as in [2], is to define

$$\sigma^{\gamma^*+p\to X} = \frac{2\pi^2\alpha_{\text{e.m.}}}{4mK}\left[\varepsilon_\lambda^{\mu'}\varepsilon_\lambda^{*\nu'}W_{\mu'\nu'}\right], \tag{103}$$

with

- $K = \sqrt{\nu^2 + Q^2}$ (Gilman convention), which is the photon momentum in the target rest frame
- $K = \nu - Q^2/(2m) = \nu(1-x)$ (Hand convention), which is constructed such that the $\gamma^* + p$ center of mass energy $W^2 = 2m\nu - Q^2 + m^2$ satisfies the relation that it would for a scattering of a proton at rest with a real photon with energy K, namely $W^2 = m^2 + 2mK$.

All of these conventions become the same in the high energy, i.e. $x \to 0$ limit.

With a little bit more time, we would derive the virtual photon cross sections more carefully, some of this is done as an exercise. Here, I will just quote the result (flux convention $K = \nu$ and proton mass neglected $m = 0$ here):

We define

$$\sigma_L^{\gamma^*+p\to X} = \frac{4\pi^2\alpha_{\text{e.m.}}}{Q^2}(F_2 - 2xF_1) \tag{104}$$

$$\sigma_T^{\gamma^*+p\to X} = \frac{4\pi^2\alpha_{\text{e.m.}}}{Q^2}2xF_1 \tag{105}$$

$$\sigma_{\text{tot}}^{\gamma^*+p\to X} = \sigma_L^{\gamma^*+p\to X} + \sigma_T^{\gamma^*+p\to X} = \frac{4\pi^2\alpha_{\text{e.m.}}}{Q^2}F_2 \tag{106}$$

This finally explains why the combination $F_2 - 2xF_1$ is called the longitudinal structure function F_L. Correspondingly $2xF_1$ can be called F_T. Note that the transverse cross section is defined as an average *per polarization*, not the sum of two polarizations. Thinking about this as a $\gamma^* p$ scattering such a definition does not really make sense. However, this is the natural convention for the ep scattering. A single transverse polarization state of the photon contributes to the total DIS cross section with a factor $1/2$ compared to a longitudinal one, because the integral over the scattered electeon angle gives a factor $1/2$: although there are 2 transverse polarizations, the "total cross section" (appropriate for ep) is approximately the sum of one "average" transverse and one longitudinal photon polarization.

In Eq. (87) we can see that the total *electron-proton* cross section is proportional to just F_2 in the limit $y \to 0$. This is the limit of the electron emitting a *soft* photon (we will discuss soft gluons in Sec.9), which carries only a small fraction of its momentum. In this limit, the photon does not really know about the spin of the electron, and the DIS cross section truly factorizes into a photon emission and a photon-proton cross section. For large y the spin of the electron matters, and the total ep cross section is not just proportional to the $\gamma^* p$ cross section, but the dependence is a bit more complicated. However, to a first approximation it is useful to think that the total DIS cross section measures F_2.

Experimentally this leads to the following situation

- $\sigma(e+p) \implies \sim F_2(x,Q^2)$
- Vary y (i.e. s) for fixed $x, Q^2 \implies F_L$

The easy thing to measure, and therefore much more precisely known, is the "reduced cross section" which is, to a first approximation ($y \ll 1$) essentially F_2. For this it is enough to measure the ep cross section with one collision energy s. To disentangle F_2 and F_L (i.e. F_2 and F_1 separately), one needs to vary the beam energy s in order measure cross sections at the same x, Q^2 and different y, and then extract F_L from the (not particularly strong) y dependence. This is known as the *Rosenbluth separation*, and is more difficult to perform: both because it requires more statistics, and because for different y at the same x, Q^2 the electron goes into a different parts of the detector with resulting different systematical uncertainties.

Exercise: Photon polarizations do not interfere

Take the kinematics of DIS as the Breit frame:

$$k = (E_B, \kappa\cos\varphi, \kappa\sin\varphi, Q/2) \tag{107}$$
$$k' = (E_B, \kappa\cos\varphi, \kappa\sin\varphi, -Q/2) \tag{108}$$

contract two different polarization vectors out of

$$\varepsilon_1^\mu = (0,1,0,0) \quad \varepsilon_2^\mu = (0,0,1,0) \quad \varepsilon_L^\mu = (1,0,0,0) \tag{109}$$

with the leptonic tensor $L_{\mu\nu}$ and check whether integrating over the angle φ gives zero. (Experimental exercise, lecturer has not worked this out in detail...)

7.4 Dipole picture

γ^* is a hadron

Now let us move to a different Lorentz frame, i.e. the target (hadron/nucleus) rest frame. Cross sections are Lorentz-invariant, but the physical picture of the scattering is not. In fact, in this frame, and at small x, we think of the γ^* as a kind of a hadron, which itself consists of quarks and gluons. DIS (and even $\gamma\gamma$-scattering) are now hadronic processes. In particular one would expect the high energy behavior of the cross sections to be similar to hadronic scattering, just scaled by a factor of $1/137$ (or $(1/137)^2$) (see Fig. 1) because at some point γp (and respectively $\gamma\gamma$) involve one (or two) additional electromagnetic vertices. In fact I want to convince you that the γ^* with a largish Q^2 is the theorist's favorite hadron it is a QCD bound state; but unlike the proton, its partonic structure can be fully understood in perturbation theory!

Let us now look at the kinematics of DIS in the target rest frame

Target Rest Frame TRF

$x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$

$$P^\mu = (\overset{0}{m}, \overset{\perp}{\mathbf{0}}, \overset{z}{0}) \implies (\overset{+}{m/\sqrt{2}}, \overset{-}{m/\sqrt{2}}, \overset{\perp}{\mathbf{0}}) \tag{110}$$
$$q^\mu \approx (\overset{0}{W^2/(2m)}, \overset{\perp}{\mathbf{0}}, \overset{z}{\sqrt{(W^2/2m)^2+Q^2}}) \implies (\overset{+}{q^+}, \overset{-}{-Q^2/(2q^+)}, \overset{\perp}{\mathbf{0}}) \tag{111}$$

High energy: $q^+ \approx W^2/(\sqrt{2}m) \to \infty$

γ^* wave $e^{-i(q^+x^- + q^-x^+)}$

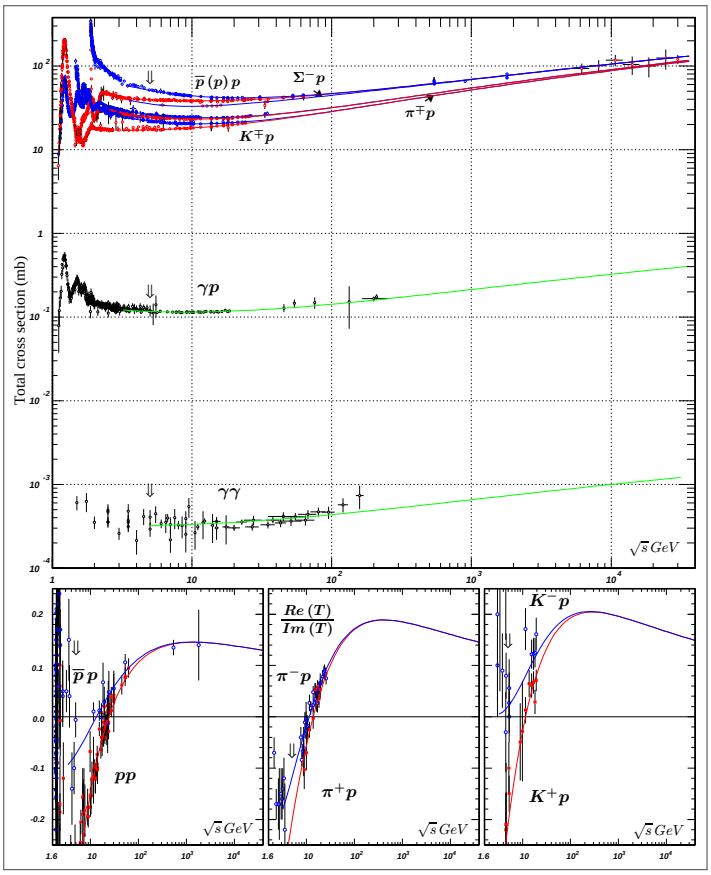
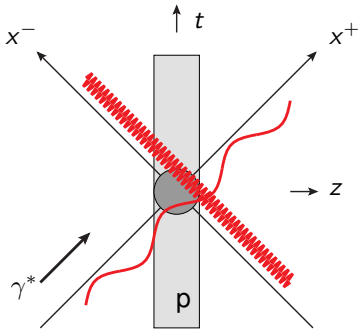


Figure 1: pp, γp, and $\gamma\gamma$ total cross sections vs $\sqrt{s}$.

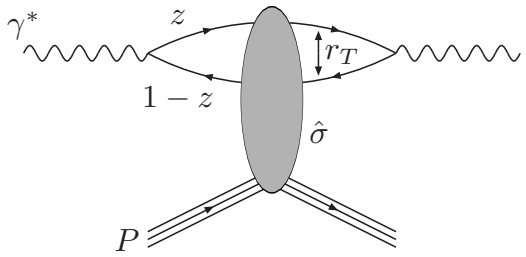


- Accurate x^-
- Target "shockwave" in x^+

In particular, since the light cone time x^+ evolution of the high energy γ^* happens with the very light cone energy q^-, the γ^* cannot change into a hadronic final state *inside* the proton. Instead, the scattering is dominated by contributions where the γ^* fluctuates into hadrons before the collision. This hadronic component of the γ^* state then scatters off the target. This physical picture of high energy scattering is not specific to DIS, but common to all high energy scattering in gauge theory; this was first understood by Bjorken, Kogut and Soper [16] who look at various QED processes.

In the case of DIS, recall that we are now assuming that the target is described by a gluon field. The photon does not couple to this field directly. Instead, it has to fluctuate into a hadronic state before the interaction. The simplest (i.e. lowest order in perturbation theory) such hadronic state is a quark-antiquark dipole. This leads to the so called

Dipole picture of DIS



High energy: factorize

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}\,\mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(\mathbf{r},z)_{T,L}\right|^2 2\int \mathrm{d}^2\mathbf{b}\mathcal{N} \tag{112}$$

- "light cone wave function" $\psi^{\gamma^*\to q\bar{q}}(\mathbf{r},z)_{T,L}$
- "dipole amplitude" $\mathcal{N}$
- **r**= dipole size, **b**= impact parameter

The justification for this factorized formula is given by the high energy kinematics: since the scattering is instantaneous (in x^+), the fluctuation of the photon into a quark-antiquark pair must happen before the target (in the target rest frame).

Note that we are here using the optical theorem: the $\gamma^* p$-cross section is calculated as twice the imaginary part of the elastic forward $\gamma^* p$-scattering amplitude. Correspondingly the dipole-target cross section is twice the dipole-target elastic scattering amplitude. In this context often (but not consistently), "dipole amplitude" refers to the dimensionless amplitude $\mathcal{N}$, and "dipole cross section" to twice the amplitude integrated over the impact parameter:

$$\sigma_{q\bar{q}} = 2\int \mathrm{d}^2\mathbf{b}\mathcal{N} \tag{113}$$

Recall: Note on "S-matrix" and optical theorem

In QFT in general, the scattering amplitude is defined in terms of the S-matrix, from which one separates the trivial identity and an imaginary unit to define the scattering matrix or T-matrix as

$$S_{fi} = \langle f|\,\hat{S}\,|i\rangle = 1 + iT_{fi} \tag{114}$$

The usual way to state the optical theorem is

$$\sigma_{tot} = 2\mathrm{Im}T_{ii} \tag{115}$$

Now we are assuming (as is the case in lowest order in QCD perturbation theory, and also to a good approximation experimentally) that at high energy the scattering matrix is purely imaginary. Thus our (forward) scattering amplitude $\mathcal{N}$ is related to the T-matrix as

$$\mathcal{N} \equiv \mathrm{Im}T_{ii} \tag{116}$$

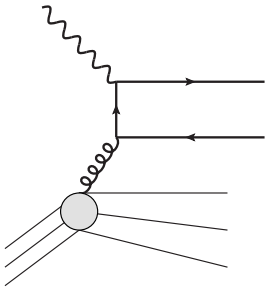
and the diagonal element of the S-matrix is given by

$$S_{ii} = \delta_{ii} - \mathcal{N} + i\mathit{mag}, \tag{117}$$

where we are all the time neglecting the imaginary part of the diagonal elements of the S-matrix, i.e. the real part of the diagonal element of the T-matrix.

In the infinite momentum frame, in the case scattering off a dilute gluon field that can be pictured as one gluon exchange, the same process would look like this:

Same process in IMF:



In the usual parton model/IMF thinking this process is formally higher order in α_s. It can however be argued to dominate at small x since the gluon distribution $xg(x,Q^2)$ is so large compared to the quark distribution. However, this kind of a picture cannot describe valence quarks, or large x physics more generally.

Without any more detailed calculation right nowwe will now just state that the information of the target color field in the CGC picture is built from two Wilson lines, one for the quark and another one, hermitian conjugate, for the gluon. In the eikonal approximation the quark and antiquark stay at the same coordinate when passing through the target color field. Thus the dipole cross section is needed in coordinate space, it depends ont he transverse coordinate separation of the quark and the antiquark. In general the Wilson lines in (37) depend also on the longitudinal coordinate x^- (they do not depend on x^+ since this is a quantity that is integrated over). However, here we again refer to the timescale argument above: since the probe has a high q^+, it has a very high resolution in x^-. In the strict high energy limit this means that we are probing the degrees of freedom in the target at a fixed, instantaneous x^-, which we can take to be 0 here. Thus in practice we do not worry about an x^--dependence in the Wilson line, and instead consider it as a function of only the transverse coordinate.

Dipole amplitude in the CGC

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c}\,\mathrm{tr}\,V(\mathbf{x})V^\dagger(\mathbf{y}) \tag{118}$$

$V(\mathbf{x})$ = familiar Wilson line (37)

Next, we will try to understand how to calculate the "virtual photon light cone wavefunction."

8 The photon light cone wave function

8.1 Light Cone Perturbation Theory

The classic reference is the Brodsky, Pauli, Pinsky review [13]. For a slightly shorter review of the basics see [18]. The introduction of the Kovchegov&Levin book [1] provides a concise introduction to what we really need, unfortunately with a different metric. We use here the covariant normalization (unlike Brodsky et al. [13]), to understand the normalization it is better to look at [19] (for loop calculations see however the remark in Sec. II D of Ref. [20]) or [21]. The terms "instant form" (for ordinary equal-time quantization), "front form" (for light cone, i.e. light front, quantization) and "point form" (quantization with a proper time as time coordinate) date back to P. Dirac [22].

In the Heisenberg picture of time evolution in Quantum Mechanics, the operators are time dependent, with a time dependence generated by the Hamiltonian operator. Their quantum nature is expressed by the commutation relations of operators

Usual QM

$$\left[\hat{A}(t),\hat{B}(t')\right]\Big|_{t=t'} = \ldots \tag{119}$$

The essential point is that we know these commutation relations (like $[\hat{x}(t),\hat{p}(t)] = i\hbar$) at *equal time*. Being able to calculate the commutation relations at unequal time $t \neq t'$ is something that can only be done once one has solved the time dependence of the theory, not something that is known from the start.

Since quantum field theory is Lorentz-covariant, one can boost to any coordinate system moving at a velocity $v < 1$. The commutation relations, equations of motion etc. will look the same in this new frame

with a different time coordinate. However, one cannot boost to exactly light velocity $v = 1$. For high energy scattering problem the physical situation is such that the natural "time" coordinate would be a lightlike variable corresponding to the trajectory of a massless particle. But formulating quantum field theory for such a variable can not be obtained by a continuous transformation from the normal "instant form" quantum mechanics. In stead, one has to start from the begining and separately *define* a quantization where the "time" coordinate is a light-like variable, conventionally taken to be x^+:

LC quantization QM

$$\left[\hat{A}(x^+), \hat{B}(y^+)\right]\Big|_{x^+=y^+} = \dots \tag{120}$$

LC Hamiltonian $\hat{P}^-$

The Hamiltonian of the theory is, by definition, the operator that generates translations in time. Since x^+ is now "time," the conjugate operator $\hat{P}^-$ is the light cone Hamiltonian.

Digression: Poincaré generators

The Poincaré group is 10 dimensional, generated by the 4 generators of translations $\hat{P}^\mu$ and the 6 rotations and Lorentz boosts (the 6 elements of the antisymmetric matrix $\hat{J}^{\mu\nu}$. These generators are divided into kinematical and dynamical ones. Kinematical generators are "easy," they do not change the time coordinate (leave the equal-time hypersurface intact); thus in Lorentz-invariant theory they are symmetries that are conserved in time. The generators that do mix different equal-time hypersurfaces are called "dynamical;" they are related to how the system changes in time and are much more complicated to understand. In instant form quantization there are 6 kinematical generators: 3 spatial translations $\hat{P}^i$ and 3 spatial rotations $\hat{J}^{ij}$ with $i, j = 1, 2, 3$. States of a Lorentz-invariant quantum mechanical system (like bound states that can otherwise have complicated interactions) are all eigenstates of these generators (momemtum and angular momentum eigenstates). The 4 other generators are dynamical (the Hamiltonian $\hat{P}^0$ and the 3 boosts $\hat{J}^{0i}$), and for example the behavior of a bound state under a boost is very complicated. In light cone quantization, on the other hand, there are 7 kinematical generators ($\hat{P}^+$, $\hat{P}^j$, $\hat{J}^{ij}$, $\hat{J}^{i+}$ and $\hat{J}^{+-}$ with $i, j = 1, 2$. Strictly speaking the longitudinal boost $\hat{J}^{+-}$ acts on the LC time as $x^+ \to e^\omega x^+$, so it does not leave constant x^+ hyperplanes invariant. Nevertheless it does leave the $x^+ = 0$ hyperplane invariant so in practice it behaves as a kinematical generator. This leaves only 3 dynamical generators: the Hamiltonian $\hat{P}^-$ and $\hat{J}^{i-}$. Just counting generators LCPT is "more symmetric" than instant form quantization. The downside is that these symmetries are somewhat nontrivial combinations of rotations and boosts.

Consider for example a simple bound state of positronium e^+e^- in QED. In instant form quantization it is described by a total 3-momentum of the bound state and the spins of the particles. The Hamiltonian is nontrivial, and boosting in any of the 3 dimensions mixes up the spins, angular momenta and different Fock states in complicated ways, e.g. what in the rest frame just looks like an interacting e^+e^- state will in a boosted frame also have a $e^+e^-\gamma$ component. The same positronium in light front quantization will be charactarized by a total 3-momentum $\hat{P}^+$, $\hat{P}^j$, and the angular momentum in the longitudinal direction $\hat{J}^{ij}$. It will be invariant under "Galileian" boosts in the 2 transverse directions $\hat{J}^{i+}$ and transform trivially under longitudinal boosts $\hat{J}^{+-}$. The two rotations $\hat{J}^{i-}$ that mix transverse and longitudinal directions will, however, be very complicated just like Lorentz-boosts are for instant form quantization.

Let us discuss (without all the details) what actually happens when one starts to quantize field theory. One of main advantage of LCPT is that it is possible to canonically quantize even nonabelian gauge theory. Thus we will here stick with canonical quantization, starting with a scalar field. We write the field operator for the noninteracting theory as a linear superposition of creation and annihilation operators. There are two equivalent ways to approach canonical quantization. We could start from the Lagrangian, derive the canonical conjugates to the fields (generalized coordinates) of the theory, and then impose Heisenberg commutation operators between the field operators and the conjucate operators. Or we could just write down a momentum mode decomposition of the fields, and impose the commutation/anticommutation operators on the coefficients, which become creation and annihilation operators. These are equivalent and we use them interchangeably. I am here assuming that the reader is familiar with classical Lagrangian/Hamiltonian mechanics and how their concepts are used to quantize the theory.

