Lecture 6: Error propagation

General formula

Dealing with correlations

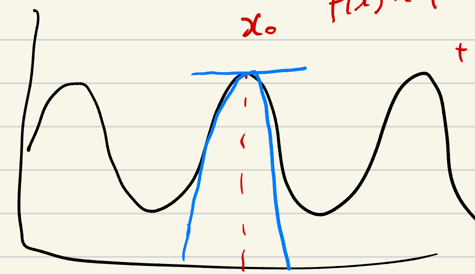
The covariance matrix

Error propagation

- Error propagation used to calculate uncertainty on function $f(x_i)$ given inputs x_i
- Very useful in experimental physics, where $f(x_i)$ is your result.
- General formula for single input is:

$$\sqrt{V(f)} = \sigma(f) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\bar{x}} \sigma(x)$$

$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x_0} x + \left.\frac{d^2f}{dx^2}\right|_{x_0} x^2$



- This is derived by doing a Taylor expansion of the function around value x_0

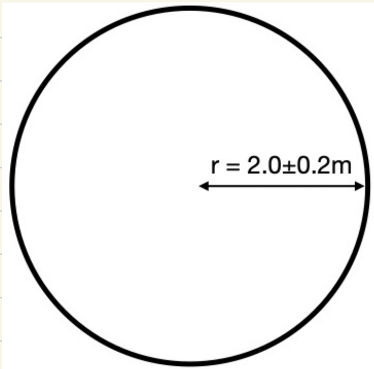
$$f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0} (x - x_0) + \cancel{\frac{1}{2!}\left.\frac{\partial^2 f(x)}{\partial^2 x}\right|_{x=x_0} (x - x_0)^2}$$

$$\cancel{+\frac{1}{3!}\left.\frac{\partial^3 f(x)}{\partial^3 x}\right|_{x=x_0} (x - x_0)^3} + \ldots$$

If $(x-x_0)$ small, neglect higher terms.

Example: circumstance/area of a circle

- Measure radius of circle - what is uncertainty on circumference/area?



$$\sqrt{V(f)} = \sigma(f) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\bar{x}} \sigma(x)$$

$$\frac{dC(r)}{dr} = 2\pi$$ Calculate derivative

$$\sigma(C) = 2\pi\sigma(r) = 1.3m \longrightarrow C = 12.6 \pm 1.3m$$

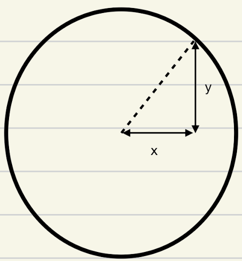
Area: $\frac{dA(r)}{dr} = 2\pi r$ $\quad \sigma(A) = 2\pi r\sigma(r) = 2.5m^2$ $\quad A = 12.6 \pm 2.5m^2$

Multiple inputs

- With multiple inputs, need to consider correlation

$$\sigma(f) = \sqrt{\left|\frac{\partial f}{\partial x}\right|^2 \cdot \sigma_x^2 + \left|\frac{\partial f}{\partial y}\right|^2 \cdot \sigma_y^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \cdot cov(x,y)}$$

$i=j=1 \qquad i=j=2 \qquad i \neq j$



x=0.8±0.1(stat)±0.1(syst) m

y=0.7±0.1(stat)±0.1(syst) m

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

What is correlation between x and y measurements?

$$\sigma_r = \sqrt{\frac{x^2}{r^2} \cdot \sigma_x^2 + \frac{y^2}{r^2} \cdot \sigma_y^2 + 2\frac{xy}{r^2} \cdot 0.5\sigma_x\sigma_y}$$

$$\sigma_r = \sqrt{\frac{0.8^2}{1.06^2} \cdot 0.14^2 + \frac{0.7^2}{1.06^2} \cdot 0.14^2 + 2\frac{0.7 \cdot 0.8}{1.06^2} \cdot 0.5 \cdot 0.14 \cdot 0.14} = 0.17\,m$$

$$\boxed{r = 1.06 \pm 0.17\,m.}$$

Ratios and control measurements

- Imagine we have two measurements which are systematically dominated.

