New Bases for Feynman Integrals

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work with **Giulio Gambuti**, **Pavel Novichkov**, *and* Lorenzo Tancredi [2311.16907];
with **Leonardo de la Cruz** *and* **Pavel Novichkov** [2410.18014];
with Stefano De Angelis, **Rourou Ma**, **Zhihou Wu**, *and* Yang Zhang [2508.04394].
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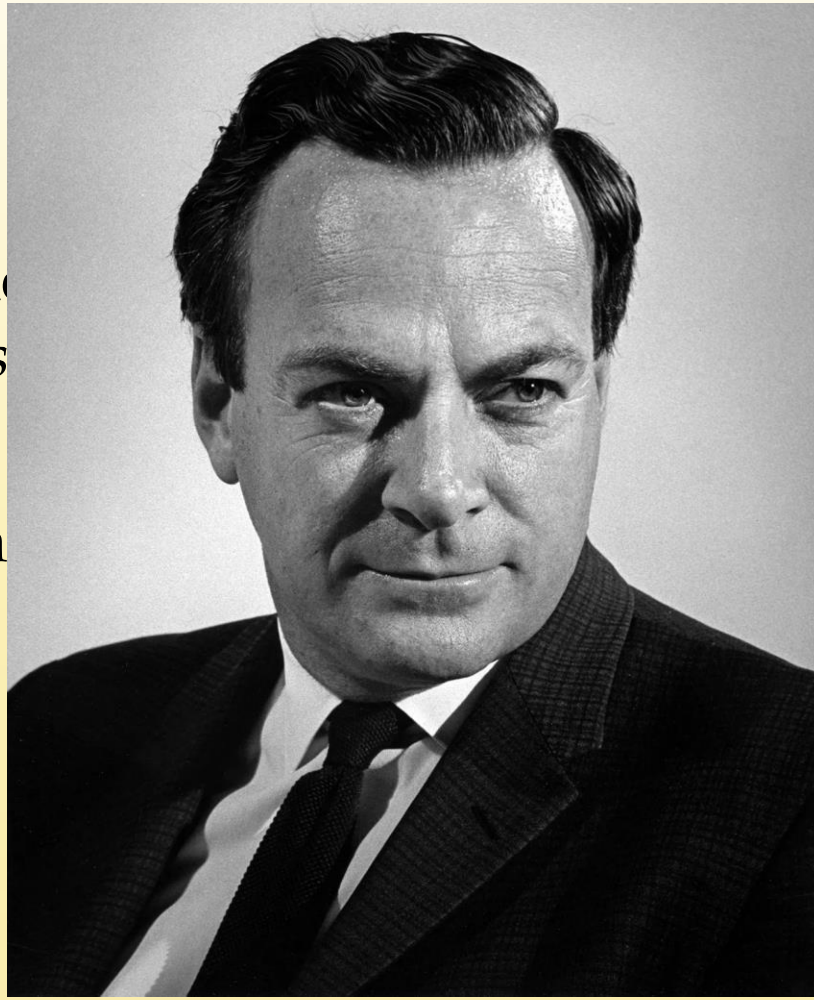
Feynman Integrals



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Goal

- Select Feynman integrals based on degree of divergence

- Hope: lead to simpler and more transparent representations

- First step: classify and organize **finite** integrals

Henn, Peraro, Stahlhofen, & Wasser; von Manteuffel, Panzer, & Schabinger; Agrawal, Jones, von Manteuffel

$\Rightarrow$ Agrawal's talk

- Mostly gloss over fine print
  - look at locally IR-finite integrals (doable strictly in $D = 4$)
  - UV convergent by power counting ("strongly UV convergent")

One-Loop Example

- Unitarity basis: box, triangle, bubble
- Box and Triangle have $\frac{1}{\epsilon^2}$ divergence (IR)
- Bubble has $\frac{1}{\epsilon}$ divergence (UV)
- Can trade $D = 4$ box ($\frac{1}{\epsilon^2}$ divergence) for $D = 6$ box (finite)

$$\mathrm{Box}^{D=4} = c_0 \mathrm{Box}^{D=6} + \sum c_i^{(3)} \, \mathrm{Tri}_i$$