Scalar field

$$\hat{\phi}(x) = \int \overbrace{\frac{d^2\mathbf{k}\, dk^+}{2p^+(2\pi)^3}}^{\equiv \widetilde{dk}} \left[e^{ik\cdot x} a^\dagger(\bar{k}) + e^{-ik\cdot x} a(\bar{k})\right] \tag{121}$$

$$\bar{k} \equiv (k^+, \mathbf{k}) \quad k^- \equiv \frac{\mathbf{k}^2 + m^2}{2k^+} \tag{122}$$

$$[a(\bar{p}), a^\dagger(\bar{q})] = 2p^+(2\pi)^2\delta^{(3)}(\bar{p}-\bar{q}) \quad \Longleftrightarrow \quad [\hat{\phi}(x), \overbrace{\partial_-\hat{\phi}(y)}^{=\Pi(x)}]\Big|_{x^+=y^+} = i\delta^{(3)}(\bar{x}-\bar{y}) \quad \bar{x} \equiv (x^-, \mathbf{x}) \tag{123}$$

This is just the normal commutation relation

$$[\hat{\phi}(x^+, \bar{x}), \hat{\Pi}(x^+, \bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y}) \tag{124}$$

between a field and its conjugate variable, which is obtained by taking a derivative of the action (which is the integral of the Lagrangian density) with respect to the time derivative of the field:

$$\Pi(x) \equiv \frac{\delta \mathcal{S}}{\delta(\partial_+\phi(x))} = \frac{\delta}{\delta(\partial_+\phi(x))}\int \mathrm{d}^4y\,\{\partial_+\phi\partial_-\phi - \partial_i\phi\partial_i\phi - V(\phi)\} = \partial_-\phi(x). \tag{125}$$

The difference with respect to equal time quantization is that now the conjugate variable to the field is not the time derivative of the field, but a spatial derivative (x^+ is the time coordinate, x^- is a spatial coordinate). Now let us see what happens to fermions. The field operator in terms of creation and annihilation operators is the usual

$$\sum_s \int \widetilde{\mathrm{d}k}\big[\overbrace{d^\dagger(\bar{k})}^{\text{creates antifermion}} v_s(k)e^{ik\cdot x} + \overbrace{b(\bar{k})}^{\text{annihilates fermion}} u_s(k)e^{-ik\cdot x}\big] \tag{126}$$

$$\{b(\bar{p}), b^\dagger(\bar{q})\} = \{d(\bar{p}), d^\dagger(\bar{q})\} = 2p^+(2\pi)^2\delta^{(3)}(\bar{p}-\bar{q}) \tag{127}$$

These are the normal anticommutation relations for the particle and antiparticle creation and annihilation operators. So far this looks just like in usual equal time quantization. Now let us ask the same question as for the scalar field: what is the conjugate momentum for the fermion field ψ? In instant form quantization it is

$$\Pi(x) = \frac{\delta}{\delta(\partial_0\psi)}\int \mathrm{d}^4x(i\overline{\psi}\not{\partial}\psi + \dots) = \frac{\delta}{\delta(\partial_0\psi)}\int \mathrm{d}^4x(i\psi^\dagger\partial_0\psi + \dots) = i\psi^\dagger(x). \tag{128}$$

Here we use $\overline{\psi} \equiv \psi^\dagger\gamma^0$ and $(\gamma^0)^2 = 1$. So there is a perfectly nice conjugate variable for all spinor components of the field. Also all spinor components of the field have a nice equation of motion:

$$\partial_0\Pi(x) = i\partial_0\psi^\dagger(x) = \frac{\delta}{\delta\psi(x)}\int \mathrm{d}^4y(i\overline{\psi}(y)\not{\partial}\psi(y) + \dots) = \dots, \tag{129}$$

which is just (the Hermitian conjugate of) the Dirac equation.

For light cone quantization, on the other hand, the conjugate to the fermion field is

Fermion

$$\Pi(x) = \frac{\delta}{\delta(\partial_+\psi(x))}\int \mathrm{d}^4y(i\overline{\psi}\not{\partial}\psi + \dots) = \frac{\delta}{\delta(\partial_+\psi(x))}\int \mathrm{d}^4y(i\psi^\dagger\gamma^0\gamma^+\partial_+\psi + \dots)$$
$$= i\psi^\dagger(x)\gamma^0\gamma^+ = i(\gamma^0\gamma^+\psi(x))^\dagger. \tag{130}$$

$$\gamma^+\gamma^+ = \frac{1}{2}\{\gamma^+, \gamma^+\} = 0 \tag{131}$$

$$1 = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(1+\gamma^0\gamma^3\right) + \frac{1}{\sqrt{2}}\left(1-\gamma^0\gamma^3\right)\right] = \overbrace{\frac{\gamma^0}{\sqrt{2}}\gamma^+}^{=\mathbb{P}^+} + \overbrace{\frac{\gamma^0}{\sqrt{2}}\gamma^-}^{=\mathbb{P}^-} \tag{132}$$

- Good component $\mathbb{P}^+\psi$, has $\partial_+\mathbb{P}^+\psi$
- Bad component $\mathbb{P}^-\psi \implies$ constraint $\mathbb{P}^-\psi = \frac{1}{\partial_-}(...)\,\mathbb{P}^+\psi$

$$\not{\partial} = i\gamma^+\partial_+ + \dots \tag{133}$$

One can easily check that $\mathbb{P}^\pm$ are *projection operators*, i.e. they satisfy $\mathbb{P}^+\mathbb{P}^+ = \mathbb{P}^+$, $\mathbb{P}^-\mathbb{P}^- = \mathbb{P}^-$ and $\mathbb{P}^+\mathbb{P}^- = 0$. This means that they divide the 4-dimensional complex space of Dirac spinors into two subspaces. Only one of these subspaces is an actual dynamical variable that has an equation of motion: this can in fact be seen directly from the Dirac equation, where only $\gamma^+\psi$ has a time derivative.

Now the problem is that while our original Dirac spinor had 4 components, there are only 2 conjugate variables. Another way of saying the same thing is to notice that in the Dirac equation the derivative operator $\not{\partial}$ only has a time derivative in the combination $\gamma^+\partial_+$, which does not act on the component $\mathbb{P}^-\psi$. The two components of a Dirac spinor are called:

- The good component $\mathbb{P}^+\psi = \psi^+$. The good component is a proper time-dependent variable. Classically it has an equation of motion, and quantum mechanically it is described by a field operator that has a Heisenberg equation of motion.
- The bad component $\mathbb{P}^-\psi = \psi^-$. It does not have an equation of motion that would tell us how it evolves in time, thus it is not a dynamical variable. Instead, its Dirac equation is a constraint equation: at every value of time we can always solve this constraint equation to obtain ψ^- as a function of the actual dynamical field ψ^+. Because the Dirac operator is $\not{\partial} = \gamma^-\partial_- + \dots$, the constraint equation is a differential equation $\partial_-\psi^- = \dots$ and must be solved by inverting the (spatial) derivative ∂_-. Much ink has been spilled over the boundary conditions necessary for performing this inversion, but we will not dwell on this further here [23]. In the diagrammatical rules of LCPT, this results in rules that have inverse powers of the momentum k^+.

To obtain the Hamiltonian of the theory, which we will write down in a moment, one must eliminate the dependent, non-independent, degree of freedon ψ^-. Thus, it will only involve good components of the field ψ, including in terms with inverse powers of the spatial derivative ∂_-. These terms are known as *instantaneous interactions*. A peculiarity is that in loop calculations, the field strength renormalization is different for the two components [24].

A very similar thing happens with gauge bosons, although in contrast to fermions there are not only complications but also advantages from light cone quantization.

Gauge boson

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{134}$$

$$E^\mu = \frac{\delta}{\delta(\partial_+ A^\mu)}\mathcal{L} \implies \text{no } \partial_+ A^- \tag{135}$$

So for gauge fields we face an analoguous situation as for the quarks, and one component of the gauge field does not have a conjugate variable, or an equation of motion that would give its time dependence. It should therefore be solved in terms of the other components by a constraint equation. To get this equation, let us look at the $+$ component of the equation of motion for the gauge field

$$[D_-, F^{-+}] + [D_i, F^{i+}] = J^+. \tag{136}$$

$$\overbrace{A^+ = 0}^{\text{choice}} \implies \overbrace{(\partial_-)^2 A^- = [D_i, F^{i+}] - J^+}^{\text{constraint}}, \tag{137}$$

Here J^+ is the color current from the fermions and possible other colored particles in the theory. The important point is that in general it is difficult to solve the constrained field component A^- from Eq. (136) because of the nonlinear terms like $[A^+, [A^-, A^+]]$. However, in the specific gauge $A^+ = 0$ there are no such

terms, and A^- can be solved. Similarly as for the fermions, we now solve A^- from Eq. (137), and substitute it back into the Lagrangian/Hamiltonian, which will then only depend on the transverse components of the gauge field $A_i,\ i = 1, 2$.

- If we look at $-$-component of equation of motion, it has more complicated nonlinear terms in A^-, e.g. $[A^-, \partial_- A^-] \sim J^-$. However, the J^+ current is the one corresponding to a conserved charge (it is the time component of a charge current 4-vector), and it is the one that is expressed in terms of the good components of the spinors: $J_a^+ = \overline{\psi}\gamma^+ t^a \psi$.
- The textbook quantization of QED in Coulomb gauge is done in a similar way. One chooses the gauge $\nabla \cdot \mathbf{A} = 0$, and then expresses the non-dynamical degree of freedom A^0 as a constraint equation. Here one problem is that for nonabelian theory it is in fact in general not possible to fix the gauge $\nabla \cdot \mathbf{A} = 0$. Also in nonabelian gauge theory the constraint, which should come from the time- i.e. 0-component of the equation of motion $[D_\mu, F^{\mu 0}] = J^0$ has terms like $[A^i, [A^i, A^0]] \sim J^0$ which make it much more difficult to try to solve the constraint A_0 in terms of the dynamical fields of the theory. Thus canonical quantization is usually only done for QED, and QCD is quantized by path integrals.

In summary, we have a light cone gauge light cone quantized theory, where there are only two physical polarization states, expressed (or at least expressable) in terms of two transverse component of the gauge potential. We can write for them a decomposition in terms of creation and annihilation operators

$$A^i(x) = \sum_{\lambda=\pm} \int \widetilde{\mathrm{d}k} \left[(\varepsilon^i_{(\lambda)})^* e^{ik\cdot x} a^\dagger_{(\lambda)}(\bar{k}) + \varepsilon^i_{(\lambda)} e^{-ik\cdot x} a_{(\lambda)}(\bar{k}) \right]. \tag{138}$$

If one uses circular polarization states (which is convenient), the polarization vectors $\varepsilon^i_{(\lambda)}$ can be taken as given in Eq. (151).

We can now also write down the Hamiltonian of the theory. Let us do this for QED, for simplicity. It is

$$\hat{P}^- = \hat{P}^-_0 + \hat{P}^-_{\mathrm{int}} \tag{139}$$

$$\hat{P}^-_0 = \int \mathrm{d}^3\bar{x} \left[\overline{\psi} \frac{m^2 + (i\boldsymbol{\nabla}_\perp)^2}{2i\partial_-} \gamma^+ \psi + \frac{1}{2} A^i \boldsymbol{\nabla}^2_\perp A^i \right] \tag{140}$$

$$\hat{P}^-_{\mathrm{int}} = \int \mathrm{d}^3\bar{x} \left[e\overline{\psi}\tilde{A}\psi + e^2 \overline{\psi}\gamma^+\psi \frac{1}{(i\partial_-)^2} \overline{\psi}\gamma^+\psi + e^2 \overline{\psi}\tilde{A}\frac{\gamma^+}{i\partial_-}\tilde{A}\psi \right] \tag{141}$$

$$\tilde{A}^\mu = (0, \tilde{A}^- = -\frac{1}{\partial_-}\partial_i A^i, \mathbf{A}) \tag{142}$$

Here we have introduced the notation $\tilde{A}^\mu$ for the vector potential in the $A^+ = 0$ gauge and with a A^- given by the the gauge field part of the constraint equation (137). The current part of the constraint equation (137) ($A^- \sim J^+/(\partial_-)^2$) is written separately as the second term in the interaction (141): it generates a four-fremion interaction (mediated by an "instantaneous photon" exchange, a Coulomb-like photon with only the A^- polarization). In the third term we write $\tilde{A}\gamma^+\tilde{A}$, but because $A^+ = 0$ and $\gamma^+\gamma^+ = 0$, in fact only the transverse components of the vector potential appear. This term is a fermion-antifermion-2-photon interaction, mediated by an "instantaneous fermion". It comes from the term $\bar{\psi}\gamma^-\partial_-\psi \sim \bar{\psi}^-\gamma^-\partial_-\psi^-$ part of the fermion Lagrangian, with both ψ^- and $\bar{\psi}^-$ expressed in terms of the good component ψ^+ via the Dirac equation, which also involves the gauge field.

In the Hamiltonian (139) the spinor and gauge fields should be thought of as linear superpositions of creation and annihilation operators according to Eqs. (126) and (138). Now in principle we are ready to start calculating light cone wave functions.

8.2 Virtual photon light cone wave function

Let us now go through the calculation of the wave function $\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}, z)$ needed for the DIS cross section in the dipole picture, Eq. (166). The most important thing to remember is that this is a purely QED process, and therefore very well understood.

The concept of a light cone wave function makes sense in the framework of Light Cone Perturbation Theory (LCPT). In LCPT one starts fron the standard Lagrangian of the theory, and quantizes it by postulating

canonical commutation relations at equal light cone time x^+ among interaction picture (i.e. time-dependent) field operators. One then works in "old fashioned" quantum mechanical perturbation theory (in stead of the "modern" covariant theory of Feyman et al). This is a framework that is used to understand the partonic structure of hadronic bound (or more generally composite) states as they are measured in high energy scattering processes. In the case of the dipole picture of DIS the "bound state" that we are quantizing is the virtual photon. We will not go into any details here, for a review see [13, 1] and for the notations and technical details of the calculation done here (and its extension to NLO) see [20].

The basic idea of the LCPT calculations here is that in perturbation theory we know the free particle Fock states, for a photon $|\gamma^*\rangle_0$, $|q\bar{q}\rangle_0$, $|q\bar{q}g\rangle_0$ etc. Physical particles are eigenstates of the interacting theory; they can always be written as linear superpositions of the free particle states

$$|\gamma^*\rangle = (1+\dots)|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^*\to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \dots \tag{143}$$

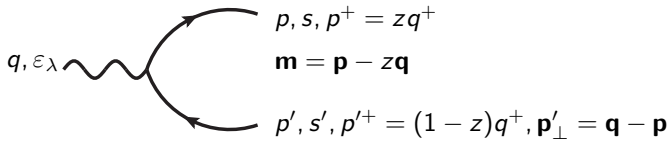
It is the coefficient functions of this expansion that are called *light cone wave functions*. In quantum mechanical perturbation theory the first perturbative correction to the ground state $|0\rangle$ wavefunction is

$$|0\rangle \quad \Longrightarrow \quad |0\rangle + \sum_n \frac{\langle n|\,\hat{V}\,|0\rangle}{E_0 - E_n}\,|n\rangle\,, \tag{144}$$

LCPT: $E \implies k^-$

Here V is the interaction part of the Hamiltonian. The energy denominator $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to n; it suppresses short-lived fluctuations whose energy differs by a large amount from the unperturbed state. The LCPT "energy" is the operator generating translations in light cone time x^+, i.e. k^-. The energy denominator corresponds to the propagator of covariant theory, integrated over the k^- momentum using the pole. The matrix elements $\langle n|\,\hat{V}\,|0\rangle$ are vertices in Feynman rules: the interaction terms in field theory are operators that change the number of particles.

Now let us briefly review what goes into the calculation of the virtual photon to quark-antiquark wavefunction. For this we need to evaluate the energy denominator and the matrix element in Eq. (144). This can be represented by the following diagram:



The matrix element in the general perturbation theory formula (144) is (now the state "$|0\rangle$" is the leading order state: the virtual photon)

$$\langle q(\bar{p}), \bar{q}(\bar{p}')|\left[\int \mathrm{d}^3\bar{x}\, e\overline{\psi}\tilde{A}\psi\right]|\gamma(\bar{q},\lambda)\rangle \tag{145}$$

At this order we just need the first, "normal interaction," term from the Hamiltonian (141). Ths incoming and outgoing states are obtained by acting on the vacuum with creation operators:

$$|\gamma(\bar{q},\lambda)\rangle = a^\dagger_\lambda(\bar{q})\,|0\rangle \tag{146}$$

$$\langle q(\bar{p},s), \bar{q}(\bar{p}',s')| = d^\dagger_{s'}(\bar{p}')b^\dagger_s(\bar{p})\,|0\rangle \tag{147}$$

Now all we need is the field operators in terms of the creation and annihilation operators, Eqs. (126) and (138), and the commutators of these operators. When the creation/annihilation operators act on the incomint/outgoing state, they just give momentum delta functions setting the momenta in the integrals (126) and (138) to be the same as the incoming/outgoing particle momenta. What is left over is the photon polarization vector in (138) and the spinors in (126). The integral over space in the interaction matrix element (145) gives an overall momentum conservation delta function. We are left with

$$\langle q(\bar{p}), \bar{q}(\bar{p}')|\left[\int \mathrm{d}^3\bar{x}\, e\overline{\psi}\tilde{A}\psi\right]|\gamma(\bar{q},\lambda)\rangle = (2\pi)^3\delta^{(3)}(\bar{q}-\bar{p}-\bar{p}')ee_f\,\bar{u}_s(p)\not{\epsilon}_\lambda v_{s'}(p') \tag{148}$$

$$s, s' = \pm\frac{1}{2}; \quad \lambda = 0 = L, \quad \lambda = \pm 1 = T \tag{149}$$

Here e is the elactromagnetic coupling constant and e_f the fractional charge of the quarks, $e_d = -1/3$ for down and $e_u = 2/3$ for up quarks. The spinor matrix element $\bar{u}_s(p)\not{\varepsilon}_\lambda v_{s'}(p')$ can be evaluated explicitly. For this we need in principle the explicit expressions for the spinors and the transversely polarized virtual photon polarization vector in the LC gauge is given by

$$\varepsilon^\mu_{\lambda=\pm}(q) = (0, \frac{\mathbf{q}\cdot\boldsymbol{\varepsilon}_\lambda}{q^+}, \boldsymbol{\varepsilon}_\lambda) = (0, 0, \boldsymbol{\varepsilon}_\lambda), \tag{150}$$

where the circularly polarized transverse polarization vectors can be taken as

$$\boldsymbol{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \end{pmatrix} \tag{151}$$

In practice one does not need to do these by hand, the needed elements can be found e.g. in [25]). Explicitly evaluating the matrix elements between the Dirac spinors (these are independent of the representation of the γ-matrices,

$$\bar{u}_s(p)\not{\varepsilon}_{\lambda=\pm1} v_{s'}(p') = \delta_{s,-s'}\frac{2}{\sqrt{z(1-z)}}\left(z\delta_{\lambda,2s} - (1-z)\delta_{\lambda,-2s}\right)\boldsymbol{\varepsilon}_\lambda\cdot\mathbf{m} + \delta_{s,s'}\delta_{\lambda,2s}\frac{\sqrt{2}m}{\sqrt{z(1-z)}}. \tag{152}$$

Here we clearly have the separated quark helicity conserving vertex which is independent of the mass, and the helicity-flip matrix element that is proportional to the mass.

A longitudinally polarized virtual photon is strictly speaking not a part of the Fock space of the theory. In stead it is part of an additional "instantaneous" vertex that arises from explicitly solving non-dynamical constraints in light cone quantization. However, for the purposes of this calculation we can in practice define a longitudinal polarization vector for a virtual photon in the $\epsilon^+ = 0$ light cone gauge

$$\varepsilon^\mu_{\lambda=0}(q) = (0, \frac{\sqrt{Q^2-\mathbf{q}^2}}{q^+}, \frac{\mathbf{q}}{\sqrt{Q^2-\mathbf{q}^2}}) \underset{\mathbf{q}=\mathbf{0}}{=} (0, \frac{Q}{q^+}, 0), \tag{153}$$

where we chose $\mathbf{q} = 0$. Using this we get

$$\bar{u}_s(p)\not{\varepsilon}_{\lambda=0} v_{s'}(p') = -2Q\sqrt{z(1-z)}\delta_{s,-s'} \tag{154}$$

The other ingredient needed is the energy denominator: the difference between the (on-shell!) energies of the incoming and the outgoing states $(q^- - p^- - p'^-)^{-1}$. It can be simplified as

$$q^- - p^- - p'^- = -\left(\frac{Q^2}{2q^+} + \frac{\mathbf{p}^2+m^2}{2zq^+} + \frac{(\mathbf{p}-\mathbf{q})^2+m^2}{2(1-z)q^+}\right) = \frac{\overbrace{Q^2z(1-z)+m^2}^{\equiv\varepsilon^2}+\mathbf{m}^2}{-2q^+z(1-z)}. \tag{155}$$

Note how the momentum conservation delta function in (148) nicely combines with the $2q^+$ from the energy denominator into a Lorentz-invariant combination $2q^+(2\pi)^3\delta^{(3)}(\bar{q}-\dots)$.

Note that neither the energy denominator nor the vertex do not depend on $\mathbf{p}, \mathbf{p}'_\perp, \mathbf{q} = \mathbf{p}+\mathbf{p}'_\perp$ separately, but only though the combination $\mathbf{m} \equiv \mathbf{p} - z\mathbf{q}$. From now on we will take $\mathbf{q} = 0$ so that $\mathbf{m} = \mathbf{p}$. This is a general and crucial fact that significantly helps simplify the calculations. This momentum should be understood as the appropriate relative transverse momentum between the quark and the antiquark. It has an interpretation in terms of the $2+1$-dimensional Galileian invariance of the theory, which we will not go into here. So now we have our final result:

$$\psi^{\gamma^*\to q\bar{q}}(\mathbf{p}, \mathbf{p}'_\perp, z) = -2q^+(2\pi)^3\delta^{(3)}(\bar{q}-\dots)\frac{z(1-z)ee_f\bar{u}_s(p)\not{\varepsilon}_\lambda v_{s'}(p')}{\varepsilon^2+\mathbf{m}^2}. \tag{156}$$

The eikonal scattering with the target happens such that the transverse *coordinates* of the quark and antiquark stay constant. Thus we need the Fourier transforms of the wave functions from transverse momenta to transverse coordinates. The transformed wavefunction is called the *mixed space* wavefunction: it is in transverse coordinate and longitudinal momentum space.

Mixed space wave function

$$\psi^{\gamma^*\to q\bar{q}}(\mathbf{x},\mathbf{y},z)=\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\int\frac{\mathrm{d}^2\mathbf{p}'_\perp}{(2\pi)^2}\psi^{\gamma^*\to q\bar{q}}(\mathbf{p},\mathbf{p}'_\perp,z)\,e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{p}'_\perp\cdot\mathbf{y}} \tag{157}$$

Note that due to the transverse momentum conserving delta function in the full momentum space wave-function (156), one of the two momentum integrals in the wave function is trivial. The Fourier-transforms of polynomials in $\mathbf{p}$ divided by the energy denominator (155) are modified Bessel functions (exercise).