A = 10.0±0.1(stat)±1.0(syst) [cm]　　　B = 20.0±0.2(stat)±2.0(syst) [cm]

- Both affected by the same multiplicative systematic with fractional uncertainty.

$A = x a$　　$B = x b$　　1cm × number of dilimtes.

$C = \frac{a}{b}$

- What is uncertainty on the ratio? $C = \frac{A}{B}$

$$\frac{\sigma(C)}{C} = \sqrt{\left(\frac{\sigma(A)}{A}\right)^2 + \left(\frac{\sigma(B)}{B}\right)^2 - 2\frac{cov(A,B)(2)}{AB(20\times10)}}$$

- $\sigma(A) = \sqrt{0.1^2 + 1^2}$　　$\sigma(B) = \sqrt{0.2^2 + 2^2}$, $cov(A,B) = \sigma_f^2 AB = 0.1^2 \times 10 \times 20 = 2\ cm^2$

$$\frac{\sigma(C)}{C} = \sqrt{0.1005^2 + 0.4005^2 - 2\cdot\frac{1.2}{200}} = \frac{\sqrt{2}}{100} = 1.4\%$$

Useful formulae (Wikipedia)

Function	Variance	Standard Deviation
$f = aA$	$\sigma_f^2 = a^2\sigma_A^2$	$\sigma_f = \|a\|\sigma_A$
$f = aA + bB$	$\sigma_f^2 = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$	$\sigma_f = \sqrt{a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}}$
$f = aA - bB$	$\sigma_f^2 = a^2\sigma_A^2 + b^2\sigma_B^2 - 2ab\,\sigma_{AB}$	$\sigma_f = \sqrt{a^2\sigma_A^2 + b^2\sigma_B^2 - 2ab\,\sigma_{AB}}$
$f = AB$	$\sigma_f^2 \approx f^2\left[\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 + 2\frac{\sigma_{AB}}{AB}\right]$ [11][12]	$\sigma_f \approx \|f\|\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 + 2\frac{\sigma_{AB}}{AB}}$
$f = \frac{A}{B}$	$\sigma_f^2 \approx f^2\left[\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 - 2\frac{\sigma_{AB}}{AB}\right]$ [13]	$\sigma_f \approx \|f\|\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 - 2\frac{\sigma_{AB}}{AB}}$
$f = aA^b$	$\sigma_f^2 \approx \left(abA^{b-1}\sigma_A\right)^2 = \left(\frac{fb\sigma_A}{A}\right)^2$	$\sigma_f \approx \left\|abA^{b-1}\sigma_A\right\| = \left\|\frac{fb\sigma_A}{A}\right\|$
$f = a\ln(bA)$	$\sigma_f^2 \approx \left(a\frac{\sigma_A}{A}\right)^2$ [14]	$\sigma_f \approx \left\|a\frac{\sigma_A}{A}\right\|$
$f = a\log_{10}(bA)$	$\sigma_f^2 \approx \left(a\frac{\sigma_A}{A\ln(10)}\right)^2$ [14]	$\sigma_f \approx \left\|a\frac{\sigma_A}{A\ln(10)}\right\|$
$f = ae^{bA}$	$\sigma_f^2 \approx f^2(b\sigma_A)^2$ [15]	$\sigma_f \approx \|f\,(b\sigma_A)\|$
$f = a^{bA}$	$\sigma_f^2 \approx f^2(b\ln(a)\sigma_A)^2$	$\sigma_f \approx \|f(b\ln(a)\sigma_A)\|$
$f = a\sin(bA)$	$\sigma_f^2 \approx [ab\cos(bA)\sigma_A]^2$	$\sigma_f \approx \|ab\cos(bA)\sigma_A\|$
$f = a\cos(bA)$	$\sigma_f^2 \approx [ab\sin(bA)\sigma_A]^2$	$\sigma_f \approx \|ab\sin(bA)\sigma_A\|$
$f = a\tan(bA)$	$\sigma_f^2 \approx [ab\sec^2(bA)\sigma_A]^2$	$\sigma_f \approx \|ab\sec^2(bA)\sigma_A\|$
$f = A^B$	$\sigma_f^2 \approx f^2\left[\left(\frac{B}{A}\sigma_A\right)^2 + (\ln(A)\sigma_B)^2 + 2\frac{B\ln(A)}{A}\sigma_{AB}\right]$	$\sigma_f \approx \|f\|\sqrt{\left(\frac{B}{A}\sigma_A\right)^2 + (\ln(A)\sigma_B)^2 + 2\frac{B\ln(A)}{A}\sigma_{AB}}$
$f = \sqrt{aA^2 \pm bB^2}$	$\sigma_f^2 \approx \left(\frac{A}{f}\right)^2 a^2\sigma_A^2 + \left(\frac{B}{f}\right)^2 b^2\sigma_B^2 \pm 2ab\frac{AB}{f^2}\sigma_{AB}$	$\sigma_f \approx \sqrt{\left(\frac{A}{f}\right)^2 a^2\sigma_A^2 + \left(\frac{B}{f}\right)^2 b^2\sigma_B^2 \pm 2ab\frac{AB}{f^2}\sigma_{AB}}$