This isolates all IR divergences in triangles

$I_4 \left[G \begin{pmatrix} \ell k_1 k_2 k_4 \\ \ell k_1 k_2 k_4 \end{pmatrix} \right]$ is finite (and in fact $\propto \mathrm{Box}^{D=6}$)

Analytic Strategy

Gambuti, Novichkov, Tancredi, DAK

- Derivable
- Proceed topology by topology
- Solve Landau equations in mixed representation:
  - for all loops i, $\sum_{d=1}^{N} \alpha_d \frac{\partial}{\partial \ell_i} \mathrm{Den}_d = 0$
  - for all denominators d, $\alpha_d \mathrm{Den}_d = 0$
  - at least one α_d strictly positive, all nonnegative
  - Kinematics-independent
  - subtleties for nonplanar integrals
- Each solution is a singular surface

Scaling

Anastasiou & Sterman (2018)

- Soft surface $\ell = 0$:

$$\ell = \lambda^2 Q$$

- Collinear surface $\ell = \alpha\, k$:
  - (k an external massless momentum)

$$\ell \sim x\, k + \lambda^2 Q\, \eta + \lambda Q\, \hat{\ell}_\perp$$

where $\eta^2 = 0,\, \eta \cdot k \neq 0,\, k \cdot \hat{\ell}_\perp = 0 = \eta \cdot \hat{\ell}_\perp$

- Determine scaling of any rational function of ℓ

Procedure

- Start with all factors
  - degree 2: $\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2$
  - degree 1: $\ell_i \cdot k_{1,2,4}$
- Build all numerators of fixed degree, *e.g.*

$$c_1\ \ell_1 \cdot \ell_2 + c_2\ \ell_1 \cdot k_4\, \ell_2 \cdot k_1 + c_3\ (\ell_1 \cdot k_2)^2 + \cdots$$

- For each singular surface from Landau equations, require coefficients of term(s) with singular scaling to vanish

$\rightarrow$ Linear equation(s) for the c_i

- Solve system for each degree
- Require UV finiteness by power counting (Weinberg rule: each loop, and overall)

Independent Numerators

- How many are there (cumulative)?
- 31 singular surfaces for planar double box
- Polynomials

Max Order in ℓ	**1**	**2**	**3**	**4**	**5**
Finite	**0**	**2**	**18**	**89**	**247**

- Not all truly independent

  Poly(ℓ_i) (Finite numerator) (subject to UV power-counting)
- Mathematical structure: ideal (before UV power-counting)
- "truncated ideal" (linear space) after

Independent Numerators

- Appropriate technology: Gröbner bases

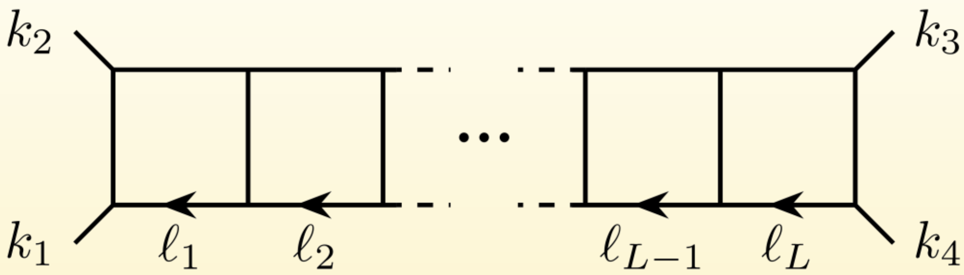
Max Order in ℓ	1	2	3	4	5
Finite	0	2	18	89	247
Independent New	0	2	4	4	0

- Define (van Neerven & Vermaseren)