Exercise: Fourier-transform of massive propagator

Calculate

$$\int_{-\infty}^{\infty}dk\frac{e^{ikx}}{k^2+m^2}.$$

This can be done the easy way using the theorem of residues; there is probably also a hard way without it. Then calculate the two dimensional transform needed in the virtual photon-dipole wave function

$$\int d^2\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2+m^2}$$

and compare the behavior at large $r=|\mathbf{r}|$. One way is to integrate over the angle first, which leaves you with the Bessel $J_0(|\mathbf{k}||\mathbf{r}|)=J_0(kr)$. The integral over k then gives a modified Bessel K_0. Mathematica will do this k-integral for you. The way to do this without mathematica is to do a Schwinger parametrization of the denominator $1/A=\int_0^\infty \mathrm{d}te^{-tA}$, then do the Gaussian integral over $\mathbf{k}$ and then recognize an integral representation of the Bessel K_0 (see e.g. 10.32.10 in https://dlmf.nist.gov/10.32).

We then leave out the overall longitudinal momentum conservation $2q^+(2\pi)\delta(q^+-\dots)$ (see discussion in Sec. 11) to form what is called the "reduced wave function". By then squaring, and summing over spins and colors we can write (see e.g. Sec. III of [20])

$$\left|\psi_T^{\gamma^*\to q\bar{q}}\right|^2 = \frac{\alpha_{\mathrm{e.m.}}}{4\pi^2}N_c e_f^2 2\left(\left[z^2+(1-z)^2\right]K_1^2(\varepsilon r)+m_f^2K_0^2(\varepsilon r)\right) \tag{158}$$

$$\left|\psi_L^{\gamma^*\to q\bar{q}}\right|^2 = \frac{\alpha_{\mathrm{e.m.}}}{4\pi^2}2N_c e_f^2 4Q^2z^2(1-z)^2K_0^2(\varepsilon r), \tag{159}$$

$$\varepsilon^2=Q^2z(1-z)+m^2 \tag{160}$$

Here $r=|\mathbf{x}-\mathbf{y}|$. These are now squares of the wave functions needed to calculate the dipole picture DIS cross section in Eq. (166).

Unfortunately there are a couple of different conventions for different factors. The overall $2q^+$ can be in different places, but it is easier to spot because it must vanish from the final result (the cross section is boost invariant but q^+ is not, so it must vanish). More tricky are factors of 4π. The result here requires an integration over the final phase space (see Eq. (270))

$$\int\frac{\mathrm{d}p^+}{4\pi p^+}\frac{\mathrm{d}p'^+}{4\pi p'^+}2q^+(2\pi)\delta(2q^+-p^+-p'^+)=\int_0^1\frac{\mathrm{d}z}{4\pi}\frac{1}{z(1-z)}, \tag{161}$$

where we absorb the 4π into $\alpha_{\mathrm{e.m.}}=e^2/(4\pi)$. This comes with a $1/(2\pi)^2$ coming together with the Bessel function squared, and an N_c from the sum over colors. For the longitudinal case there is a 2 for the sum over spins with $(2Q)^2z(1-z)$ from the vertex (154) squared and $(z(1-z))^2$ originating from the energy denominator squared. For the transverse case the $1/(z(1-z))$ from the vertex squared cancels a part of the $(z(1-z))^2$ from the energy denominator squared, and the phase space (161) cancels the rest. There is a 2^2 from the prefactor of the vertex (152), together with a $1/2$ because we are averaging over polarizations (see Sec. 7), and the rest of the vertex (152) summed over polarizations gives $z^2+(1-z)^2$.

To summarize, we have understood the DIS process as a virtual photon γ^* scattering on target at rest. In the high energy limit this process factorizes into two parts:

- γ^* fluctuates into $q\bar{q}$: this is a well understood QED process

- The $q\bar{q}$ scatters off the color field of the target: this is the part that tells us about the color field of the target.

Some side remarks about this process:

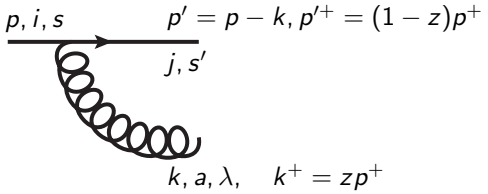
- The modified Bessel function K_n decreases exponentially in its argument. This means that the typical dipole transverse size contributing to the scattering process is $r \sim 1/Q$. This gives a concrete meaning to our earlier statement that Q should be thought of as a transverse momentum or resolution scale in the scattering.
- We used, in Eq. (166), the optical theorem expressing the total cross section as $2\mathcal{N}$, where $\mathcal{N}$ is, in our notation, the imaginary part of the scattering amplitude. But the quantity that we really are using to describe the target is the elastic scattering amplitude. Thus we get a direct connection between inclusive scattering (total cross section) and exclusive scattering, where the target indeed stays intact. In DIS the exclusive scattering process is referred to as Diffractive DIS, and it can be independently measured. In the parton picture inclusive and exclusive DIS are different processes described by independent, separate parton distributions. But in the dipole picture one calculates the cross sections for both from the same scattering amplitude, i.e. Wilson line correlator. This enables one to make nontrivial predictions for DIS cross sections that are not possible in the parton model.
- Another way to state the content of the eikonal approximation is that we are assuming that fixed-size dipoles are the basis of states that diagonalizes the imaginary part of the T-matrix. In general, the fact that in the high energy/eikonal approximation particles fly through target at fixed $\mathbf{x}$ does not imply zero momentum transfer! Rather it should be understood that even if they do get a momentum transfer, their energy is so high (and/or the longitudinal extent of the target so short, these statements are Lorentz-transforms of each other) that the transverse momentum transfer is not enough to modify the transverse position of the particle during the time that it spends inside the target.

9 BK equation

9.1 Soft gluon radiation

As discussed above, the dipole picture of DIS represents just the leading order in a QCD perturbation theory expansion. The next correction comes in, when one adds a gluon, either to make a loop correction, or to interact with the target. We shall here consider the kinematical limit of these processes where the gluon is very soft because, as we will see, it contributes to the cross section as a large logarithm of the energy or x. When we want to study what happens at very small x, we have to resum these logarithms of $\alpha_s \ln 1/x$. This resummation is performed using the Balitsky-Kovchegov (BK) equation. Our purpose here is to derive this equation. In order to do this it is enough to consider the radiative correction, where the dipole emits a gluon, and this gluon then interacts with the target nucleus. The loop corrections can, in the soft gluon limit but not generally, be deduced indirectly from a unitarity argument.

NLO: radiate soft gluon $z \ll 1$



Light cone wavefunction

$$\psi^{q\to qg}(z,\mathbf{k}) = \frac{1}{\frac{\mathbf{p}^2}{2p^+} - \frac{\mathbf{k}^2}{2k^+} - \frac{\mathbf{p}'^2_\perp}{2p'^+}} \bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p) \tag{162}$$

Similarly as for the $\gamma^* \to q\bar{q}$ wavefunction in Sec. 7, we need the matrix element, which can be calculated in any representation of the γ-matrices. The full matrix element (now for massless quarks for simplicity) is

$$\bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p) = \frac{-2gt^a_{ji}}{z\sqrt{1-z}}(\delta_{\lambda,2s} + (1-z)\delta_{\lambda,-2s})\mathbf{q}\cdot\boldsymbol{\varepsilon}^*_\lambda, \quad \mathbf{q} = \mathbf{k} - z\mathbf{p} \tag{163}$$

We will here be interested only in the high energy limit where the quark longitudinal momentum p^+ is large, so that there is a lot of phase space for emission of gluons with a much smaller k^+. We will therefore assume that the emitted gluon is soft, and take $z \to 0$. In this limit the emitted gluon light cone energy k^- is large and dominates the energy denominator in (162). Also the emission vertex simplifies, in particular we see that it becomes the same for both of the gluon polarization states $\lambda = \pm 2s$. This is very general feature at high energy limit and related to what we have already discussed in the context of the eikonal approximation: the interaction of the gauge soft gauge field with the hard particles cares only about one vector: that of the hard particle, and not e.g. the spin states of the quark or gluon. In the soft limit also the relative momentum $\mathbf{q} = \mathbf{k} - z\mathbf{p} \to \mathbf{k}$

In *soft* limit $z \to 0$:

$$\psi^{q\to qg}(k^+, \mathbf{k}) \approx \frac{-2zp^+}{\mathbf{k}^2}\frac{-2gt^a_{ji}}{z}\mathbf{k}\cdot\boldsymbol{\varepsilon}^*_\lambda = \frac{4gt^a_{ji}p^+}{\mathbf{k}^2}\mathbf{k}\cdot\boldsymbol{\varepsilon}^*_\lambda \tag{164}$$

From this one can calculate (somewhat sketchily) a "probability to emit a gluon" by squaring the wave-function and integrating over a Lorentz-invariant phase space for gluon emission, which exhibits the typical gauge theory logarithmic divergences for both soft $z \to 0$ and collinear $\mathbf{k} \to \mathbf{0}$ gluon emission.

$$\mathrm{d}P_{q\to qg} = \left|\psi^{q\to qg}(k^+, \mathbf{k})\right|^2 \frac{\mathrm{d}k^+\,\mathrm{d}^2\mathbf{k}}{2k^+(2\pi)^3} \sim \frac{\mathrm{d}z}{z}\frac{\mathrm{d}^2\mathbf{k}}{\mathbf{k}^2} \quad \left(\sum_{\lambda=\pm1}\varepsilon_i\varepsilon^*_j = \delta_{ij}\right) \tag{165}$$

soft $\frac{\mathrm{d}z}{z}$

collinear $\frac{\mathrm{d}^2\mathbf{k}}{\mathbf{k}^2}$

Out of these divergences the collinear one, corresponding to the emission of very large (transverse) wavelength gluons will cancel when we consider the emission from a color neutral dipole, because the emissions from the quark and the gluon must destructively interfere in this limit. (In fact, in the calculation we are going to do, the collinear divergence is an artefact of taking $z \to 0$ and is not actually there at finite z. Nevertheless there is a cancellation between the emission from the quark and the antiquark, and in some sense the physics becomes more transparent if we take $z \to 0$ here and let this spurious divergence appear, to be then canceled later...)

The soft divergence, on the other had, does not cancel. In stead it is something that we must absorb into a redefinition of what exactly we mean by the target. To visualize this it is conventional to draw diagrams where the vertical axis corresponds to rapidity. At the top is the probe photon with a large q^+ and at the bottom the target, with the smallest p^+, i.e. the largest p^-, with the small momentum fraction $x \sim p^+/q^+$. The large logarithmic contributions come from integrating over the longitudinal momentum of the emitted gluons over the whole rapidity interval. Since the phase space integral is logarithmic, each gluon contributes a factor $\sim \alpha_s \ln 1/x$, which we are assuming to be of order one. Therefore we must resum the emission of arbitrary numbers of these gluons.

Exercise: Fourier-transform of the Weizsäcker-Williams kernel

Fourier-transform the essential part of the LC wave function for emitting a soft gluon:

$$\int d^2\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\boldsymbol{\varepsilon}\cdot\mathbf{k}}{\mathbf{k}^2}.$$

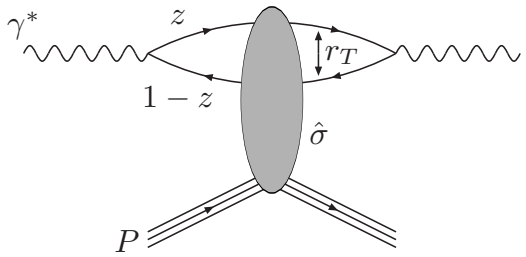
This can be done analytically (without mathematica!) by first integrating over the angle, which gives a Bessel function J_1 that is the derivative of J_0; thus the radial integral is easy. Note that there are two independent azimuthal angles, those of $\boldsymbol{\varepsilon}$ and $\mathbf{r}$. The coordinate space expression is known as the Weizsäcker-Williams kernel: it gives the electric or magnetic field at position $\mathbf{r}$ of a point charge moving at the speed of light in the z direction at $\mathbf{r} = 0$.

9.2 Idea of a renormalization group equation

This resummation is done by using a renormalization group equation, in this case the BK equation. To start with, we have to introduce some separation scale in rapidity $y \sim \ln k^+$. Now whatever is at a higher k^+ will be called a part of the probe photon, and all that is at a lower p^+ we decide to call a part of the target. The separation scale now appears as a cutoff in the integral over the longitudinal momentum fraction z. Now the

smart thing to do is to choose in the end the separation scale so close to the rapidity of the probe q^+ that the integral $\sim \alpha_s \int_{z_{\text{cutoff}}} dz/z$ is small. Then we do not have a large logarithm left, but this comes at the expense of our parametrization of target depending on this cutoff. The advantage is that now we can calculate how exactly the target depends on this cutoff: this dependence is precisely the RGE equation that tells us how the cross section depends on the energy.

We start from the leading order dipole picture cross section



$$\sigma^{\gamma^* p}_{T,L} = \int \mathrm{d}^2\mathbf{r}\,\mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(\mathbf{r},z)_{T,L}\right|^2 2\int \mathrm{d}^2\mathbf{b}\overbrace{\left(1-\frac{1}{N_c}\,\mathrm{tr}\,V(\mathbf{x})V^\dagger(\mathbf{y})\right)}^{\mathcal{N}^{q\bar{q}+p}} \tag{166}$$

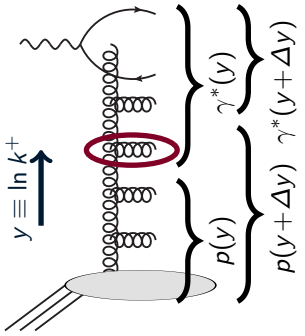
$$\left|\psi^{\gamma^*\to q\bar{q}}_T\right|^2 = \frac{\alpha_{\text{e.m.}}}{4\pi^2} N_c e_f^2 2\left(\left[z^2+(1-z)^2\right]K_1^2(\varepsilon r) + m_f^2 K_0^2(\varepsilon r)\right) \tag{167}$$

$$\left|\psi^{\gamma^*\to q\bar{q}}_L\right|^2 = \frac{\alpha_{\text{e.m.}}}{4\pi^2} 2N_c e_f^2 4Q^2 z^2(1-z)^2 K_0^2(\varepsilon r), \tag{168}$$

$$\varepsilon^2 = Q^2 z(1-z) + m^2 \tag{169}$$

Here $r = |\mathbf{x}-\mathbf{y}|$.

Now we want to see what happens to this cross section when we emit a gluon.



If we now look at an individual gluon (at y) the cross section has to be the same whether this gluon is a part of the probe γ^* Fock state, or a part of the target p. The equality of the cross section gives us an equation, which is the BK evolution equation.

$$\begin{aligned}\sigma^{\gamma^* p} &= \overbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_y \otimes 2\mathcal{N}_y^{q\bar{q}+p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_y \otimes 2\mathcal{N}_y^{q\bar{q}g+p} + \ldots}^{\text{gluons up to } y \text{ part of proton}} \\ &= \underbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_{y+\Delta y} \otimes 2\mathcal{N}_{y+\Delta y}^{q\bar{q}+p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_{y+\Delta y} \otimes 2\mathcal{N}_{y+\Delta y}^{q\bar{q}g+p} + \ldots}_{\text{gluons up to } y+\Delta y \text{ part of proton}}\end{aligned} \tag{170}$$

9.3 The Balitsky-Kovchegov equation: a short derivation

Let us now put this idea into practice. To derive the BK equation one needs to

- Calculate $\psi^{\gamma^* \to q\bar{q}g}(z)$
- Take soft gluon limit $z \to 0$;
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q\bar{q}$ cross section

The full $\psi^{\gamma^* \to q\bar{q}g}(z)$ wavefunction is a rather complicated object (see e.g. [20]), because we need to go to second order in perturbation theory. Somewhat schematically, it will be

$$\psi^{\gamma^* \to q\bar{q}g} = \underbrace{\frac{\langle q\bar{q}g|\,\hat{P}^-_{\text{int}}\,|q\bar{q}\rangle}{k^-(\gamma^*) - k^-(q\bar{q}g)}}_{\approx \frac{\langle q\bar{q}g|\hat{P}^-_{\text{int}}|q\bar{q}\rangle}{-k^-(g)} = \psi^{q\to qg}} \frac{\langle q\bar{q}|\,\hat{P}^-_{\text{int}}\,|\gamma\rangle}{k^-(\gamma^*) - k^-(q\bar{q})} \approx \tag{171}$$

The complicated thing is that the energy denominators are always calculated with respect to the incoming γ^* state. Thus the wavefunction does not actually factorize into a product of a $\gamma^* \to q\bar{q}$ and a $q/\bar{q} \to q/\bar{q} + g$ wavefunction. However, we do not need to deal with all this complication here, due to two things.

- In a somewhat miraculous manner, when one Fourier-transforms to transverse coordinate space, the $\gamma^* \to q\bar{q}g$ LCWF *does* in fact factorize into a somwehat modified $\gamma^* \to q\bar{q}$ wavefunction (that knows about the gluon emission), and a completely unmodified $q/\bar{q} \to q/\bar{q} + g$ wavefunction. I have yet to find a clear interpretation for why this is the case. Somehow, in momentum space information propagates forward: later splittings know about the earlier ones because the energy denominators are always calculated with respect to the original incoming state. In transverse coordinate space this information propagates backwards: the last splitting (the gluon emission) does not know about the history (the $\gamma^* \to q\bar{q}$ splitting), but the earlier splittings know about the future (the part of the wavefunction related to the $\gamma^* \to q\bar{q}$ splitting knows that a gluon will be emitted later).
- The soft gluon limit $z \to 0$ makes things even simpler. As we discussed (see Eq. (164)) in this limit the energy denominator is completely dominated by the energy of the gluon $k^-(\gamma^*) - k^-(q\bar{q}g) \approx -k^-(g)$ and therefore does not care about what the gluon was emitted from. In this limit the wavefunction factorizes in both momemtum and coordinate space.

Let us now move on with the calculation in coordinate space. We need to Fourier-transform the gluon emission wave function, which becomes:

$$\psi^{q\to qg}(k^+, \mathbf{r}) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} \psi^{q\to qg}(k^+, \mathbf{k}) = -i2p^+ \frac{2gt^a_{ji}}{2\pi} \frac{\boldsymbol{\varepsilon}\cdot\mathbf{r}}{\mathbf{r}^2} \delta_{s,s'} \tag{172}$$

Now we need to coherently sum the emissions from the quark and the antiquark, where for the latter there is a relative minus sign in the emission vertex.

$$\mathbf{r} = \mathbf{x} - \mathbf{y} \quad \mathbf{r}'_\perp = \mathbf{x} - \mathbf{z} \quad \mathbf{z} - \mathbf{y} = \mathbf{r} - \mathbf{r}'_\perp \tag{173}$$

We are being sloppy with the exact factors from the phase space. However, up to overall constants we can now write the Fock state wavefunction of the virtual photon as

$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{x},\mathbf{y}} C(\mathbf{r}=\mathbf{x}-\mathbf{y})\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r})_{\text{LO}}\,|q_i(\mathbf{x},z)\bar{q}_i(\mathbf{y},1-z)\rangle$$
$$+\int_{z,\mathbf{x},\mathbf{y},\mathbf{z}} \psi^{\gamma^*\to q\bar{q}}_{\text{LO}}(z,\mathbf{r})\int\frac{\mathrm{d}z'}{4\pi z'}\frac{-i2g}{2\pi}t^a_{ji}\left[\frac{(\mathbf{x}-\mathbf{z})\cdot\varepsilon}{(\mathbf{x}-\mathbf{z})^2}-\frac{(\mathbf{y}-\mathbf{z})\cdot\varepsilon}{(\mathbf{y}-\mathbf{z})^2}\right]$$
$$\times\,|q_i(\mathbf{x},z)\bar{q}_j(\mathbf{y},1-z)g_a(\mathbf{z},z')\rangle\,, \quad (174)$$

Here we have, as discussed above, taken the $\gamma^* \to a\bar{q}g$ wavefunction to be just a product of photon splitting and gluon emission wavefunctions, which is only true because we are taking the gluon longitudinal momentum z' to be small.