General case with multiple inputs

$i=j$

$$V(f) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) \cdot cov(x_i, x_j)$$

$$\left|\frac{df}{dx_i}\right|^2 \sigma(x_i)$$

- When i=j simplifies to $\left|\frac{\partial f}{\partial x_i}\right|^2 V(x_i)$

$$\begin{aligned}
V(f) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) \cdot cov(x_i, x_j) \\
&= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) \times \begin{pmatrix} V(x_1) & cov(x_1,x_2) & \cdots & cov(x_1,x_n) \\ cov(x_1,x_2) & V(x_2) & \cdots & cov(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ cov(x_1,x_n) & cov(x_2,x_n) & \cdots & V(x_n) \end{pmatrix} \times \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{pmatrix} \\
&= \underbrace{G}_{\text{Jacobian}} \times \underbrace{V_x}_{\text{Covariance matrix} \equiv \text{Error matrix}} \times \underbrace{\tilde{G}}_{\text{Jacobian}}
\end{aligned}$$

The covariance matrix

- Sometimes called the error matrix, a highly important quantity in data analysis.
- Defines uncertainties of a set of measurements along with their correlations.
- For example, if we have independent inputs xi:

$x_i = \{8.2\pm0.1s, 8.1\pm0.1s, 8.0\pm0.1s\}$

$$V_x = \begin{pmatrix} 0.01s^2 & 0 & 0 \\ 0 & 0.01s^2 & 0 \\ 0 & 0 & 0.01s^2 \end{pmatrix}$$

$V_{x_{ij}} = \mathrm{cov}(x_i, x_j) = 0$ when $i \neq j$

- If uncertainties dominated by common additive systematic:

$x_i = \{8.2\pm0.1s, 8.1\pm0.1s, 8.0\pm0.1s\}$

$$V_x = \begin{pmatrix} 0.01s^2 & 0.01s^2 & 0.01s^2 \\ 0.01s^2 & 0.01s^2 & 0.01s^2 \\ 0.01s^2 & 0.01s^2 & 0.01s^2 \end{pmatrix}$$

$\mathrm{cov}(x_1, x_2) = \sigma_{x_1}\sigma_{x_2}$

Multiple functions

- What if you are measuring multiple things?

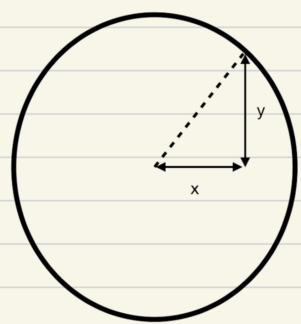
↳ Now have set of functions $f_i(x_j)$

$$V_f = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_n} \end{pmatrix}}_{G} \times \underbrace{\begin{pmatrix} V(x_1) & \cdots & cov(x_1, x_n) \\ \vdots & \ddots & \vdots \\ cov(x_1, x_n) & \cdots & V(x_n) \end{pmatrix}}_{V_x} \times \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_N}{\partial x_n} \end{pmatrix}}_{\tilde{G}}$$

Cov matrix

- Jacobian vector becomes a matrix

- This becomes useful in examples such as coordinate transformation.



x=0.8±0.1(stat)±0.1(syst) m

y=0.7±0.1(stat)±0.1(syst) m

$x, y \rightarrow r, \theta$

When does error propagation break down?