To be a bit more precise let us also write the exact expressions for the longitudinal phase space integrals here. These require a bit of care to be done correctly. The Lorentz-invariant phase space element for the integrations is

$$\frac{\mathrm{d}k^+}{4\pi k^+}, \quad (175)$$

but the factor $2k^+$ is not there in the momentum conservation delta function: each vertex yields a

$$2\pi\delta\left(\sum p^+\right). \quad (176)$$

For the quark-antiquark dipole the phase exact expression for the phase space integral is

$$\text{“}\int_z\text{”} \equiv \int\frac{\mathrm{d}p^+}{4\pi p^+}\frac{\mathrm{d}p'^+}{4\pi p'^+}(2\pi)\delta(p^+ + p'^+ - q^+) = \frac{1}{2q^+}\int\frac{\mathrm{d}z}{4\pi z(1-z)}. \quad (177)$$

For the quark-antiquark-gluon state we have in (174) already assumed that the gluon is soft $k^+ = z'q^+ \ll p^+, p'^+, q^+$. This has enabled us to factor out the kinematics of the leading order vertex; this is possible only in the soft limit. For the full kinematics the whole expressions are a bit more complicated [20]. For the term where the gluon is emitted from the quark, the phase space integral (denoting p'' the momentum of the quark before emitting the gluon) is

$$\text{“}\int_{z,z'}\text{”} \equiv \int\frac{\mathrm{d}p^+}{4\pi p^+}\frac{\mathrm{d}p'^+}{4\pi p'^+}\frac{\mathrm{d}p''^+}{4\pi p''^+}\frac{\mathrm{d}k^+}{4\pi k^+}(2\pi)\delta(p''^+ + p'^+ - q^+)(2\pi)\delta(p^+ + k^+ - p''^+)$$
$$= \frac{1}{(q^+)^2}\int\frac{\mathrm{d}z}{4\pi}\frac{\mathrm{d}z'}{4\pi}\frac{1}{z}\frac{1}{1-z-z'}\frac{1}{z+z'}\frac{1}{z'} \approx \frac{1}{q^+}\int\frac{\mathrm{d}z}{8\pi z(1-z)}\frac{1}{zq^+}\int\frac{\mathrm{d}z'}{8\pi z'}.$$
$$= \text{“}\int_z\text{”}\times\frac{1}{2p^+}\int\frac{\mathrm{d}z'}{4\pi z'}, \quad (178)$$

where we separated out an explicit $1/2p^+$ to recover the normal Lorentz-invariant phase space $\mathrm{d}z'/(4\pi z')$. Likewise, when the gluon is emitted from the antiquark we get, using this time p'' for the momentum of the antiquark before the emission

$$\text{“}\int_{z,z'}\text{”} \equiv \int\frac{\mathrm{d}p^+}{4\pi p^+}\frac{\mathrm{d}p'^+}{4\pi p'^+}\frac{\mathrm{d}p''^+}{4\pi p''^+}\frac{\mathrm{d}k^+}{4\pi k^+}(2\pi)\delta(p^+ + p''^+ - q^+)(2\pi)\delta(p'^+ + k^+ - p''^+)$$
$$= \frac{1}{(q^+)^2}\int\frac{\mathrm{d}z}{4\pi}\frac{\mathrm{d}z'}{4\pi}\frac{1}{z}\frac{1}{1-z-z'}\frac{1}{1-z}\frac{1}{z'} \approx \frac{1}{q^+}\int\frac{\mathrm{d}z}{8\pi z(1-z)}\frac{1}{(1-z)q^+}\int\frac{\mathrm{d}z'}{8\pi z'}.$$
$$= \text{“}\int_z\text{”}\times\frac{1}{2p'^+}\int\frac{\mathrm{d}z'}{4\pi z'}, \quad (179)$$

Here the explicit factors $\frac{1}{2p^+}$, $\frac{1}{2p'^+}$ in Eqs. (179) and (178) cancel against the $2p^+$ in Eq. (172) and the corresponding $2p'^+$ in the emission wavefunction from the antiquark, and therefore do not appear in Eq. (174).

It is important to note that the $\int_{z'}$-integral is divergent because of the phase space $\int_{z'} \sim \int \mathrm{d}z'/z'$. We have also added a coefficient $C(\mathbf{r})$ in front of the leading order term, because the normalization of this term needs to be adjusted in order to keep the wavefunction normalized. We can determine this coefficient fron the requirement that the normalization of the wave function with the extra gluon, i.e. square of the dipole

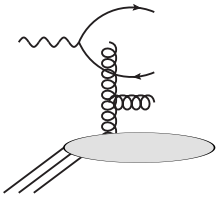
term with the normalization constant plus the square of the $q\bar{q}g$ term, are the same as the square of the dipole term at lowest order. Note that here squaring involves also one sum over the colors of the quarks, which yields a factor N_c for the dipoles. We get the condition

$$N_c\,|C(\mathbf{r})|^2 = N_c - \frac{(2g)^2}{(2\pi)^2}\frac{1}{4\pi}t^a_{ij}t^a_{ji}\int\frac{dz'}{z'}\int d^2\mathbf{r}'_\perp\sum_{\lambda=\pm1}\left|\frac{(\mathbf{x}-\mathbf{z})\cdot\varepsilon_\lambda}{(\mathbf{x}-\mathbf{z})^2}-\frac{(\mathbf{y}-\mathbf{z})\cdot\varepsilon_\lambda}{(\mathbf{y}-\mathbf{z})^2}\right|^2 \quad (180)$$

$$= N_c - \frac{\alpha_s}{\pi^2}\frac{{N_c}^2-1}{2}\Delta y\int d^2\mathbf{r}'_\perp\frac{\mathbf{r}^2}{{\mathbf{r}'_\perp}^2(\mathbf{r}-\mathbf{r}'_\perp)^2} \qquad \sum_{\lambda=\pm1}\varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*}=\delta_{ij} \quad (181)$$

Putting these together, and remembering that we now redefine our scattering amplitude so that the scattering off the dipole and the dipole-gluon system from the old target becomes the same as the scattering of just a dipole off a new, renormalized, target.

$$\mathcal{N}^{y+\Delta y}_{q\bar{q}} = \mathcal{N}^{y}_{q\bar{q}} + \frac{\alpha_s}{\pi^2}\frac{{N_c}^2-1}{2N_c}\int_y^{y+\Delta y} d\ln 1/z'\int d^2\mathbf{r}'_\perp\frac{\mathbf{r}^2}{{\mathbf{r}'_\perp}^2(\mathbf{r}-\mathbf{r}'_\perp)^2}\left[\mathcal{N}^{\ln 1/z'}_{q\bar{q}g}-\mathcal{N}^{\ln 1/z'}_{q\bar{q}}\right] \quad (182)$$

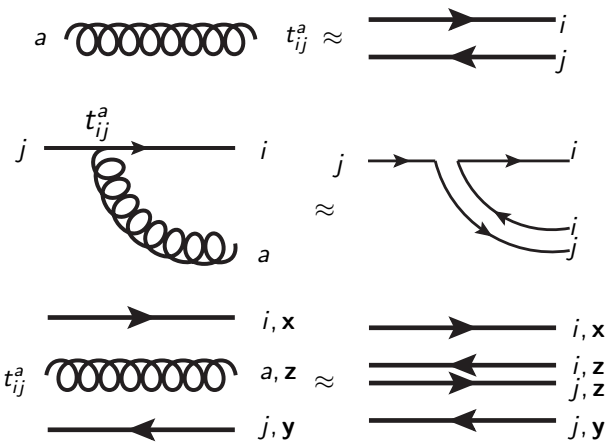


What this equation now tells us is that the dipole scattering on new target $\mathcal{N}^{y+\Delta y}_{q\bar{q}}$ consists of

- A dipole scattering off original target $\mathcal{N}^{y}_{q\bar{q}}$
- A contribution where the dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off the target and
- A correction to the normalization of original dipole (There are now less dipoles in γ^*)

Now we are almost there. We are looking for an evolution equation for the scattering amplitude $\mathcal{N}_{q\bar{q}}$: but enocuntered new quantity $\mathcal{N}_{q\bar{q}g}$, which needs to be related to $\mathcal{N}_{q\bar{q}}$. Let us do this in the large N_c approximation. At large N_c a gluon is approximately the same as a quark-antiquark pair (note: not a color neutral dipole, in fact a general quark-antiquark pair is $1/{N_c}^2$ parts dipole and $({N_c}^2-1)/{N_c}^2$ parts octet, i.e. gluon, and at large N_c we can forget the color neutral dipole). So when we have a quantum state $|q(\mathbf{x})\bar{q}(\mathbf{y})g(\mathbf{z})\rangle$, we can approximate it by $|q(\mathbf{x})\bar{q}(\mathbf{z})q(\mathbf{z})\bar{q}(\mathbf{y})\rangle$

- $N_c\to\infty \implies g\approx q\bar{q}$ (not dipole!)
- ${N_c}^2-1$ gluon colors $\approx {N_c}^2$ $q\bar{q}$ colors.



Now, instead of $\mathcal{N}_{q\bar{q}g}$, we need $\mathcal{N}_{q\bar{q}q\bar{q}}$; the amplitude for the simultaneous scattering of two dipoles. We can obtain it with the following argument:

- $\mathcal{N}$ is a really a *scattering probability*;
- $S = 1 - \mathcal{N}$ is a probability *not to scatter*

For two dipoles no scattering means that neither dipole scatters, from which we deduce

$$S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}} \tag{183}$$

In order to justify this argument we need to make an additional *mean field* approximation, namely that the scattering probabilities of the two dipoles are independent of each other, which enables the factorized form (183). In fact this mean field approximation can be argued to hold in the large N_c limit, which we are already assuming here.

Thus the scattering probability for a two-gluon system is

$$\mathcal{N}_{q\bar{q}q\bar{q}} = 1 - S_{q\bar{q}q\bar{q}} = 1 - (1 - \mathcal{N}_{q\bar{q}})(1 - \mathcal{N}_{q\bar{q}}) \tag{184}$$

With this argument we end up with the approximation

$$S = 1 - \mathcal{N} \quad \& \quad S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}} \tag{185}$$

$$\implies \mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{y})g(\mathbf{z})) \approx \mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{z})) + \mathcal{N}(q(\mathbf{z})\bar{q}(\mathbf{y})) - \mathcal{N}(q(\mathbf{x})\bar{q}(\mathbf{z}))\mathcal{N}(q(\mathbf{z})\bar{q}(\mathbf{y})). \tag{186}$$

The argument leading to this can also be stated in another way: in order to calculate the probability for a scattering to happen, one must add the probabilities that one of the two dipoles scatters, and then subtract the case that both dipoles scatter, since this would otherwise get counted twice. What we are really doing here is using the Mueller dipole formalism [26], where all of QCD at high energy is formulated in terms of dipoles that can split into other dipoles.

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2 - 1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$
$$\times \left[\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{z}) + \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z},\mathbf{y}) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{z})\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z},\mathbf{y}) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x},\mathbf{y})\right] \tag{187}$$

Differentially in Δy

$$\partial_y \mathcal{N}(\mathbf{r}) = \frac{\alpha_s N_c}{2\pi^2}\int \mathrm{d}^2\mathbf{r}'_\perp \frac{\mathbf{r}^2}{\mathbf{r}'_\perp{}^2(\mathbf{r}'_\perp - \mathbf{r})^2}\left[\mathcal{N}(\mathbf{r}'_\perp) + \mathcal{N}(\mathbf{r}-\mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}'_\perp)\mathcal{N}(\mathbf{r}-\mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r})\right] \tag{188}$$

This is the BK equation (1995) [27, 28, 29]. I personally like to call this the holy grail of small x physics. It is the basic tool for a lot of practical phenomenology.

- Given initial condition $\mathcal{N}(\mathbf{r})$ at $y = y_0$ the equation predicts the scattering amplitude at larger y = smaller x = higher $\sqrt{s}$.
- If one drops the nonlinear term one gets the BFKL (Balitsky-Fadin-Kuraev-Lipatov) equation (see e.g. [30]), which was derived by summing normal Feynman diagrams for the scattering, already a longer time ago.
- The divergences of the kernel at $\mathbf{r}'_\perp \to 0$ and $\mathbf{r}'_\perp \to \mathbf{r}$ are regulated because $\mathcal{N}(0) = 0$. This feature is referred to as "color transparency": a dipole is a color neutral object as a whole, and if its size is zero it should not scatter at all.
- The equation enforces the black disk limit (unitarity) $\mathcal{N} < 1$
- For practical work the coupling α_s should not be fixed, but depend on a distance scale, which can be some combination of $\mathbf{r}, \mathbf{r}'_\perp, \mathbf{r} - \mathbf{r}'_\perp$. What, in the end, is both theoretically and practically the correct way to do this is still a matter of discussion in the field.

The equation can be solved numerically, let us look at a few features of the solution

- Small dipoles $r \lesssim 1/Q_\mathrm{s}$ scatter very little, color transparency is satisfied.
- Large dipoles $r \gtrsim 1/Q_\mathrm{s}$ scatter with probability almost one, but not more. This is *gluon saturation* in action, although in this calculation gluon saturation does not appear as some mystical complicated phenomenon, but just the obvious fact that the scattering can only happen with a probability of at most one.
- When going towards smaller x, the "front" of the solution moves towards the left. This means that the density of gluons in the target increases, justifying a posteriori our assumption that at smaller x the classical field picture of the target gets better and better.

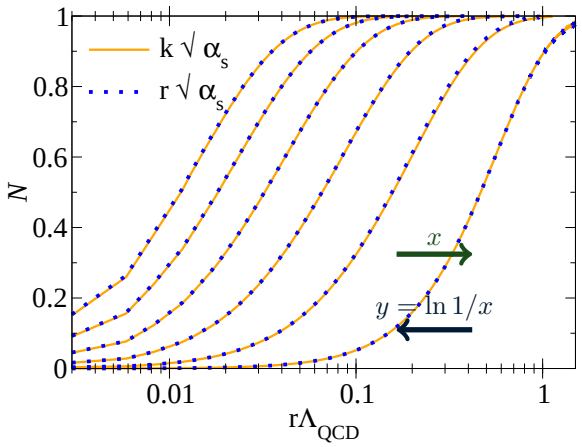


Figure 2: Solution of the BK equation from a numerical calculation. (Actually cheating, this particular plot from [31] is a solution of JIMWLK, which generalizes BK)

To set this in context remember that for the DIS cross sections F_2, F_L this solution of the BK equation has to be convoluted with the (known) γ^* wavefunction.

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{b}\,\mathrm{d}^2\mathbf{r}\,\mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(r,z)_{T,L}\right|^2 2\mathcal{N}(\mathbf{r},\mathbf{b},x) \tag{189}$$

Fits HERA data ($x < 0.01$ Q^2 moderate) with this formula work extremely well, although this depends on a specific way to model the b-dependence of the dipole amplitude.

The JIMWLK equation is a generalization of the BK equation, in the sense of giving uo the mean field approximation that we had to use. In fact, even without this approximation and at finite N_c it turns out, after some color algebra magic, that the dipole amplitude satisfies an equation that has the form (188). However, the amplitudes $\mathcal{N}$ are replaced by expectation values of the amplitudes in some distribution. In general, the expectation value of the product of amplitudes $\langle\mathcal{N}\mathcal{N}\rangle$ cannot be expressed in terms of the expectation value of the amplitude $\langle\mathcal{N}\rangle$. Thus the resulting equation does not close. One can then derive an additional evolution equation for the x-dependence of $\langle\mathcal{N}\mathcal{N}\rangle$, but this will involve a new quantity $\langle\mathcal{N}\mathcal{N}\mathcal{N}\rangle$, and so on ad infinitum. This set of evolution equations is known as the *Balitsky hierarchy*. It is equivalent to the JIMWLK equation, which is written as an evolution equation for the probability distribution of the Wilson lines from which the amplitudes are constructed. Knowing the probability distribution is equivalent to knowing all possible expectation values.

JIMWLK

$$\langle\mathcal{N}\mathcal{N}\rangle \neq \langle\mathcal{N}\rangle\,\langle\mathcal{N}\rangle \tag{190}$$

$\Longrightarrow$ need probability distribution

$$W_y\,[V_\mathbf{x}] \tag{191}$$

Exercise: BK equation in zero transverse dimensions

(Kovchegov & Levin, 4.5 b) Solve the BK equation in zero transverse dimensions:

$$\partial_y N = \alpha_s N - \alpha_s N^2, \quad N(y=0) = N_0 \ll 1 \tag{192}$$

10 Gluon saturation & CGC

10.1 Spacetime structure in 2 gauges

Let us now reflect the z-axis and consider a nucleus (in stead of the probe) moving in the $+$-direction, just to confuse the reader... This means that the color field of the nucleus has a large A^+ component.

Until now we have been describing the high energy nucleus as a classical field with one Lorentz-component (which was A^- and is now A^+), which we then developed into a path-ordered exponential or Wilson line, which is the eikonal scattering amplitude for a high energy probe passing through the nucleus. What does this picture imply in terms of the partonic content of the nucleus, i.e. the gluon distribution? For this we need some kind of a microscopical picture of where the classical field comes from. We know that it should represent small-x gluonic degrees of freedom. We also remember from our discussion of the eikonal vertex that a gluon carrying only a small longitudinal momentum fraction (a soft gluon) is essentially *classical* radiation, that only cares about color charge. This leads to the basic idea of the CGC: a classical field approximation for small x gluons, radiated by classical color charges representing the large-x degrees of freedom.

The basic idea of the CGC is that we separate the microscopic degrees of freedom inside a high energy nucleus into two things:

CGC separation of scales:

- small x: classical field
- large x: color charge

For a first simplistic picture one can think of the color charge as valence quarks, which then radiate gluons, represented by the classical field. A classical field radiated by a classical current follows the classical equation of motion. For Yang-Mills theory this is given by

$$[D_\mu, F^{\mu\nu}] = J^\nu \tag{193}$$

Note that here we are using a matrix notation, with

$$J^\mu = J^\mu_a t^a \quad F_{\mu\nu} = F^a_{\mu\nu} t^a \quad D_\mu = \partial_\mu + igA_\mu \tag{194}$$

so that the commutator of the generators

$$[t^a, t^b] = i f^{abc} t^c \tag{195}$$

provides just a short way of writing

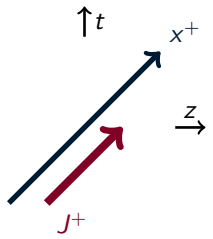
$$\partial_\mu F^{\mu\nu}_a - g f^{abc} A^b_\mu F^{\mu\nu}_c = J^\mu_a \tag{196}$$

Equation (193) involves some yet unspecified color current J^μ. The commutator in (193) refers to the fact that the gauge potential A_μ, field strength $F_{\mu\nu}$ and color current J^μ are matrices in color space. Incidentally, note that since the field strength is a commutator of covariant derivatives, $F_{\mu\nu} = -i/g[D_\mu, D_\nu]$, one can show using the Jacobi identity that the color current must be *covariantly conserved*,

$$[D_\nu, J^\nu] = 0 \tag{197}$$

in stead of the usual abelian charge conservation $\partial_\mu J^\mu = 0$. This means that the conservation of color charge actually requires J^μ to rotate in color space.

For a nucleus moving in the $+z$-direction, the classical current should be proportional to the momenta of the large-x degrees of freedom. We therefore expect that the most important (or, only nonzero) component of the current should be a J^+-component.



Now we have to somehow implement the statement that the charge is supposed to correspond to large x and the fields to small x in a spacetime picture. The consequence of this is that we can treat the color current J^+ as a very thin sheet in x^-. Sometimes one quotes a somewhat naive justification for this approximation in terms of a Lorentz-contraction of the nucleus from its rest frame thickness $2R_A$ to $\Delta z \sim 2R_A m_A/\sqrt{s}$ by the boost from the rest frame energy m_A to the large energy $\sqrt{s}$. However, the more proper argument for this assumption is the combination of the uncertainty principle with the assumption about a scale separation. The current represents large x degrees of freedom, which for a fast nucleus have a large p^+ momentum. The classical field, on the other hand, corresponds to degrees of freedom with smaller p^+. As a consequence of this the charges are better localized x^- than the field, so the field (which is what we are interested in) sees the charges as localized in x^-.

Charges are thin sheet in x^-

1. Naive explanation: Lorentz-contraction
2. Real explanation: Heisenberg $\Delta x^- \sim 1/p^+$, charges have large p^+

A similar argument must be made for the dependence of the current on the light cone time coordinate x^+. A simplistic argument states that for a fast-moving nucleus time is Lorentz-dilated so that all the internal dynamics happens very slowly in x^+. A more detailed statement of the same effect is that any probe of the small x (small p^+) degrees of freedom will have a larger $p^- \sim {p_T}^2/p^+$ than the light cone energies p^- of the color charges. Since the color charges have a small p^-, they evolve more slowly in x^+ than the fields, or an external probe of the system operating at the timescale of the small-x fields. Therefore from the point of view of the field, or of this external probe, the charges appear as slowly dependent of x^+. The extreme limit of this argument is that we can consider the charge density to be independent of the light cone time x^+, i.e. static.

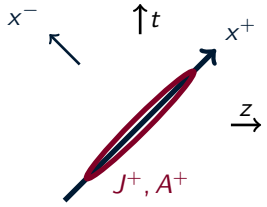
Digression: Glass

A system that is dime dependent, but whose time dependence is unnaturally slow compared to the timescales that one is probing it with, is known as a glass in statistical physics. An actual glass is a "glass" in the statistical physics sense, because glasses are non-crystalline like liquids, but nevertheless do not flow. In a loose (and not technically correct) sense one can describe a glass as a liquid that flows so slowly that on the timescales of interest it behaves like a solid.

Taking these energy- and momentum scale arguments to the extreme now arrive at the desired form for the color current as a δ-function in x^-, independent of x^+, which leaves us with a purely 2-dimensional transverse color charge density. The notation $\delta(x^-)$ should be thought of as a somewhet formal limit, we will still need to maintain the concept of path ordering (now in x^- because we switched the direction of motion of the nucleus): so we should still think of the nucleus as slightly elongated in x^-, but very thin compared to any other x^--scale in the scattering problem.

Current *static*, independent of x^+ $\Longrightarrow$ *glass*

1. Time is dilated for the nucleus
2. Charges: small p^-: slow dependence on x^+



We can now easily find a solution to the equation of motion (193), in which there is only is only one component in the gauge field, A^+, with

$$J^+(x^-, \mathbf{x}) \approx \delta(x^-)\rho(\mathbf{x}) \tag{198}$$

$$A^+(x^-, \mathbf{x}) \approx -\delta(x^-)\frac{1}{\boldsymbol{\nabla}_\perp^2}\rho(\mathbf{x}) \tag{199}$$

$$F^{+i} = \partial_i A^+ \tag{200}$$

This looks very nice, the big $+$-component of the field corresponds to a color current in the $+$-direction. This is precisely what we would expect in the physical situation of a color charged object (which the nucleus is, if you only look at it locally within a region of size $\ll 1/\Lambda_{\mathrm{QCD}}$ in the transverse plane) moving in the $+z$-direction, and generating a field A^+ that a leftmoving high energy probe would like to couple to. It is easy to see that the current is covariantly conserved, because it is independent of x^+, and simultaneously $A^- = 0$.

Note that the only nonzero component of the field strength tensor $F^{\mu\nu}$ is F^{+i}, which corresponds to F^{0i} and F^{zi}, i.e. chromoelectric and - magnetic fields perpendicular to the z-axis. This is the same as the Weizsäcker-Williams field of an electric charge boosted to the speed of light, which can be thought of as an "equivalent photon" cloud accompanying the photon.

The physical picture of "gluons as partons" requires two things

- Infinite momentum frame: we have to boost to a frame where the nucleus is moving fast, so that the partons are collinear (at least approximately) to the nucleus.
- Light cone gauge: in order to have a partonic interpretation we also have to gauge transform to *light cone gauge* $A^+ = 0$.

The choice of the light cone gauge is often just stated without any particular justification or deep reason. We can give two reasons for why we need to work in light cone gauge (this was also discussed in Sec. 8.1).

1. In order to talk about gluons as partons we need to quantize the gluon field. In high energy scattering one is probing the hadron or nucleus at an instant in light cone time, with an approximately light-like probe. This means that we want to light cone quantize the gluon field. Canonical quantization proceeds by imposing canonical commutation relations between fields $\phi(x)$ and their canonical conjugate variables. The canonical conjugate of a generalized coordinate is the derivative of the action with respect to the time derivative of the variable, so in light cone quantization

$$\pi = \frac{\delta}{\delta(\partial_+\phi)}\int \mathrm{d}^4x\mathcal{L}. \tag{201}$$

But since the QCD Lagrangian is built from the antisymmetric field tensor $F_{\mu\nu}$, the light cone time x^+ derivative of the component A^- is not present in the Lagrangian. In stead, $A^- = \ldots$ will be a constraint. We want to solve this constraint equation explicitly so that we can completely eliminate A^- from the theory and only work with physical degrees of freedom. This can be achieved using the $\nu = +$ component of the gauge field equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\nu, \tag{202}$$

where J^ν is the color current from the matter fields. Writing the $\nu = +$ component out we get

$$[D_-, F^{-+}] + [D_i, F^{i+}] = J^+. \tag{203}$$

Now by choosing the gauge $A^+ = 0$ we see that $F^{-+} = -\partial_- A^-$ is a simple function of A^- and does not even depend on its time derivative. Likewise the covariant derivative D_- is just a normal (spatial, so does not interfere with the dynamics) derivative, and altogether we have

$$(\partial_-)^2 A^- = [D_i, F^{i+}] - J^+, \tag{204}$$

from which we can solve, at any light cone time x^+ the constrained field A^- in terms of the physical degrees of freedom: A_i and the matter particle fields. Thus the light cone gauge $A^+ = 0$ conveniently enables a complete reduction of the 4 components of the gauge field to 2 dynamical degrees of freedom. This is similar to the way the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ enables one to solve A^0 as an explicit constraint in terms of the matter fields in the canonical quantization of QED.

2. In order to talk about partons, we need to be able to look at their longitudinal momentum p^+, which is some fraction of the longitudinal momentum of the proton or nucleus, $p^+ = xP^+$. Because p^+ is the conjugate variable to x^-, a parton distribution will be defined as a Fourier-transform from x^- to p^+ of a two-point function of fields. Schematically: the parton distribution for partons ϕ, characterized by some field operator $\hat{\phi}$ in a hadron h with momentum P^+ will be

$$f_\phi(x) \overset{?}{\sim} \int \mathrm{d}x^- e^{ixP^+x^-} \langle h| \hat{\phi}^\dagger(x^-)\hat{\phi}(0) |h\rangle \tag{205}$$

For the parton ϕ =gluon, the relevant operator would be $\hat{\phi} \sim F^{i+}$, where $i = 1, 2$ is connected to the polarization of the gluon. Now if the particle ϕ carries color, such a two-point function is not gauge invariant, because the gauge transformations of the field operators $\hat{\phi}^\dagger(x^-)\hat{\phi}(0)$ at different values of x^- do not cancel. Thus the proper, physical, measurable definition must include a Wilson line between the coordinates of the operators:

$$f_\phi(x) \sim \int \mathrm{d}x^- e^{ixP^+x^-} \langle h| \hat{\phi}^\dagger(x^-) \left[\mathbb{P}\exp\left\{ig\int_0^{x^-} \mathrm{d}y^- A^+(y^-)\right\}\right] \hat{\phi}(0) |h\rangle \,. \tag{206}$$

A consequence of this is that in an arbitrary gauge, the parton distribution for a parton ϕ does not have a clear interpretation as measuring just partons ϕ, but depends in a complicated way also on the gluon fields. But *if*, and only if, we choose the gauge $A^+ = 0$, the parton distribution of parton ϕ is just a two-point function of the field operator $\hat{\phi}$, and thus has (at least roughly) a physical interpretation as measuring the distribution of partons ϕ in the hadron.

So, let us now transform our field to the light cone gauge $A^+ = 0$. In addition to satisfying the requirement of a formal partonic interpretation in terms of a light cone quantization of the degrees of freedom in the nucleus, this gauge transformation will be required to calculate what happens in the collision of our nucleus with another one. We will come back to this in Sec. **??**. But already at this stage it is useful to note that the other colliding nucleus will have a color current J^-, and if our color field has a A^+-component it will cause the color current of the other nucleus to precess in color space. But if we transform our field to $A^+ = 0$-gauge, the current of the other nucleus can stay unaffected by it, leaving only the fields to interact. This is easier to deal with, in fact a major reason for using the classical field approximation in the first place is that classical fields are easier to deal with.

So let us now make this gauge transformation to $A^+ = 0$. We will keep our static approximation, nothing depends on x^+, but maintain a general x^--dependence remembering that the support of our color current in x^- is very narrow. Note that since there is no dependence on x^+, the gauge transform will not generate an A^--component. This should not be seen as a gauge choice, but as a result of the high energy kinematics.

Gauge transform:

$$A^+ \quad \Rightarrow \quad V^\dagger(\mathbf{x}, x^-)A^+V(\mathbf{x}, x^-) - \frac{i}{g}V^\dagger(\mathbf{x}, x^-)\partial_- V(\mathbf{x}, x^-) = 0 \tag{207}$$

$$A^- \quad \Rightarrow \quad -\frac{i}{g}V^\dagger(\mathbf{x}, x^-)\partial_+ V(\mathbf{x}, x^-) = 0, \text{ still} \tag{208}$$

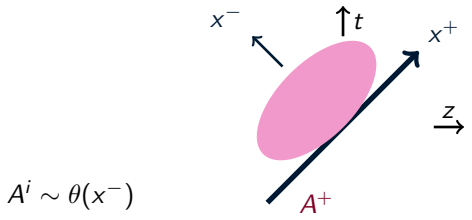
$$A^i \quad \Rightarrow \quad \frac{i}{g}V^\dagger(\mathbf{x}, x^-)\partial_i V(\mathbf{x}, x^-) \qquad \textit{transverse pure gauge} \tag{209}$$

(207) solved by

$$V(\mathbf{x}, x^-) = \mathbb{P}\exp\left[-ig\int_{-\infty}^{x^-} \mathrm{d}y^- A^+\right]$$

$\Longrightarrow$ Wilson line!

Now since $A^+(y^-)$ is very localized around $y^- = 0$, the Wilson line V jumps very rapidly from unity (independently of the transverse coordinate) to a nontrivial $\mathbf{x}$-dependent value around $y^- = 0$. Thus its derivative, which gives A^i, has a $\sim \theta(x^-)$-function like discontinuity around $x^- = 0$. Now our small-x gluons look like we would expect them to look like based on the Heisenberg uncertainty principle: they are small p^+ gluons and thus should be delocalized in x^-. This was not the case in the covariant gauge.



$A^i \sim \theta(x^-)$

The current also is gauge transformed as

$$J^\mu \to V^\dagger J^\mu V. \tag{210}$$

This, however, does not change the region of support in x^-: the charges are still localized as $\sim \delta(x^-)$ as we want. Usually practical calculations are in fact (implicitly) done in terms of the covariant gauge charge density ρ that determines the Wilson lines, even if for the gauge fields one uses light cone gauge.

10.2 McLerran-Venugopalan model

In order to be able to actually calculate things we will now need a concrete model for what the charge density ρ in Eqs. (198), (199) could look like. Such a model is provided by the Mclerran-Venugopalan (MV) model [32, 33, 34]. The MV model is not the same thing as the CGC, but it is a model that makes sense within the CGC classical field picture.

Let us briefly review the physics argument behind this model. The situation one is thinking about is that of small enough x so that the CGC spacetime picture discussed above is valid; but not necessarily asymptotically small. Instead, we are working in the limit where the nuclear mass number A is very large. This implies that at a fixed coordinate in the transverse plane there is an parametrically large number $\sim A^{1/3}$ of valence quark-like colored degrees of freedom. Since these color charges come from separate nucleons, they should be uncorrelated with each other in the longitudinal direction. Also in the transverse plane, at two different coordinates, the total color charge comes from a sum of at only partially overlapping color charges in the different nucleons, and is therefore different. Thus one can also assume the color charges at different transverse coordinates to be uncorrelated. Finally, since the color charge at one coordinate is a sum of many uncorrelated color charges, the central limit theorem states that its distribution should be Gaussian.

MV model: charge density $\rho(\mathbf{x})$ is

- stochastic, Gaussian random
- local in x^- and $\mathbf{x}$

$$\left\langle \rho^a(\mathbf{x}, x^-)\rho^b(\mathbf{y}, y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{211}$$

The importance of this model is that it is based on a very general physical argument, but is nevertheless is very explicit, enabling one to calculate a large variety of things in various approximations. It is also simple in terms of parameters. So far $\mu^2(x^-)$ is just some unknown function, but should be thought of as being rather narrowly peaked in x^-: $\mu^2(x^-) \sim \delta(x^-)$. It will turn out that the physics in the high energy limit does not care about the functional form, but everything will just depend on the integral of $\mu^2(x^-)$ over x^-, which is essentially the saturation scale

$$Q_s^2 \sim g^4 \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-). \tag{212}$$

Digression: Powers of g

There are different conventions for where to put powers of g. Here, μ^2 is a number density. From this we construct ρ as a charge density, thus $\rho \sim g\mu$. Then, from the gluon field $A^+ \sim \rho/\boldsymbol{\nabla}_\perp^2 \sim g\mu$, one calculates the Wilson line which involves $gA^+ \sim g^2\mu$. Thus in fact physics only depends on the combination $g^2\mu$ in the end.

For further developments it is important to note that one is making two independent approximations that need to be considered separately.

- We assumed that the charge density correlator is proportional to $\delta^{(2)}(\mathbf{x}-\mathbf{y})$. This, as we will see, leads to a very specific dependence of the unintegrated gluon distribution $\varphi^{\mathrm{WW}}(\mathbf{k})$ on k_T at high momentum. The resulting distribution is roughly consistent with a DGLAP-like behavior of the gluon distribution, but not for example with the solution of the BK equation at very small-x. This assumption is therefore usually thought of as appropriate for moderately small x and large A, but not parametrically small x.
- The other assumptions; namely that the charges at different x^- are uncorrelated, and that the distribution is Gaussian (which implies a specific relation between the higher 4-, 6- etc. correlators and the 2-point function (211)) are much more general. These assumptions can be used to to calculate relations between different Wilson line correlators, and in the cases where this has been checked (see e.g. [35]) this assumption has been found to be consistent with JIMWLK evolution. What this implies in practice is that one can use the assumption of locality in x^- and Gaussian correlators to calculate multiparticle correlations, and combine this with a transverse coordinate structure that is not given by $\delta^{(2)}(\mathbf{x}-\mathbf{y})$, but taken e.g. from the solution of the BK equation. This procedure or approximation is often used in the literature (see e.g. [36, 37, 38, 39, 40]). Depending on the authors it is referred to in the literature as the "nonlinear Gaussian" [35], "nonlocal Gaussian" [38] or "Gaussian truncation" [10] approach.

10.3 Dipole cross section

In order to get a more physical picture of what the MV model means, let us now use it to calculate two kinds of unintegrated gluon distributions, the dipole distribution or dipole amplitude (appearing in the hybrid formalism in Sec. 11), and the so-called Weizsäcker-Williams (WW) distribution.

Dipole

$$S(r=|\mathbf{x}-\mathbf{y}|) \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\, V(\mathbf{x})V^\dagger(\mathbf{y})\right\rangle \qquad V(\mathbf{x},x^-) = \mathbb{P}\exp\left[-ig\int_{-\infty}^{x^-} \mathrm{d}y^- A^+\right] \tag{213}$$

$$\left\langle A_a^+(\mathbf{x},x^-)A_b^+(\mathbf{y},y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)L(\mathbf{x}-\mathbf{y}) \tag{214}$$

Here we have introduced a notation $L(\mathbf{x}-\mathbf{y})$ for the two-point function of the covariant gauge field A^+, which is related to the color charge density by Eq. (199). Since our MV model color charge density has a two-point function $\sim \delta(x^- - y^-)$, also the two point function of the fields will be proportional to the same delta function. The transverse coordinate dependence is more complicated because of the 2-dimensional Laplacian, therefore we just denote it by L and come back to it later.

In order to make sense of the path ordered exponential, it is not in practice very convenient to use the path-ordered product definition (37). In stead, we will discretize the integral over x^- into many infinitesimally thin pieces of width $\Delta^- \to 0$. Now a path ordered exponential of an integral turns into a path ordered exponential of a sum, which is nicely interpreted as a path ordered product of exponentials of individual terms in the sum.

Discretize

$$x^- = n\Delta^- \quad\Longrightarrow\quad \left\langle A_{ma}^+(\mathbf{x})A_{nb}^+(\mathbf{y})\right\rangle = g^2\delta^{ab}\mu_n^2 \overbrace{\frac{1}{\Delta^-}}^{\delta(x^- - y^-)}\delta_{mn}\, L(\mathbf{x}-\mathbf{y}) \tag{215}$$

$$V(\mathbf{x}) = \lim_{N\to\infty} e^{-ig\Delta^- A_N^+(\mathbf{x})}\cdots e^{-ig\Delta^- A_n^+(\mathbf{x})}\cdots e^{-ig\Delta^- A_{-N}^+(\mathbf{x})} \tag{216}$$

Plugging the infinite produce representation of the Wilson lines into the dipole operator (213) we obtain

$$S(r) = \lim_{N\to\infty} \frac{1}{N_c} \mathrm{tr} \Big\langle e^{-ig\Delta^- A_N^+(\mathbf{x})} \cdots e^{-ig\Delta^- A_n^+(\mathbf{x})} \cdots e^{-ig\Delta^- A_{-N}^+(\mathbf{x})} \times e^{ig\Delta^- A_{-N}^+(\mathbf{y})} \cdots e^{ig\Delta^- A_n^+(\mathbf{y})} \cdots e^{ig\Delta^- A_N^+(\mathbf{y})} . \Big\rangle \quad (217)$$

Notice that if the path ordering in the Wilson line is from right to left, in the hermitian conjugate the corresponding ordering is from left to right. Thus in the trace (217) the elements at the lower limit in x^- are next to each other.

Now according to the MV model assumption the color fields at different steps in x^- are indpendent of each other. This means that the expectation value of the product is the product of expectation values, and we can start calculating the expectation values one rapidity step at the time, starting with the innermost $n = -N$. Now we also use the fact that Δ^- is small and expand

$$\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x})} e^{ig\Delta^- A_n^+(\mathbf{y})} \right\rangle \approx \Big\langle 1 - ig\Delta^- A_n^+(\mathbf{x}) + ig\Delta^- A_n^+(\mathbf{y}) + g^2(\Delta^-)^2 A_n^+(\mathbf{x}) A_n^+(\mathbf{y}) - \frac{1}{2} g^2 (\Delta^-)^2 \left(\left(A_n^+(\mathbf{x})\right)^2 + \left(A_n^+(\mathbf{y})\right)^2 \right) \Big\rangle + \mathcal{O}(\Delta^-)^{3/2} \quad (218)$$

Here the power counting combines the explicit Δ^- in the exponent with

$$A^+ \sim \frac{1}{\sqrt{\Delta^-}}, \quad (219)$$

which is to be understood as following in a r.m.s. sense from the discretized two-point function (215).

The expectation value of a single A^+ vanishes and we use (215) for the two point function:

$$\begin{aligned} &\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x})} e^{ig\Delta^- A_n^+(\mathbf{y})} \right\rangle \\ &\quad \approx 1 + g^4 (\Delta^-)^2 \frac{\mu_n^2}{\Delta^-} \Big[L(\mathbf{x}-\mathbf{y}) - \frac{1}{2} L(\mathbf{x}-\mathbf{x}) - \frac{1}{2} L(\mathbf{y}-\mathbf{y}) \Big] \overbrace{t^a t^a}^{=C_F \mathbb{I}_{N_c\times N_c}} \\ &\quad \approx \exp\left\{ g^4 \Delta^- C_F \mu_n^2 \underbrace{\Big[L(\mathbf{x}-\mathbf{y}) - \frac{1}{2} L(\mathbf{x}-\mathbf{x}) - \frac{1}{2} L(\mathbf{y}-\mathbf{y}) \Big]}_{-\Gamma(\mathbf{x}-\mathbf{y})} \right\} \mathbb{I}_{N_c\times N_c} \end{aligned} \quad (220)$$

Here we have defined a new notation Γ for the field-field correlator L minus its value at the origin. As we will see, the value of L at zero has a power law (=bad) IR divergence. This subtraction removes this leading divergence.

Now that this expectation value is just a number times the identity matrix, it commutes with all the other infinitesimal Wilson line factors in the infinite product (217). We can therefore pull it out and repeat the same procedure for the next factor (next n). Doing this one by one we finally get just a product of terms like (220), each of them proportional to the identity matrix. The overall trace now just gives a factor N_c that cancels with the normalization. We can now return from the discrete to the continuous notation as

$$S(r) = \exp\left\{ -g^4 C_F \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-) \Gamma(\mathbf{x}-\mathbf{y}) \right\} \quad (221)$$

This is now practically our final result. Note that, as advertized, the Wilson line correlator only depends on the integral of $\mu^2(x^-)$ over x^-; at this stage the functional form (which we never specified) of the dependence on x^- does not matter any more.

The only thing we have used so far from the MV model is the more general of the two approximations involved: the independence of different x^- color charges and the Gaussian expectation value [3]. We have not

[3] In fact we did not even use the Gaussian property, because we expanded in powers of Δ^- and only used the two-point function.

yet actually used the assumption that the correlator (211) is proportional to $\delta^{(2)}(\mathbf{x}-\mathbf{y})$ at all. Therefore the result (221) can be used independently of the MV model assumption. In particular, we can take a solution of the BK equation for $S(r)$, and consider (221) as the definition of the correlator $\Gamma(\mathbf{x}-\mathbf{y}) \equiv -L(\mathbf{x}-\mathbf{y})+L(\mathbf{0})$. But in order to see what the more restrictive local charge correlation assumption gives us, let us now calculate $\Gamma(\mathbf{x}-\mathbf{y})$ in the actual model, recalling that physically we expect this to be a good approximation in the limit of smallish x and very large A. In order to get the A^+ correlator from the MV model assumption for the correlator of ρ's we need to solve the Poisson equation (199). This is done by Fourier-transforming; we also need to introduce an additional IR cutoff to deal with the very divergent integrals in intermediate stages of our calculation. For now we will suppress the argument x^- for brevity, the following steps happen locally in x^-.

The covariant gauge field A^+ is related to the color charge by the Poisson equation (199). This equation is best solved by Fourier-transforming:

At each x^-

$$A^+(k_T) = \frac{1}{\mathbf{k}^2}\rho(\mathbf{k}) \tag{222}$$

To work in momentum space we also need the Fourier-transform of the color charge 2-point function (211)

$$\begin{aligned}\left\langle \rho^a(\mathbf{k})\rho^b(\mathbf{p})\right\rangle &= g^2\delta^{ab}\mu^2\int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y}e^{i\mathbf{k}\cdot\mathbf{x}+i\mathbf{p}\cdot\mathbf{y}}\delta^{(2)}(\mathbf{x}-\mathbf{y}) \\ &= g^2\delta^{ab}\mu^2(2\pi)^2\delta^{(2)}(\mathbf{k}+\mathbf{p})\end{aligned} \tag{223}$$

Combining the relation between A^+ and ρ, and the two-point function of ρ's, we get

$$\begin{aligned}\Gamma(\mathbf{x}-\mathbf{y}) &= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}(2\pi)^2\frac{\delta^{(2)}(\mathbf{k}+\mathbf{p})}{{p_T}^2{k_T}^2}\left[1-e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{p}\cdot\mathbf{y}}\right] \\ &= \frac{1}{2\pi}\int \mathrm{d}k\frac{1-J_0(kr)}{k^3} \approx \frac{1}{8\pi}r^2\ln\overbrace{\frac{1}{r\Lambda}}^{\gg 1}\end{aligned} \tag{224}$$

This integral is still IR divergent at $k=0$, but only logarithmically so. The leading power law divergence cancels [4] in the combination $\Gamma(\mathbf{x}-\mathbf{y}) \equiv -L(\mathbf{x}-\mathbf{y})+L(\mathbf{0})$. This is true generally: expectation values of nonsinglet operators are very very sick because they depend on $L(\mathbf{x}-\mathbf{y})$ alone. They diverge (or vanish) as an exponential function of a power of the IR cutoff; they do not describe meaningful physical quantities. Expectation values of singlet operators, such as the dipole that we are calculating here, depend always on the combination $\Gamma(\mathbf{x}-\mathbf{y})$ which, in the MV model, is only logarithmically divergent. One can always argue away the logarithm for coordinate space quantities like $S(r)$, but it does affect the analytical properties and therefore the momentum space behavior, so it is better to keep it for now. Note that since Λ in Eq. (224) is an IR cutoff in the k-integral, we should assume that it is always small compared to any coordinate value that we are interested in: $r\Lambda \ll 1$. The full $\Gamma(r)$ is, however finite, because of the power of r multiplying the logarithm.