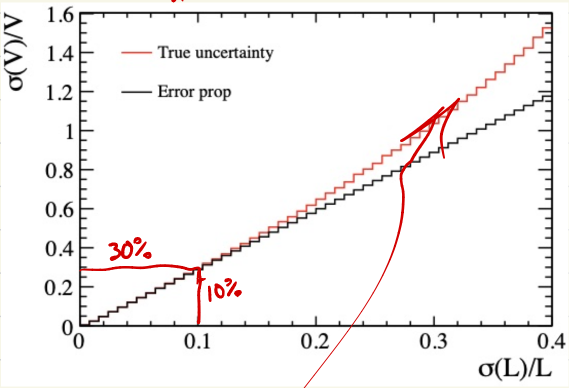
- If the relative errors of the inputs become large, can no longer drop terms in the Taylor expansion.

$\frac{dV}{dL} = 3L^2$

$$f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x - x_0) + \frac{1}{2!}\left.\frac{\partial^2 f(x)}{\partial^2 x}\right|_{x=x_0}(x - x_0)^2 + \frac{1}{3!}\left.\frac{\partial^3 f(x)}{\partial^3 x}\right|_{x=x_0}(x - x_0)^3 + \ldots$$

~~If small~~

$L = 1 \pm 0.1\,\mathrm{cm}$



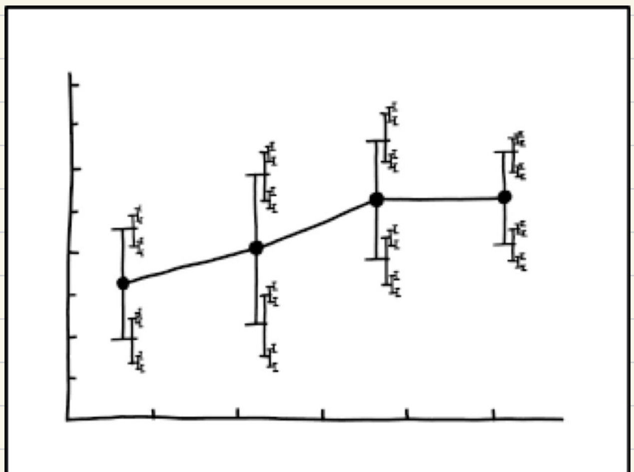
Other ways it can fail:

- Input uncertainties not Gaussian.
- No analytical formula to relate function with inputs.

Way out: Monte Carlo simulation.

Summary

- Gaussian error propagation arises from Taylor expansion of function $f(x)$.
- When dealing with multiple inputs, need to account for correlation.
- Covariance matrix encodes uncertainties and correlations of multiple variables.
- Gaussian error propagation doesn't always work!



I DON'T KNOW HOW TO PROPAGATE ERROR CORRECTLY, SO I JUST PUT ERROR BARS ON ALL MY ERROR BARS.

Lecture 7: Systematic uncertainties

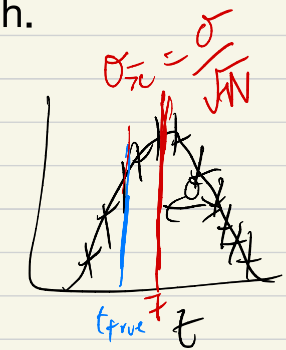
Two types of uncertainties

- **Statistical:** Deviation from true value different each time you repeat measurement with identical conditions.

  e.g: Random delay you press the start/stop button on stop watch.

- Calculate this uncertainty by looking at standard deviation of values.

- Uncertainty decreases as you increase number of measurements.



- **Systematic:** Same deviation from true value for each repetition.

  e.g: Stop watch runs fast

- Systematic effects are difficult to estimate: Require experience and understanding.

- Measurements typically quoted as: $x = \text{value} \pm \sigma_{\text{stat}} \pm \sigma_{\text{syst}}\ [\text{unit}]$

- Total uncertainty obtained by adding in quadrature $\sigma_{\text{tot}} = \sqrt{\sigma_{\text{stat}}^2 + \sigma_{\text{syst}}^2}$

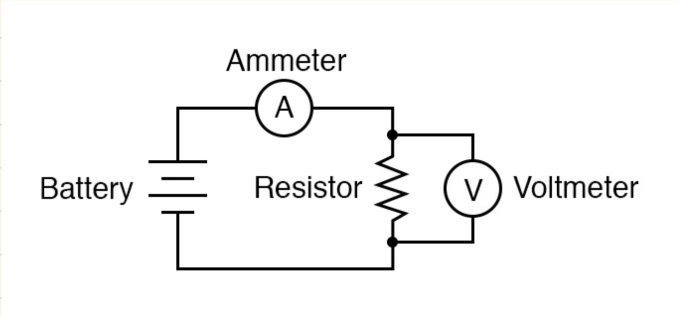
Systematic treatment

- Understand the sources of systematic uncertainty.
    - Think about inputs to measurement: What are the assumptions?
- Eliminate possible sources of systematic uncertainty.
    - Isolate environment from external sources.
    - Perform measurement relative to control value. $x_{diff} = (m_n - m_p)$
- Correct for known systematic effects.
    - Correct for environmental parameters, use calibration curves for equipment.
    - Use measurement of control value to estimate correction.
- Estimate remaining systematic uncertainty.
    - Precision of corrections
    - Effects you did not correct for.

In reality, no set procedure. That’s what makes systematics difficult.

Example: Measuring resistance of resistor.

- Read current and voltage from Ammeter/Voltmeter.



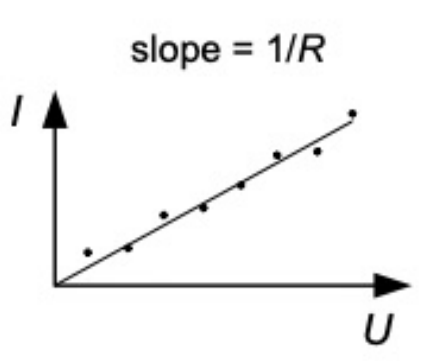
$$R = \frac{V}{I} = \frac{U}{I}$$

- Understand: Inputs: Ammeter & Voltmeter - what can go wrong?

  Offset: Non-zero current when voltage is zero.

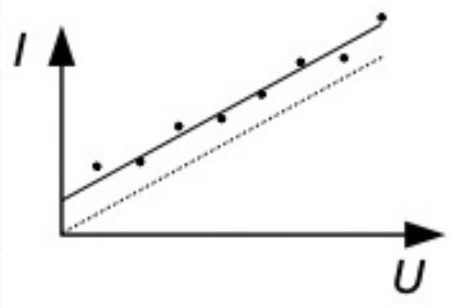
  Finite resistance of Ammeter/Voltmeter.

Eliminate: Measure values at different voltages.

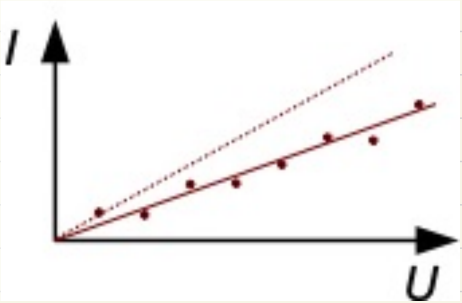


Scatter around line is statistical uncertainty due to electronic noise.

- Now insensitive to offset at zero.



What if we get wrong slope due to internal resistance? Easy to spot.



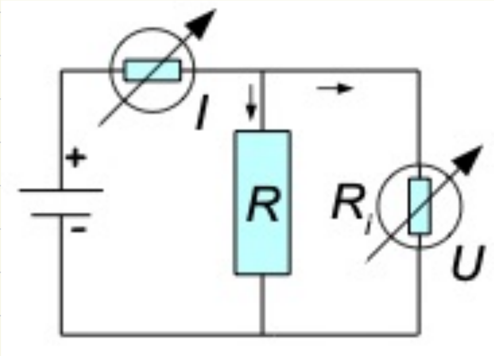
Apply correction

Measure resistance of known value and see how different the result is.