Different ways of regulating the IR divergence lead to different values for the remaining finite term. The leading log, however, is universal. Our result is for the dipole cross section is now:

$$S_{MV}(r) \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\,V(\mathbf{x})V^\dagger(\mathbf{y})\right\rangle = \exp\left\{-\frac{g^4C_F}{8\pi}\left[\int_{-\infty}^{\infty}\mathrm{d}x^-\mu^2(x^-)\right]r^2\ln\frac{1}{r\Lambda}\right\} \tag{225}$$

$$Q_s^2 \sim \frac{g^4C_F}{4\pi}\left[\int_{-\infty}^{\infty}\mathrm{d}x^-\mu^2(x^-)\right] \tag{226}$$

At least for a Gaussian correlator some anomalously large expectation values of higher orders in the series in powers of Δ^- do not matter, but for this it is enough that the higher point correlators are not too large, they don't need to be exactly Gaussian. In fact it is probably better to think of the Gaussianity as arising via the central limit theorem; a color charge that is built up of layers in x^- that are independent of each other will behave as a Gaussian.

[4] It is curious that a small-momentum, i.e. IR, divergence manifests itself in the short-distance behavior of the two-point function.

This provides a concrete link between the MV model variable μ^2 (as already promised, this only enters as the integral over x^-) and the dipole cross section, e.g. the quar-nucleus scattering or DIS cross sections. We can now make a fit to DIS data, use that to extract a value for μ^2, and apply this to calculating heavy ion collisions: this is indeed what is done in the very successful IPglasma model for the initial stage of a heavy ion collision [41].

The dipole cross section interpolates between 1 at $r = 0$ and 0 at $r \to \infty$ as it should, satisfying the expectations for a SU(3) matrix correlator. We can identify the saturation scale Q_s from the characteristic scale in this r-dependence. Here we calculated the dipole amplitude in the fundamental representation, which is why our result has the Casimir operator C_F, overall the saturation scale should have *Casimir scaling*, i.e. be proportional to the relevant Casimir. The glasma fields in the initial stage of a heavy ion collision are gluons, there the relevant scale is the adjoint Casimir $C_\mathrm{A} = N_\mathrm{c}$ which, in the MV model, is larger than the saturation scale measured e.g. in DIS. In addition to the color factor, the value of Q_s depends on the precise convention chosen to determine it from the dipole correlator (there are many ways to extract something so vaguely defined as "the" characteristic scale from a function). In the MV model specifically, it also depends on the details of how one regularized the IR divergence in the momentum integral, this adds another uncertainty factor into the interpretation of MV model calculations [42].

In momentum space, the Fourier-transform of the dipole expectation value, multiplied by ${k_T}^2$, is known as the dipole gluon distribution

$$\varphi_{\mathrm{dip.}}(k_T) \sim {k_T}^2 \int \mathrm{d}^2\mathbf{r} e^{i\mathbf{r}\cdot\mathbf{k}} S(r) \tag{227}$$

Thanks to the logarithm of r in the MV model expression, it behaves [Exercise!] like $1/{k_T}^2$ at large k_T. At small k_T, the dipole distribution behaves as $\sim {k_T}^2$ (this is easy to see, since the integral of $S(r)$ over $\mathbf{r}$ is finite, therefore $S(k_T = 0)$ is a constant and $\varphi_{\mathrm{dip.}}(k_T) \sim {k_T}^2 S(k)$).

10.4 Weizsäcker-Williams distribution

To characterize the number distribution of gluons in the nucleus one needs, as discussed above, to look at a 2-point function of the gluon field in the light cone gauge. Note that now (contrary to the dipole picture of DIS or the hybrid formulation of proton-nucleus scattering) we trying to analyze gluons, quanta, in the target nucleus, not the probe. It would be difficult to light cone quantize both at the same time because we cannot simultaneously fix the gauge to be both $A^+ = 0$ and $A^- = 0$.

This gluon number density corresponds to what is know as the Weizsäcker-Williams gluon distribution. The full calculation in the MV model was first done in [43], following a procedure very similar to the above. Repeating this would take up too much time in this lecture, so we will just quote the result below. But before that it is useful to look at the *dilute limit* of the calculation. This can be derived by assuming that $\rho \sim g^2\mu$ is small, in which case one can expand the Wilson lines to lowest order. Since in the MV model everything is expressed in terms of $A^+ \sim \rho/\boldsymbol{\nabla}_\perp^2$, the dilute limit of small $g^2\mu$ is at the same time the limit of large transverse momenta.

High $\mathbf{k}$ — small μ:

$$\frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = \varphi^{\mathrm{WW}}(\mathbf{k}) \sim \left\langle A_a^i(\mathbf{k}) A_a^i(-\mathbf{k}) \right\rangle \tag{228}$$

$$A_a^i(\mathbf{k}) = \frac{k^i}{{k_T}^2}\rho^a(\mathbf{k}) \quad \Longrightarrow \quad \left\langle A_a^i(\mathbf{k}) A_a^i(-\mathbf{k}) \right\rangle = \frac{1}{g^2} \overbrace{(2\pi)^2\delta^{(2)}(\mathbf{k}=\mathbf{0})}^{\pi R_A^2} ({N_\mathrm{c}}^2 - 1) \frac{\overbrace{g^4\mu^2}^{\sim Q_\mathrm{s}^2}}{\mathbf{k}^2} \tag{229}$$

Here we used the momentum space correlator (223) and the fact that in the dilute limit the light cone gauge transverse gauge field in (209) reduces to just a derivative of covariant gauge A^+ field. This gluon distibution reduced in a very natural way to the DGLAP-evolved integrated gluon distribution, for which the definition in terms of the quantized light cone gauge gluonic field operators leads to

$$xG(x,Q^2) = \int^{Q^2} \mathrm{d}^2\mathbf{k} \frac{\mathrm{d}N}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} \sim \pi R_A^2 ({N_\mathrm{c}}^2 - 1) \frac{Q_\mathrm{s}^2}{g^2} \ln Q^2, \tag{230}$$

i.e. a function that grows logarithmically with Q^2. This growth can be identified with the growth in the number of gluons in DGLAP evolution when increasing the resolution scale Q^2.

The result from [43] (see e.g. [44] for a mathematically clearer but more formal derivation) for the Weizsäcker-Williams gluon distribution is

WW distribution

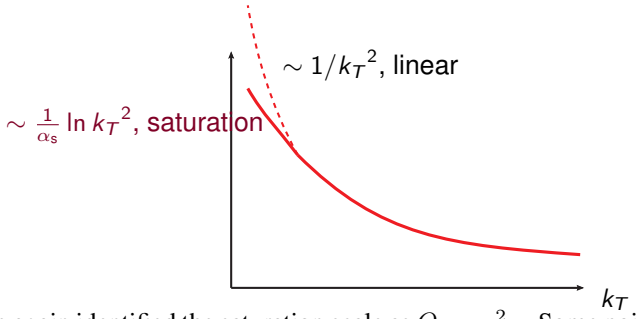
$$A^i = \frac{i}{g} V(\mathbf{x}) \partial_i V(\mathbf{x}) \tag{231}$$

$\Longrightarrow$

$$\left\langle A_a^i(\mathbf{x}) A_b^i(\mathbf{y}) \right\rangle = \delta^{ab} \frac{N_c}{g^2} \frac{\boldsymbol{\nabla}_\perp^2 \Gamma(\mathbf{x}-\mathbf{y})}{\Gamma(\mathbf{x}-\mathbf{y})} \left(\exp\left\{ -g^4 N_c \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-) \Gamma(\mathbf{x}-\mathbf{y}) \right\} - 1 \right) \tag{232}$$

$\Longrightarrow$

$$\varphi_{WW}(k_T) \sim \begin{cases} \frac{1}{\alpha_s} \ln k_T, & k_T \ll Q_s^2 \\ \frac{1}{\alpha_s} \frac{Q_s^2}{k_T^2}, & k_T \gg Q_s^2. \end{cases} \tag{233}$$



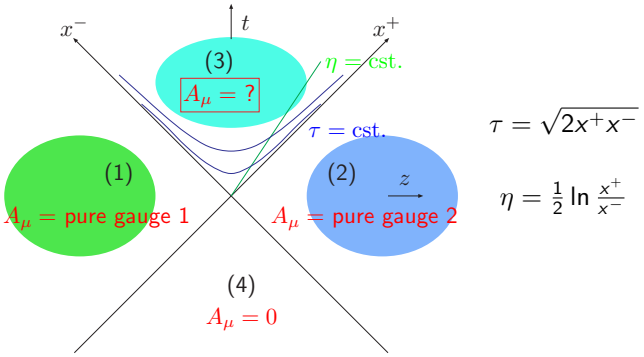
Here we have again identified the saturation scale as $Q_s \sim g^2\mu$. Some points to note about the expression (233):

- The factor in the exponent is an adjoint representation Casimir C_A, compared to the fundamental Casimir $C_F = (N_c{}^2 - 1)/(2N_c)$ in the fundamental representation dipole (225). This is what one would expect for gluons as opposed to a fundamental representation probe.
- Both the dipole gluon distribution (227) and the WW distribution (233) have (when properly normalized) the same large k_T limit. However, they look quite different at small k_T. Both exhibit saturation: the behavior changes at $k_T \lesssim Q_s$; the dipole distribution changes to $\sim k_T{}^2$ and the WW distribution to $\sim \ln k_T$. You can deduce the latter property by noting that due to the 1 inside the bracket in (231), the integral over r of the coordinate space distribution diverges logarithmically; this would be the $k_T = 0$ value of the momentum space distribution.
- In a strict operatorial sense the dipole and WW distributions are independent quantities, some cross sections depend on one, some on the other. Without evoking a model such as MV one has to measure both of them separately experimentally. The MV model (or in general the nonlinear Gaussian assumption) provides an additional physical model that allows one to relate them to each other. For a discussion explaining how these two distributions result in the small x limit from transverse momentum distributons (TMD's) that have explicitly gauge invariant operatorial definitions, see [40].
- The WW distribution is, when one writes it in terms of the saturation scale Q_s in stead of the model parameter μ, proportional to $1/g^2$ or $1/\alpha_s$. The WW distribution is the best we can do if we try to make a bona fide effort to count gluons. In the CGC field the result is that, at weak coupling $\alpha_s \to 0$, there is a nonperturbatively large number of them. This is the justification for the word *condensate* in the CGC.

10.5 Glasma

Now let us finally move to the collision of two high energy nuclei. We follow here a calculation that was originally done in Ref. [45]. We will represent both nuclei in a symmetrical way, by their own classical color currents and radiated fields, with nucleus (1) moving in the positive z-direction and nucleus (2) in the negative one.

$$J^\mu = \underbrace{\delta^{\mu+}\rho_{(1)}(\mathbf{x})\delta(x^-)}_{A^i\sim\theta(x^-)} + \underbrace{\delta^{\mu-}\rho_{(2)}(\mathbf{x})\delta(x^+)}_{A^i\sim\theta(x^+)} \tag{234}$$



Let us recall the expressions for these "transverse pure gauge" fields without the θ-functions, which we will deal with separately.

$$A^i_{(1,2)} = \frac{i}{g} V_{(1,2)}(\mathbf{x})\partial_i V^\dagger_{(1,2)}(\mathbf{x}) \tag{235}$$

$$V_{(1,2)}(\mathbf{x}) = Pe^{ig\int dx^\mp \frac{\rho(\mathbf{x},x^\mp)}{\nabla^2_\perp}} \tag{236}$$

Equations (235) and (236) provide valid solutions to the equations of motion in the regions (1) and (2) in the figure above, which are causally connected to only one of the two nuclei. Let us now derive an initial condition for the field inside the future light cone, which is causally connected to both. *Inside* the region (3) there is no current (since the color currents are proportional to delta functions)

Region (3):

- Inside: $[D_\mu, F^{\mu\nu}] = 0$; no source
- Need initial conditions at $\tau = 0$
- $\sqrt{s} \to \infty$: should be boost invariant: $A_\mu(\tau, \mathbf{x})$

Gauge choice:

- Avoid color precession: $A^- = 0$ at $x^- = 0$ where J^+ lives, + vice versa
- Boost invariant: τ, η-components

$\Longrightarrow$ Fock-Schwinger $A_\tau = \left(x^+A^- + x^-A^+\right)/\tau = 0$

The Fock-Schwinger or temporal gauge condition has the very nice property that on the x^+-axis, i.e. $x^- = 0$, it reduces to $x^+A^- = 0$, i.e. $A^- = 0$. Thus the current that lives on this light cone is not only covariantly conserved $[D_\mu, J^\mu] = \partial_+ + ig[A^-, J^+] = 0$, but in fact does not color precess and is constant $\partial_+ J^+ = 0$. This significantly simplifies our following calculation. The other advantage of the $A_\tau = 0$-condition is that in order to solve the equations of motion numerically in a Hamiltonian formalism, which is the standard thing to do, one needs to choose a temporal gauge condition. There are a few calculations of gluon production inside the forward light cone in another gauge, the covariant gauge, but this is done only in the dilute limit, not including any final state interactions [46, 47]. I am not aware of a numerical implementation of the full equations in another gauge.

Now, as discussed above, we need an initial condition for the field inside the future light cone. For the case of one nucleus we could have two components equal to zero, one as a gauge choice and the other one as a result of our current being independent of the light cone time x^+. In the case of two nuclei there is a dependence on both x^- and x^+ in the problem, and we cannot have this "accidental" vanishing of one component of the gauge field any more. So we can only make one component of the gauge field vanish with a gauge choice, but our solution inside the future light cone will have nonzero values for the other three. With $A_\tau = 0$, this third component is the longitudinal field orthogonal to it, namely A_η or $A^\eta = -A_\eta/\tau^2$.

In the following we will use the latter, because A^η is finite at $\tau = 0$, and consequently $A_\eta(\tau = 0) = 0$.

Since the currents exist only on the light cones, the field can have discontinuities across the light cone. We can write an ansatz for the solution in terms of θ functions. Inserting this into the equations of motion will yield δ-functions arising from derivatives of the θ-functions. We can determine the initial conditions for the field at $\tau = 0^+$ by requiring that the coefficients of these δ-functions vanish. This is enough to determine the initial condition, we do not need the terms that do not have a $\delta(x^\pm)$.

$$\text{Ansatz: } A_i = \overbrace{A_i^{(1)}(\mathbf{x})\theta(-x^+)\theta(x^-) + A_i^{(2)}(\mathbf{x})\theta(x^+)\theta(-x^-)}^{\text{known}} + A_i^{(3)}(\mathbf{x},\tau)\theta(x^+)\theta(x^-) \quad (237)$$

$$A^\pm = \pm\theta(x^+)\theta(x^-)x^\pm A^\eta(\mathbf{x},\tau) \quad (238)$$

$$[D_\mu, F^{\mu\nu}] = J^\nu \quad \text{match} \quad \delta(x^\pm) \quad (239)$$

$\delta(x^-)\delta(x^+)$ in:

$$J^i = \partial_-\partial_+ A^i + \cdots = \delta(x^-)\delta(x^+)(-A_i^{(1)}(\mathbf{x}) - A_i^{(2)}(\mathbf{x}) + A_i^{(3)}(\mathbf{x})) = 0 \quad (240)$$

To get a contribution like $\delta(x^-)\delta(x^+)$ we need two derivatives $\partial_-\partial_+$ acting on the theta functions $\theta(\pm x^-)\theta(\pm x^+)$. When acting on the transverse component A^i such derivatives appear in the $J^i = 0$ equation. In principle we have derivatives like $\partial_-\partial_+$ acting on $\theta(\pm x^-)\theta(\pm x^+)$ in $A^\pm$ in the equations with $J^\pm$. However, in this case the contribution is killed by the additional $x^\pm$ in the relation between $A^\pm$ and A^η, which gives terms like $\delta(x^-)\delta(x^+)x^\pm = 0$. We now have our first equation.

Next, we have to start looking at terms like $\delta(x^\pm)\theta(x^\mp)$, where one derivative is acting on a theta function and other derivatives on something else. There are such terms in the $J^i = 0$ equation too, but it turns out they do not give us new constraints on the fields in region (3). Instead, we have to look at the equations for $J^\pm$. Since $+$ and $-$ are symmetric, we can look at either J^+ or J^-; let us look at the first one. Now $J^+ \sim \delta(x^-)$, so we want to be looking for terms $\sim \delta(x^-)$ in the equation for J^+. There are several:

$\delta(x^-)\theta(x^+)$ in J^+ equation

$$\partial_- F^{-+} = \left[\partial_-\partial_+(\overbrace{x^+\theta(x^+)\theta(x^-)}^{A^+}) - \partial_-\partial_-(\overbrace{-x^-\theta(x^+)\theta(x^-)}^{A^-})\right] A^\eta(\mathbf{x},\tau) \quad (241)$$

$$= 2\delta(x^-)\theta(x^+)A^\eta(\mathbf{x},\tau) + \ldots \quad (242)$$

$$\partial_i(\overbrace{\partial_- A_i}^{F^{i+}}) = \delta(x^-)\theta(x^+)\partial_i(\overbrace{-A_i^{(2)}(\mathbf{x}) + A_i^{(3)}(\mathbf{x})}^{A_i^{(1)}}) = J^+|_{x^+>0} - \ldots \quad (243)$$

$$ig[A_i, \overbrace{\partial_- A_i}^{F^{i+}}] = ig[\overbrace{\theta(-x^-)A_i^{(2)} + \theta(x^-)A_i^{(3)}}^{A_i^{(2)}+\theta(x^-)A_i^{(1)}}, \overbrace{-A_i^{(2)}(\mathbf{x}) + A_i^{(3)}(\mathbf{x})}^{A_i^{(1)}(\mathbf{x})}]\delta(x^-)\theta(x^+) \quad (244)$$

$$= ig[A_i^{(2)}, A_i^{(1)}(\mathbf{x})]\delta(x^-)\theta(x^+) + \ldots \quad (245)$$

Here we have only considered the $\theta(x^+)$ part of the $x^- = 0$ light cone, and thus the $A_i^{(1)}$ part of the ansatz (237) does not appear immediately, but only after eliminating $A_i^{(3)}$ using the first matching condition (240). The $\ldots$ on the r.h.s. of Eq. (243) are a reminder of the fact that the equation of motion for the nucleus (1) also has a contribution from the nonlinear term $\sim [A_i^{(1)}, \partial_- A_i^{(1)}]$; as discussed in Sec. (10) one needs to spread out the δ function in x^- to treat this term properly, which we are not doing here. This does not, however generate a contribution that would affect the initial condition for A^η. In relating F^{i+} to A_i note both the antisymmetry of $F^{\mu\nu}$ and the Lorentz signature $A_i = -A^i$.

Requiring that all the equations are satisfied leads us to the initial conditions for the glasma fields first derived in Ref. [45]:

$$A_i^{(3)}|_{\tau=0} = A_i^{(1)} + A_i^{(2)} \quad (246)$$

$$A^\eta|_{\tau=0} = ig\left[A_i^{(1)}, A_i^{(2)}\right] \quad (247)$$

11 Hybrid formalism for pA collisions

11.1 Forward rapidity in proton-nucleus collisions

Now we have developed the concept of measuring the gluon field in a dense target by measuring the scattering amplitude of a simple elementary colored probe going through this target. What would be a possible scattering experiments for doing this? It will turn out in the end that, in many ways, the cleanest such a process is deep inelastic scattering. However, it is easier for us to start with another process that looks even simpler at first, which is single inclusive hadron production in proton-nucleus collisions at forward rapidity. For the purposes of our discussion here this is indeed the simplest process. When doing actual phenomenology or higher order calculations it has complications that DIS does not have (collinear divergences, parton distributions, fragmentation functions, real-virtual cancellations, ...), but we will not encounter them in this first, introductory discussion.

Experimental data that we might want to address could be something like the nuclear modification ratio R_{pA}. Some experimental measurements for this quantity from ALICE [48] and LHCb [49, 50] are shown in in Fig. 3.