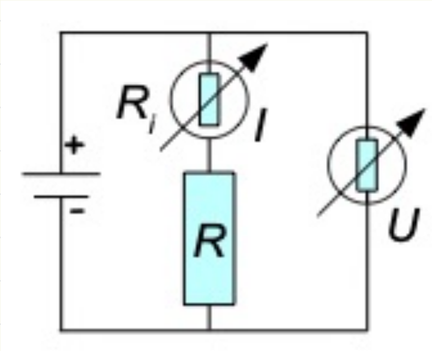
Correct result by difference to control measurement.

Vary circuit setup to estimate finite resistance of ammeter/voltmeter.

Estimate systematic based on variation.



$R < \frac{U}{I}$



$R > \frac{U}{I}$

Example: Measuring length of pendulum with ruler

- Bad physicist: Ignore thermal expansion of ruler.
- OK physicist: Forgets to measure temperature in the room.

    Tries to estimate temperature. Uncertainty on estimate becomes systematic uncertainty.

- Good physicist: Measures temperature, corrects for thermal expansion. Uncertainty on correction: systematic.
- Also good physicist: Realise that thermal expansion of ruler is negligible compared to other uncertainties.

This is also fine but need intuition for this!

Additive vs multiplicative systematics

The way in which systematics enter numerically changes their impact.

Consider a constant offset to the resistance, which biases the measurement.

$$R_{meas} = R_{True} + R_{syst} \qquad \sigma_{syst} = \sigma_{R_{syst}}$$

Systematic uncertainty in this case is then additive, and the systematic will add quadratically to the remaining uncertainty.

Another possible case is where the resistance is multiplied by an input with an associated systematic.

$$R_{meas} = \frac{V}{I_{meas}} \qquad \frac{\sigma_{syst}}{V} = \frac{\sigma_V}{V} = \sigma_f$$

Fractional uncertainty on the input.

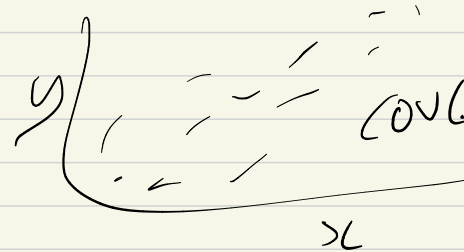
In the second case the systematic is multiplicative.

Correlation in presence of common systematic uncertainty.

- If systematic effect common to two measurements: correlation introduced between two uncertainties.

$$x_1 = A \pm \sigma_1\,(\mathrm{stat}) \pm \sigma_{\mathrm{syst}}\,(\mathrm{syst})\ \ [\mathrm{units}]$$

$$x_2 = B \pm \sigma_2\,(\mathrm{stat}) \pm \sigma_{\mathrm{syst}}\,(\mathrm{syst})\ \ [\mathrm{units}]$$



cov(x,y) >

- If systematic independent of central value (e.g: offset) then covariance is

$$\mathrm{cov}(x_1, x_2) = \sigma^2_{\mathrm{syst}}$$ Additive case

- If systematic multiplicative, then covariance becomes

$$\mathrm{cov}(x_1, x_2) = \sigma_f^2 AB$$ Multiplicative case

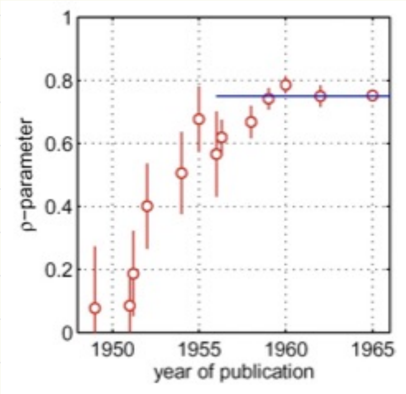
$$\sigma_f^2 = \frac{\sigma_{syst}^2}{(V_{syst\ input})^2}$$

$$A = \frac{B}{C} \qquad \frac{\sigma_A}{A} = \sqrt{\left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_C}{C}\right)^2}$$

Blinding

- Try to be objective as scientists, but difficult to avoid bias when see result.

  → When you do systematic variation, do you keep value closest expected one?



Example from muon physics: each measurement clearly not independent from previous.

Solution? Blinding: Dont look at the central value until you have frozen the analysis.

Here is an example of one during my Ph. D. All the data in the signal region is blinded.