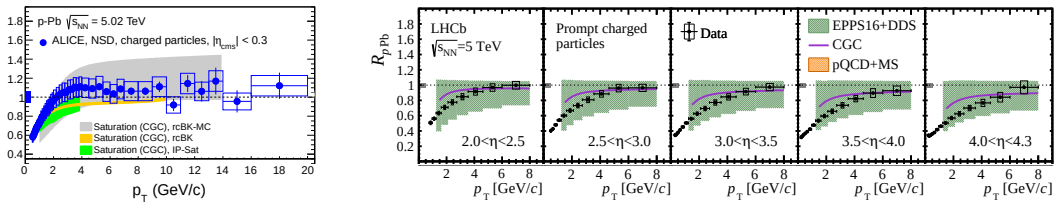
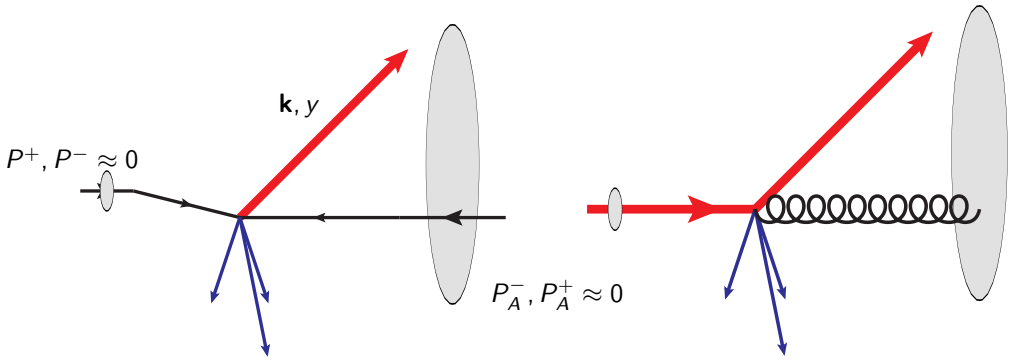


Figure 3: Nuclear modification ratio R_{pA} from ALICE [48] at midrapidity $y \approx 0$ and LHCb [49, 50] at forward rapidities.

The nuclear modification ratio R_{pA} is the ratio of a cross section for something in proton-nucleus collisions, divided by the mass number A times the corresponding cross section in proton-proton collisions. We expect saturation effects to be more important in nuclei than in protons and more important at small x than at large x. Thus a suitable place to look for them is in a collision system where a large x parton, preferably a quark, collides with small-x degrees of freedom. This, very generically, happens at forward rapidity (i.e. in the proton-going direction) in a proton-nucleus collision. To see this let us take a quick look at the kinematics. We consider a proton moving in the positive z direction, with a large momentum P^+ and very small $P^- = m_p^2/(2P^+)$. It collides with a heavy ion in the negative z direction, with a large P_A^- and a small $P_A^+ = m_A^2/(2P_A^-)$. Usually with nuclear collisions one calculates energies per nucleon, and thus the center of mass collision energy per nucleon is $s = (P_A + P)^2/A \approx 2P \cdot (P_A/A) = 2P^+P_A^-/A$. For simplicity let us assume that we are in the proton-nucleon center of mass frame. This means that the longitudinal momentum of the proton (P^+) is the same as the longitudinal momentum per nucleon in the nucleus (P_A^-/A). At the LHC, this is not exactly the frame where the detectors sit, but close enough for the purposes of this discussion; for a more careful calculation for LHC data one needs to take this small difference into account better.

We now produce and measure a particle with transverse momentum $\mathbf{k}$ at rapidity y, with y large. Four-momentum must be conserved. Since the proton has very little p^-, all the k^- of the produced particle must come from the nucleus. Similarly all the k^+ of the produced particle must come from the proton. If the produced particle is massless, we calculate its plus and minus momenta as follows.



$$\text{C.M.S} \implies P^+ = \frac{P_A^-}{A} = \sqrt{\frac{s}{2}} \tag{248}$$

$$k^+ = \frac{1}{\sqrt{2}} k_T e^y = \frac{k_T e^y}{\sqrt{2}P^+} P^+ = \frac{\overbrace{k_T e^y}^{x_p \lesssim 1}}{\sqrt{s}} P^+ \tag{249}$$

$$k^- = \frac{1}{\sqrt{2}} k_T e^{-y} = \frac{k_T e^{-y}}{\sqrt{2}(P_A^-/A)} \frac{P_A^-}{A} = \underbrace{\frac{k_T e^{-y}}{\sqrt{s}}}_{x_A \ll 1} \frac{P_A^-}{A} \tag{250}$$

The fractions x_p and x_A are the fractions of the proton and nucleus longitudinal momenta carried by the produced particle. We see that when one is looking in the proton-going direction, where y is large, the process requires a large x_p and a small x_A. To a first approximation, the degrees of freedom at large x_p in the proton are valence quarks, and the ones at small x_A in the nucleus gluons. Thus, the process of particle production at forward rapidity in proton-nucleus collisions provides us a simple example of a dilute probe interacting with the color field of a dense nucleus. We shall next proceed to calculating the cross section for such a process.

11.2 Quark-nucleus scattering

To quantify the scattering of dilute probes on a dense gluon field target in a systematical way, we need to specify more precisely the theoretical framework that we use. In this context, the best way to do this is to use Light Cone Perturbation theory. A classic review is provided by the review of Brodsky, Pauli and Pinsky [13]. This review is not, however, very convenient for the kind of calculations we do here: the notations are different, and the discussion does not systematially use the concept of the Light Cone Wave Function (LCWF). A more useful reference for our calculation is the Kovchehov-Levin book [1], and the papers on DIS in the dipole picture [51, 52, 19, 20, 21, 53, 54]. We will, as in these papers, use a covariant normalization for the states (unlike the review [13]) and the Kogut-Soper metric for light cone variables introduced in Sec.4 (unlike the book [1]). A classic paper where the treatment of scattering off a classical background field is introduced is [16], which provides the more formal field theoretical justification for what we are doing here.

In our conventions, quark momentum eigenstates are normalized as

$$\langle q(\mathbf{p}, p^+, \lambda_p, i_p)| \, |q(\mathbf{q}, q^+, \lambda_q, i_q\rangle = 2p^+(2\pi)^3 \delta(p^+ - q^+) \delta^{(2)}(\mathbf{p} - \mathbf{q}) \, \delta_{i_p, i_q} \delta_{\lambda_p, \lambda_q}, \tag{251}$$

where $\lambda_{p,q}$ is the *light front helicity* (spin, for the purposes of our discussion here), and $i_{p,q}$ the color, in the fundamental representation. For brevity of notation we will not write out the helicity explicitly for now. Because the eikonal interaction with the target preserves helicity, we will not write it out explicitly from now on.

Our incoming state for the quark is a single quark in a specific momentum state. The eikonal interaction, as we recall, preserves the transverse coordinate of the particle. We therefore need to Fourier transform to

coordinate space. The Fourier-transforms and coordinate space states are defined as

$$|q(\mathbf{p},p^+,i)\rangle = \int \mathrm{d}^2\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{x}} |q(\mathbf{x},p^+,i)\rangle \tag{252}$$

$$|q(\mathbf{x},p^+,i)\rangle = \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} e^{-i\mathbf{p}\cdot\mathbf{x}} |q(\mathbf{p},p^+,i)\rangle \tag{253}$$

$$\langle q(\mathbf{x},p^+,i_p)| |q(\mathbf{y},q^+,i_q)\rangle = 2p^+(2\pi)\delta(p^+-q^+)\delta^{(2)}(\mathbf{x}-\mathbf{y})\,\delta_{i_p,i_q} \tag{254}$$

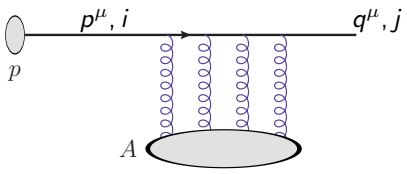
The state $|\mathrm{in}\rangle = |q(\mathbf{p},p^+,i)\rangle$ is the state of the incoming particle just before the shockwave of the target color field, i.e. at light cone time $x^+ \to 0^-$. As we have discussed, the effect of the color field shock wave is to color rotate the particle. This means that the state vector (bra) of the scattered particle in coordinate space is rotated by the Wilson line. We denote this rotation as the effect of the "scattering operator" $\hat{S}$.

Digression: Interaction picture

To be more specific, we are, as is common in scattering theory, using the Interaction or Dirac picture of time evolution. This means that we split the Hamiltonian of our system into two parts. The free part, corresponding to the kinetic energy k^- of the partons in the probe, causes a time evolution of the operators of the theory. The interaction part of the Hamiltonian, on the other hand, generates a time dependence in the state vector of the system. In this case, the interaction part consists of the interaction with the background field. Later, we will also include the interactions between different Fock states of the probe, most importantly gluon emission, in the interaction part of the Hamiltonian.

$$|\mathrm{in}\rangle = |q(\mathbf{p},p^+,i)\rangle = \int \mathrm{d}^2\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{x}} |q(\mathbf{x},p^+,i)\rangle \quad x^+ < 0 \tag{255}$$

$$\hat{S}\,|q(\mathbf{x},q^+,i)\rangle = \sum_j [V(\mathbf{x})]_{ji}\,|q(\mathbf{x},q^+,j)\rangle \quad x^+ > 0 \tag{256}$$



Now we know what is the quantum state of the outgoing particle. To calculate the scattering amplitude, we need to project out the final state that we are measuring from the outgoing particle state. Now, because our background field does not (in the eikonal approximation) depend on x^-, it does not carry any longitudinal momentum p^+. Thus the scattering amplitude will be proportional to a delta function $\delta(p^+ + q^+)$. As is customary in all field theory calculations, this delta function is factorized out. This is done because to calculate the cross section we will need to square the amplitude. The square of the delta function becomes, in the calculation relating the square of the matrix element to the cross section, a single delta function the cross section level times a dimensional factor that is needed to match the correct interpretation of the cross section as a number of scatterings per unit time. It is also customary to factor out a factor $2q^+(2\pi)$ and an imaginary unit together with the delta function. This is just a convention (in field theory one writes $S = 1 + iT$), but helps to keep the terminology straight, since one often ends up discussing the meaning of the "real" and "imaginary" parts of the scattering amplitude separately. We also need to subtract the original incoming state, i.e. subtract identity from the interaction operator $\hat{S}$: when there is no background field and $\hat{S} = \hat{1}$, the scattering amplitude should be zero.

This leads us to define the invariant scattering amplitude $\mathcal{M}$ in the following way from the matrix element between the incoming and outgoing states. Given the explicit expression (256) it is straightforward to calculate the amplitude for our particular case of quark-color field scattering.

$$2q^+(2\pi)\delta(p^+-q^+)i\overbrace{\mathcal{M}(i,\mathbf{p}\to j,\mathbf{q})}^{\text{invariant amplitude}} \equiv \langle q(\mathbf{q},q^+,j)|\,(\hat{S}-1)\,|q(\mathbf{p},p^+,i)\rangle \tag{257}$$

$$\langle q(\mathbf{q},q^+,j)|\,(\hat{S}-1)\,|q(\mathbf{p},p^+,i)\rangle$$
$$= \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y} e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}}\,\langle q(\mathbf{y},q^+,j)|\,(\hat{S}-1)\,|q(\mathbf{x},p^+,i)\rangle$$
$$= \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y} e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}}\,([V(\mathbf{x})]_{ji}-\delta_{ij})\,2p^+(2\pi)\delta(p^+-q^+)\delta^{(2)}(\mathbf{x}-\mathbf{y}) \quad (258)$$

$$\mathcal{M}(i,\mathbf{p}\to j,\mathbf{q}) = \int \mathrm{d}^2\mathbf{x} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\,([V(\mathbf{x})]_{ji}-\delta_{ij}) \quad (259)$$

Note that we now, again, have the interpretation of the Wilson line V as the *eikonal scattering amplitude of a dilute probe off the target color field*.

Now, the square of the invariant amplitude, multiplied by the invariant phase space element and a momentum conserving delta function, is just the cross section. There are some factors of (2π) that, conventionally, go along with the momenta. In the case of light cone coordinates, there are additional factors of $2p^+$ that always go together with integrals or delta functions in p^+ to turn them into Lorentz-invariant phase space integrals (see Eq. (10)). To make a properly normalized cross section, we have to average over incoming colors i and sum over outgoing ones j. At this point let us also take the transverse momentum of the incoming quark to be zero, since we are interested in collinear high-x quarks coming from a high energy proton.

(Set $\mathbf{p}=\mathbf{0}$)

$$\mathrm{d}\sigma = \frac{1}{N_\mathrm{c}}\sum_{i,j} 2p^+(2\pi)\delta(p^+-q^+)\,|\mathcal{M}(i,\mathbf{p}\to j,\mathbf{q})|^2\,\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{\mathrm{d}q^+}{4\pi q^+} \quad (260)$$

$$\frac{1}{N_\mathrm{c}}\sum_{i,j}|\mathcal{M}(i,\mathbf{0}\to j,\mathbf{q})|^2 = \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y} e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}$$
$$\times\frac{1}{N_\mathrm{c}}\left[\mathrm{tr}\,V(\mathbf{x})V^\dagger(\mathbf{y})\overbrace{-\mathrm{tr}\,V(\mathbf{x})-\mathrm{tr}\,V^\dagger(\mathbf{y})+N_\mathrm{c}}^{\sim\delta^{(2)}(\mathbf{q})}\right] \quad (261)$$

In simplifying the Wilson lines it is useful to note that $[V(\mathbf{x})]^*_{ji} = [V^\dagger(\mathbf{x})]_{ij}$. Note that out of the four terms resulting from the square of the amplitude, the last three will give a delta function in the produced quark transverse momentum, after integration over $\mathbf{x}$ or $\mathbf{y}$. While they need to be there to preserve unitarity, we shall promptly disregard these terms as not physically interesting, since in the experimental situation we will be measuring quarks that actually scatter off the transverse plane

The crucial, general formulae here are Eq. (257) defining the invariant amplitude in terms of the incoming and outgoing states, and Eq. (270) which tells us how to calculate the cross section from this amplitude. These equations are valid for any process, the other equations just for the particular case of quark-nucleus scattering. These general formulae look quite similar to the corresponding ones in covariant field theory ("instant form," i.e. equal-t quantization), apart from the fact that only longitudinal momentum is conserved and some factors $2p^+$ are needed to make things Lorentz-covariant. Contrary to conventional Feynman perturbation theory, transverse momentum, and consequently k^-, are now not conserved for the "particles" that we are quantizing, because these particles are scattering off an external classical field that can also contribute momentum. To my knowledge, these general expressions were first written down in the very important paper of Bjorken, Kogut and Soper [16].

Let us now perform a final step and see how one goes from quark-nucleus scattering to proton-nucleus scattering. The quarks come from a collinear distribution in the proton $q(x,Q^2)$, and we must integrate over the momentum fraction x. The energy of the incoming quark is $p^+ = xP^+$, and we can insert an identity in terms of an integral over p^+ with a delta function. We change the variable for the longitudinal momentum from q^+ to the rapidity by $\mathrm{d}y = \mathrm{d}q^+/q^+$. This leads us to

$$\frac{\mathrm{d}\sigma^{p+A\to q+A}}{\mathrm{d}^2\mathbf{q}\,\mathrm{d}y} = \int \mathrm{d}x q(x,Q^2) \int \mathrm{d}p^+ \delta(p^+ - xP^+)\; q^+ \overbrace{\frac{\mathrm{d}\sigma^{q+A\to q+A}}{\mathrm{d}^2\mathbf{q}\,\mathrm{d}q^+}}^{\sim\delta(q^+-p^+)}$$
$$= \int \mathrm{d}x q(x,Q^2) \overbrace{\int \mathrm{d}p^+ \delta(p^+ - xP^+) q^+ \delta(p^+ - q^+)}^{q^+\delta(q^+-xP^+)=x\delta(x-q^+/P^+)}$$
$$\times \frac{1}{(2\pi)^2} \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y} e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{N_\mathrm{c}} \mathrm{tr}\, V(\mathbf{x}) V^\dagger(\mathbf{y}) \quad (262)$$

Hybrid formula for quark production

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{q}\,\mathrm{d}y} = \frac{1}{(2\pi)^2} x q(x,\mu^2) \int \mathrm{d}^2\mathbf{x}\,\mathrm{d}^2\mathbf{y} e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \overbrace{\frac{1}{N_\mathrm{c}} \mathrm{tr}\, V(\mathbf{x}) V^\dagger(\mathbf{y})}^{\text{dipole operator}} \quad (263)$$

Now we have obtained a key result! This is our first example of how to measure the gluon field of a dense target. Of course, there are caveats. Firstly, one does of course not measure quarks in a collider experiment. There are typically two things one could do to actually connect (263) to experiment.

1. Measure jets. Experimentally, a jet is a collimated (close in angle) set of high-ish momentum particles. These are then "clustered" together into jets, and one is interested in the total momentum of the jet. One should think of a jet a direct product of a high momentum parton being produced in the primary collision. Jets are where perturbation theory and experiment meet, jets cross sections are well defined (infrared and collinear safe) observables. In a theory calculation a jet is a leading order a one-parton state, and at higher orders in perturbation theory also a 2-, 3- etc particle state. Thus a jet is in some sense the closest one can get experimentally to single quark production, but with the understanding that the relation between a jet and the quarks and gluons that create it must be systematically improved order by order in perturbation theory. The inconvenient thing about jets for our purpose is that experimentally identifying a jet as being separate from the background of the other particles produced in a hadronic collision, the jet must have a sufficiently high transverse momentum. This means that it will be typically at higher x and, more importantly, removed from the saturation region of $p_T \sim Q_\mathrm{s}$.

2. The other option is to measure individual hadrons. In this case one needs a way to calculate the transformation of the produced quark to hadrons. Typically this is done by using "fragmentation functions," which are extracted from data (in particular e^+e^- →hadrons) and satisfy their own version of the DGLAP evolution equation.

In addition to relating the quark to a measurable observable, one of course also needs to know where to get the dipole amplitude. One option for this is to use a model: later in these lectures we will discuss the McLerran-Venugopalan (MV) model [33, 32, 34] for the classical color fields, and calculate the dipole amplitude from this model. We can also derive renormalization group equations that allow one to predict, from a weak coupling QCD calculation, the dependence of the dipole amplitude on the collision energy. These go by the name of the Balitsky-Kovchegov (BK) [27, 29, 28] or JIMWLK [55, 56, 57, 58, 59, 60, 61, 62] equations. One an also, together with a model, parametrization of the BK/JIMWLK equations, extract the dipole cross amplitude from Deep Inelastic Scattering data.

To address the nuclear modification ratio R_{pA} we need to have a dipole operator separately for protons and for nuclei. We then calculate the cross sections using our hybrid formula, and take the ratio. This is the way the "CGC" curves in the LHCb plot have been obtained.

We will do an explicit calculation of the dipole operator in a specific model, the MV model, later. But even without such a calculation, it is possible to discuss the physics of gluon saturation in terms of the dipole operator.

Gluon saturation

- $\mathbf{x} = \mathbf{y} \implies \frac{1}{N_c} \left\langle \mathrm{tr}\, V(\mathbf{x}) V^\dagger(\mathbf{y}) \right\rangle = 1$
- $|\mathbf{x} - \mathbf{y}| \to \infty \implies \frac{1}{N_c} \left\langle \mathrm{tr}\, V(\mathbf{x}) V^\dagger(\mathbf{y}) \right\rangle \to 0$

$\implies \exists$ length scale $|\mathbf{x} - \mathbf{y}| \sim 1/Q_s$ s.t. $\frac{1}{N_c} \left\langle \mathrm{tr}\, V(\mathbf{x}) V^\dagger(\mathbf{y}) \right\rangle \sim \frac{1}{2}$ Q_s = Saturation scale

Perturbation theory in a theory with point-like particles (partons in the target) would always give you a power-law dependence. But a power law dependence on $|\mathbf{x}-\mathbf{y}|$ cannot be true at all values: it is not possible to satisfy the group theory constraints on possible values of the trace of an SU(N_c) matrix with a pure power law. Here the expectation values $\langle\rangle$ refer to some, quantum or classical, average over the states of the target.

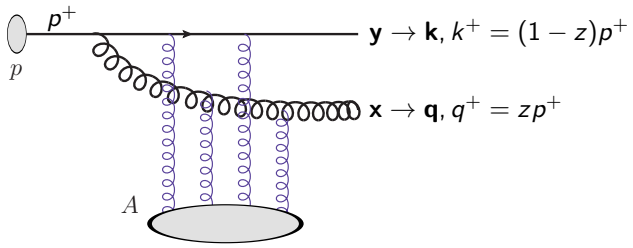
11.3 Going to higher orders

Without doing an actual calculation here, let us discuss what happend when we go to higher order in perturbation theory. Recall that we all already working to all orders in the classical field. Our probe, the quark, on the other hand is a quantum mechanical particle that we understand in perturbation theory. Starting from the single quark state, the next order in perturbation theory is when this quark emits a gluon. The emission can then happen either before or after the target. In the former case the quark-gluon system must pass thrugh the target color field. In stead of a single fundamental representation Wilson line this process will result in the product of an adjoint representation Wilson line for the gluon and a fundamental one for the quark. The adjoint representation Wilson line can be related to the fundamental one by

$$(V_{\mathrm{adj}})^{ab} = 2\,\mathrm{tr}\, t^a V t^b V^\dagger. \tag{264}$$

If one uses this relation to express everyting in terms of the fundamental representation, one ends up with three Wilson lines in the scattering amplitude. Squaring the amplitude, this means that we have operators made up of up to six Wilson lines in the cross section. We can express the result in terms of a $q + A \to q + g + X$ cross section, first derived in this formalism in Ref. [63]. Schematically, the expression looks like this:

Emit a gluon



$$\frac{\mathrm{d}\sigma^{qA\to qgX}}{\mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{k}} \propto \int_{\mathbf{x},\bar{\mathbf{x}}_\perp,\mathbf{y},\bar{\mathbf{y}}_\perp} e^{-i\mathbf{q}\cdot(\mathbf{x}-\bar{\mathbf{x}}_\perp)}\, e^{-i\mathbf{k}\cdot(\mathbf{y}-\bar{\mathbf{y}}_\perp)}\, \mathcal{F}(\bar{\mathbf{x}}_\perp - \bar{\mathbf{y}}_\perp, \mathbf{x}-\mathbf{y})$$
$$\left\langle \hat{Q}(\mathbf{y}, \bar{\mathbf{y}}_\perp, \bar{\mathbf{x}}_\perp, \mathbf{x})\, \hat{D}(\mathbf{x}, \bar{\mathbf{x}}_\perp) - \hat{D}(\mathbf{y}, \mathbf{x})\hat{D}(\mathbf{x}, \bar{\mathbf{z}}_\perp) - \hat{D}(\mathbf{z}, \bar{\mathbf{x}}_\perp)\hat{D}(\bar{\mathbf{x}}_\perp, \bar{\mathbf{y}}_\perp) \right.$$
$$\left. + \frac{C_F}{N_c}\hat{D}(\mathbf{z}, \bar{\mathbf{z}}_\perp) + \frac{1}{{N_c}^2}\left(\hat{D}(\mathbf{y}, \bar{\mathbf{z}}_\perp) + \hat{D}(\mathbf{z}, \bar{\mathbf{y}}_\perp) - \hat{D}(\mathbf{y}, \bar{\mathbf{y}}_\perp)\right) \right\rangle_{\text{target}} \tag{265}$$

($\mathbf{z} = z\mathbf{x} + (1-z)\mathbf{y}$, $\bar{\mathbf{z}}_\perp = z\bar{\mathbf{x}}_\perp + (1-z)\bar{\mathbf{y}}_\perp$.)

$$\hat{D}(\mathbf{x}-\mathbf{y}) \equiv \frac{1}{N_c}\,\mathrm{tr}\, V(\mathbf{x})V^\dagger(\mathbf{y}) \quad \hat{Q}(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}) \equiv \frac{1}{N_c}\,\mathrm{tr}\, V(\mathbf{x})V^\dagger(\mathbf{y})V(\mathbf{u})V^\dagger(\mathbf{v}) \tag{266}$$

Some things can be pointed out about these equations. The function $\mathcal{F}(\bar{\mathbf{x}}_\perp - \bar{\mathbf{y}}_\perp, \mathbf{x}-\mathbf{y})$ includes the dynamics of the vertex of gluon emission from the quark. We could calculate it from LCPT even in these lectures, if we had a bit more time. The transverse coordinates $\mathbf{y}$ and $\mathbf{x}$ are the coordinates of the quark and the gluon in the amplitude (remember that in the eikonal approach the coordinates stay fixed when

propagating through the background field). Correspondingly, $\bar{\mathbf{y}}_\perp$ and $\bar{\mathbf{x}}_\perp$ are the coordinates of the quark and the gluon in the conjugate amplitude. The difference between the coordinates in the amplitude and the conjucate, e.g. $\mathbf{x} - \bar{\mathbf{x}}_\perp$ is Fourier-conjugate to the momentum f the produced particle. There are diagrams where the gluon is emitted either before or after the classical gluon field. This sum of two diagrams in the amplitude is then squared, yielding even more terms. The gluon emission vertex includes a fundamental representation generator t^a. When expressing the adjoint Wilson lines in terms of funamental ones, one gets more fundamental representation generators. One then sums over all colors of final state particles, and averages over colors of the incoming quark. These color summations turn into traces of Wilson lines. One then finally gets rid of the generators t^a by the Fierz identity:

$$(t^a)_{ij}(t^a)_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\delta_{jk} - \frac{1}{N_\mathrm{c}}\delta_{ij}\delta_{k\ell}\right), \tag{267}$$

which still increases the number of terms.

This is a wonderful result, that leads to further kinds of physics in two different ways.

- One can use it to study correlations between produced hadrons or jets. Particularly interesting are correlations in the azimuthal angle, between particles that have relatively low momenta, not too much larger than the saturation scale. Here, the esential thing for the physics is that the produced particles receive, in addition to the back-to-back momentum that they have in recoiling against each other, intrinsic momentum from the classical field. In this case the larger saturation scale of the nucleus leads to a larger momentum being transfered from the target, which leads to a washing out of the back-to-back azimuthal correlation dictated by momentum conservation if there is no moemntum transfer from the target. Observing such an effect in two-particle correlations would be a direct probe of the natura of the saturation scale as a transverse momentum scale, and thus very valuable. There are hints of such a nuclear effect in the dihadron correlations observed at RHIC, but there is not yet a unanimity of the interpretation of these observations.

- One can also take the qg cross section result and integrate over the momentum of one of the particles. This procedure gives a part of the NLO correction to single particle production. Note that an integration over a momentum sets the coordinates of the corresponding particle in the amplitude and the conjugate amplitude to be equal. This significantly simplifies the Wilson line operator expression. A particularly important class of NLO corrections are the ones that are enhanced by some large logarithm. In this case, there are two relevant ones:

  - Collinear divergences: there can be an infrared divergence in the integral over the tranverse momentum of either of the two partons. This divergence can be regularized, and then absorbed into the DGLAP-evolution of the quark distribution in the leading order cross section. To my knowledge, this was first demonstrated in [64]. The fact that there is a collinear divergence at NLO that can be absorbed into DGLAP evolution of the parton distribution function in the LO cross section expression is the real "proof" of the hybrid factorization formula, at least at this order.
  - Rapidity divergences (or soft divergences). There is also a divergence in the limit of $z \to 0$. This divergence has to be absorbed into the BK/JIMWLK evolution of the Wilson lines of the target. That this happens was demonstrated in [65, 66] (see also [67, 68]).

  Taking into account not only the logaritmically enhanced contributions but the full expression is needed for the full NLO calculation of single inclusive particle production in the hybrid formalism.

12 Diffractive DIS at NLO

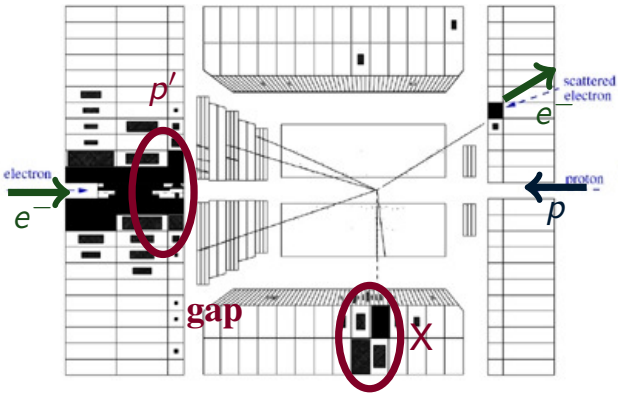
12.1 Physical situation: diffractive DIS

In this section we will look at an example of an advanced one-loop calculation of a physical process at one loop accuracy, using LCPT. The calculations tend to become lengthy, and the point is not to reproduce here the details of a calculation that takes a year for a talented graduate student and is impossible for a senior professor. Rather, the aim is to discuss key conceptual and technical issues that can come up in such

a calculation: how to set it up, how to evaluate the diagrams, what kind of divergences come up in the calculation and how they should be treated.

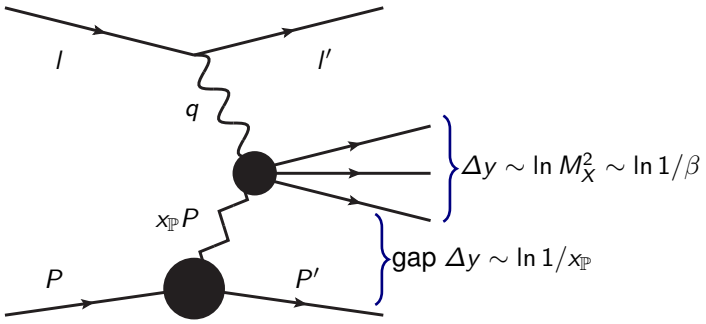
We will here concentrate on the calculation of the diffractive DIS structure function, following [69]. Inclusive diffraction ís a key observable for the EIC. In particular, the very robust and perhaps counterintuitive expectation from the dipole picture is that the diffractive over total cross section ratio should be larger in nuclear than proton targets. At HERA roughly 15% of the scattering events are diffractive. Experimentally the signal for a diffractive DIS event is a "rapidity gap" between the virtual photon remnants and the target.

Diffractive DIS



The kinematics of the diffractive process is a bit more differential than for inclusive DIS. Basically there are two additional variables: the momentum transfer to the target and the invariant mass of the final state.

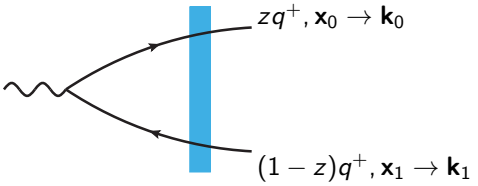
- The virtuality Q^2 is the same as for all DIS events
- Momentum transfer $t = (P - P')^2$
- Diffractive system mass M_X^2, $\beta = Q^2/(Q^2 + M_X^2)$
- Gap size $x_{\mathbb{P}}$, target evolution rapidity $\sim \ln 1/x_{\mathbb{P}}$



The fact that there is a large rapidity interval separating the two systems implies that the interaction happened via a color neutral "pomeron" exchange.

12.2 LO calculation

The leading order amplitude can be described by the following diagram:



Here the blue bar represents the shockwave target, the action of the Wilson line operator $\hat{S}$. Let us now turn this into equations. The incoming state is a virtual photon with a helicity λ, and the outgoing state a normalized color singlet quark-antiquark state

$$|\text{in}\rangle = |\gamma^*(q^+,\lambda)\rangle_D = (1+\dots)\,|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0\,, \tag{268}$$

$$|\text{out}\rangle = |q\bar{q}\rangle_{D,\underline{0}} = \frac{\delta_{ij}}{\sqrt{N_c}}\,|q(i,\mathbf{k}_0,zq^+)\bar{q}(j,\mathbf{k}_1,(1-z)q^+)\rangle_0 + \dots \tag{269}$$

Here the photon to quark-antiquark wavefunction is given in (156). The cross section is a generalization of Eq. (270) to the a general final state multiparticle phase space d [P.S.], in terms of an invariant amplitude which is obtained from the matrix element of the scattering operator between the incoming and outgoing states, stripped from the overall plus momentum conservation delta function. The experimental situation corresponds to measuring the invariant mass of the outgoing state and the momentum transfer, corresponding to the total transverse momentum of the final state. In equations this can be implemented as a delta function in these variables

$$\mathrm{d}\sigma = 2p^+(2\pi)\delta(p^+ - q^+)\,|\mathcal{M}|^2\,\mathrm{d}\,[\text{P.S.}] \tag{270}$$

$$2q^+(2\pi)\delta(p^+_{\text{in}} - p^+_{\text{out}})i\mathcal{M} \equiv \langle\text{out}|\,(\hat{S}-1)\,|\text{in}\rangle \tag{271}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}{M_X}^2\,\mathrm{d}t} = \int \mathrm{d}\,[\text{P.S.}]\,\delta\left({M_X}^2 - \frac{(z\mathbf{k}_1-(1-z)\mathbf{k}_0)^2}{z(1-z)}\right)\delta\left(t+(\mathbf{k}_0+\mathbf{k}_1)^2\right)\frac{\mathrm{d}\sigma}{\mathrm{d}\,[\text{P.S.}]} \tag{272}$$

For the final result, the nontrivial calculation is that one needs to Fourier transform back and forth in order to evaluate the scattering operator in transverse coordinate space, but then relate these transverse coordinates to the momentum space variables for the final state.

$$\frac{\mathrm{d}\sigma^{\mathrm{D}}_{\lambda,\,q\bar{q}}}{\mathrm{d}{M_X}^2\,\mathrm{d}|t|} = \frac{N_c}{4\pi}\int_0^1 \mathrm{d}z \int_{\mathbf{x}_0\mathbf{x}_1\bar{\mathbf{x}}_0\bar{\mathbf{x}}_1} \mathcal{I}^{(2)}_{\Delta}\mathcal{I}^{(2)}_{M_X} \times \sum_{f,h_0,h_1}\left(\widetilde{\psi}_{\gamma^*_\lambda\to q_{\bar{0}}\bar{q}_{\bar{1}}}\right)^\dagger\left(\widetilde{\psi}_{\gamma^*_\lambda\to q_0\bar{q}_1}\right)\quad \mathcal{N}^\dagger_{q\bar{q}}(\bar{\mathbf{x}}_0-\bar{\mathbf{x}}_1)\mathcal{N}_{q\bar{q}}(\mathbf{x}_0-\mathbf{x}_1) \tag{273}$$

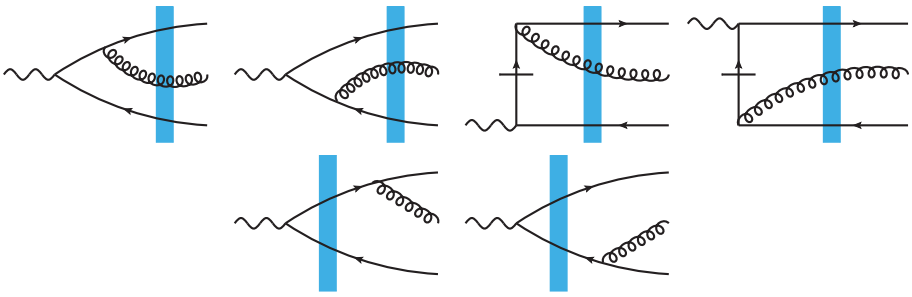
$$\mathcal{I}^{(2)}_{M_X} = \int\frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\delta(\mathbf{l}^2 - z(1-z)M_X{}^2)e^{i\mathbf{l}\cdot(\mathbf{x}_{\bar{1}\bar{0}}-\mathbf{x}_{01})} \tag{274}$$

$$\mathcal{I}^{(2)}_{\Delta} = \int\frac{\mathrm{d}^2\Delta}{(2\pi)^2}\delta(\Delta^2-|t|)e^{i\Delta\cdot(z\mathbf{x}_{\bar{0}0}-(1-z)\mathbf{x}_{\bar{1}1})} \tag{275}$$

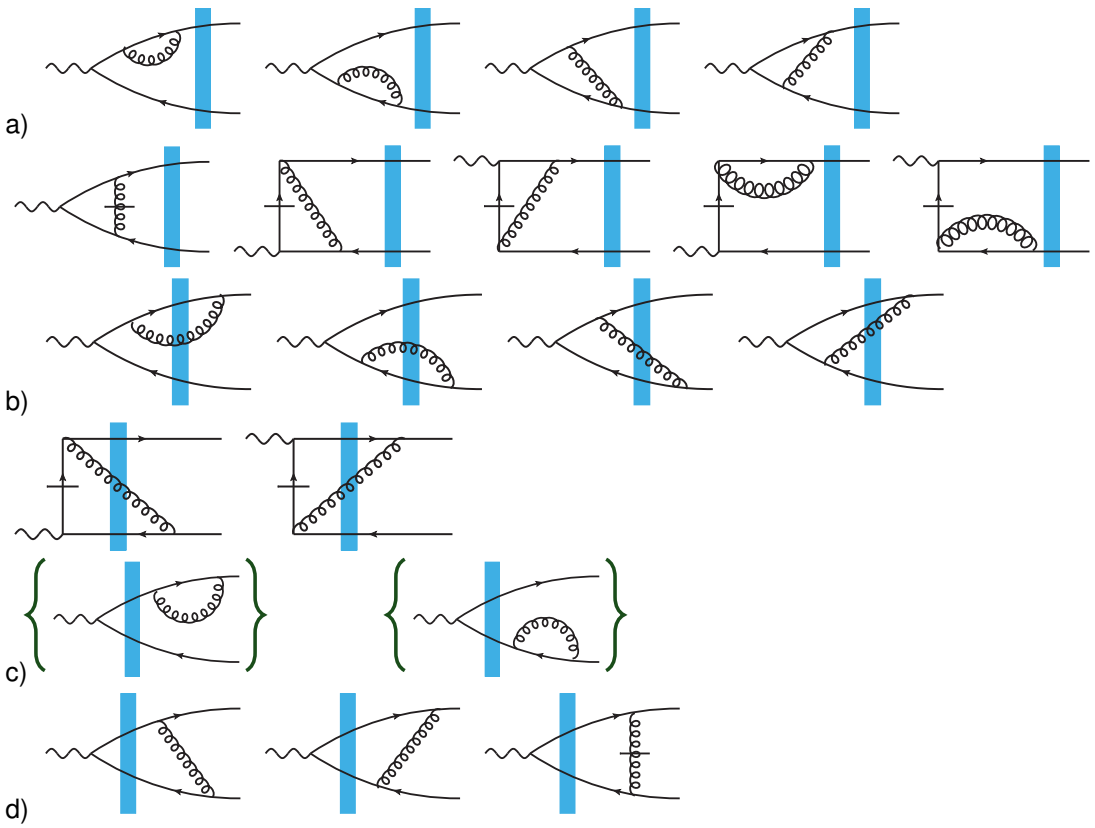
12.3 NLO corrections

For the radiative

Radiative:



The contributions from the diagrams with a gluon emission before the target form a separate finite subset of the NLO correction, which includes the leading large Q^2 contribution. They were calculated already earlier in [70]. The diagrams with an emission after the target, and the interferences between emissions before and after have infrared divergences that cancel against the virtual contributions.



a) Vertex corrections: known 1-loop $\gamma \to q\bar{q}g$ wavefunction

b) Gluon crosses shockwave, but not the cut:

- Loop corrections to amplitude, tree level wavefunctions
- 3-point operator of Wilson lines
- BK/JIMWLK evolution of LO amplitude

c) Wave function renormalization corrections for outgoing particles: for massless quarks these are zero in dimensional regularization, due to canceling IR and UV poles.

d) Final state interactions

For other calculations with basically the same structure of virtual corrections see e.g. [71, 72, 73, 74]

I will not write down here explicit expressions for the different contributions to the NLO cross section. However, it is instructive to look at what kind of divergences appear and what they correspond to physically. Our regularization procedure is the following:

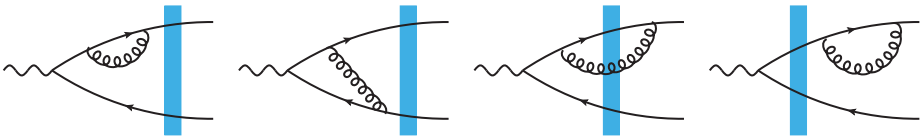
Regularization

- $\mathrm{d}^{2-2\varepsilon}\mathbf{k} \implies \frac{1}{\varepsilon}$, collinear or UV
- $k^+ > \alpha$, $\alpha \to 0$ $\implies 1/\alpha, \ln^2 \alpha, \ln \alpha$

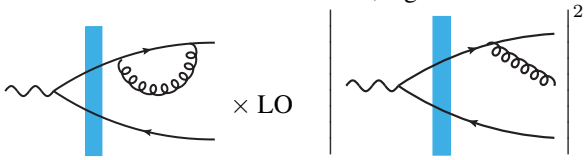
Divergences

- $\mathrm{d}^{2-2\varepsilon}\mathbf{k} \implies \frac{1}{\varepsilon}$, collinear or UV
- $k^+ > \alpha$, $\alpha \to 0$ $\implies 1/\alpha, \ln^2 \alpha, \ln \alpha$

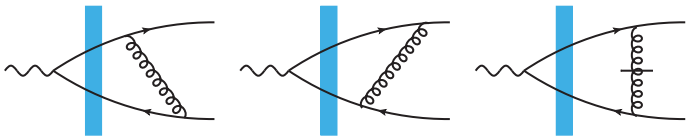
1. UV $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon}\ln\alpha$ divergences appear in virtual corrections to the cross section: the $\gamma^* \to q\bar{q}$ vertex, contributions where the gluon crosses the shockwave but is not measured in the amplitude, and the wavefunction renormalization constant. Some representative diagrams (to be multiplied by the complex conjugate of the leading order amplitude):



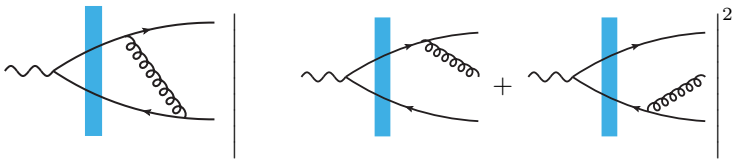
2. Collinear, i.e. infrared $\frac{1}{\varepsilon}$ divergences appear in the wavefunction renormalization at the amplitude level and final state emission contrbutions to the cross section, e.g.



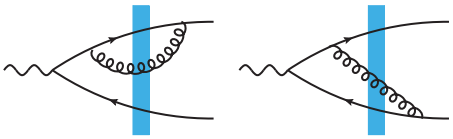
3. Linear $1/\alpha$ divergences are an artefact of the light cone gauge and LCPT formulations, which separates the instantaneous interaction in to a separate 4-particle vertex. They cancel between normal and instantaneous exchanges of gluons, which would in covariant perturbation theory in covariant gauge correspond to the same diagram:



4. $\ln^2\alpha$ from final state exchange and emission The M_X restriction matters here, for a jet cross section the pattern of cancellation for this kind of divergences would be very different.



5. The remaining $\ln\alpha$ divergence is absorbed into the BK/JIMWLK renormalization of the target Wilson lines:



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