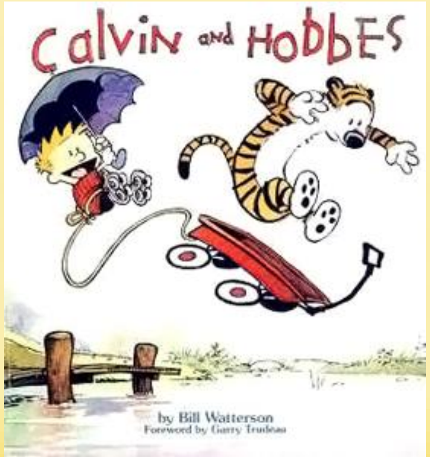
$$v_i^\mu \equiv \frac{G\begin{pmatrix} k_1 & \cdots & \mu & \cdots & k_R \\ k_1 & \cdots & k_i & \cdots & k_R \end{pmatrix}}{G\begin{pmatrix} k_1 & \cdots & k_R \end{pmatrix}}, \quad \nu_{ij} \equiv G\begin{pmatrix} l_i & k_1 & \cdots & k_R \\ l_j & k_1 & \cdots & k_R \end{pmatrix} \Big/ G\begin{pmatrix} k_1 & \cdots & k_R \end{pmatrix}$$

to get nice forms for generators

All-Loop Ladder Conjecture



Max Order in ℓ	1	2	3	4	5	6	7	8
Independent	0	2	$2L$	L^2	0	0	0	0
$\mathcal{O}(\epsilon)$ independent	0	0	0	$(3L^2-9L+8)/2$	$(L-2)(L^2-4L+5)$	$(L-2)\times$ $(L^3-9L+16)/8$	$(L-2)(L-3)\times$ $(L^2-5L+8)/4$	$(L-2)(L-3)\times$ $(L^2-5L+8)/8$



Geometric Strategy

De la Cruz, Novichkov, DAK

- Not yet derivable — lots of conjecture
- Parametric representation
- Focus on exponents of monomials

$$\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_n^{e_n}$$

$$\boldsymbol{e} \equiv (e_1, e_2, \ldots, e_n)$$

- Build on theorem of Berkesch, Forsgård, & Passare on convergence of Euler–Mellin integrals

Geometric Strategy

- Newton polytope: convex hull of all positive-weight linear combinations of all exponent vectors in a given polynomial
- BFP instructs us to look at Newton polytope of Symanzik polynomials

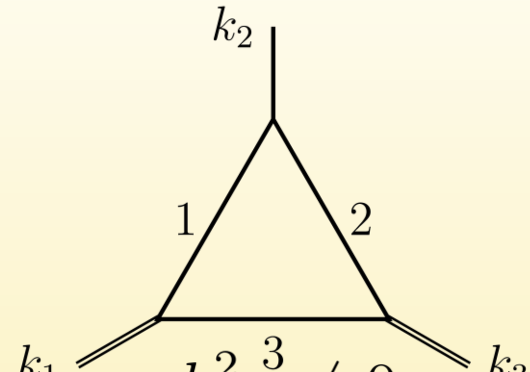
$$\text{Newton}\left(\left[\mathcal{U}^{E-\frac{D}{2}(L+1)-r}\mathcal{F}^{\frac{DL}{2}-E}\right]^{-1}\right)$$

E propagators, D dimensions, L loops, rank r

- BFP: integral of a Feynman-parameter monomial converges if
  - $\mathcal{U}$ and $\mathcal{F}$ have no zeros on faces of polytope (true for planar integrals)
  - the vector $\boldsymbol{e}+\mathbf{1}$ lies in the `relative interior' of the polytope
- Find interior with tools like NConvex, or via conjecture on generating function
- Conjecture: integral is finite iff each parameter monomial is in the relative interior

Geometric Strategy: Toy Example

- Consider the two-mass triangle



$$I_{\triangle}[\mathcal{N}(\ell)] = \int \mathrm{d}^D\ell \, \frac{\mathcal{N}(\ell)}{\ell^2\,(\ell-k_2)^2\,(\ell+k_1)^2}, \qquad k_{1,3}^2 \neq 0$$

- In parametric form

$$I_{\triangle,r}[\mathcal{N}(\ell)] = \sum_{\mathbf{m}\in B} c_{\mathbf{m}} \int \frac{\mathrm{d}^3\alpha}{\alpha_1\alpha_2\alpha_3} \delta(1-\alpha_1-\alpha_2-\alpha_3)\,\alpha^{\mathbf{m}+1}\mathcal{U}^{3-D-r}\mathcal{F}^{D/2-3}$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3, \quad \mathcal{F} = -(k_1^2\alpha_1\alpha_3 + k_3^2\alpha_2\alpha_3)$$

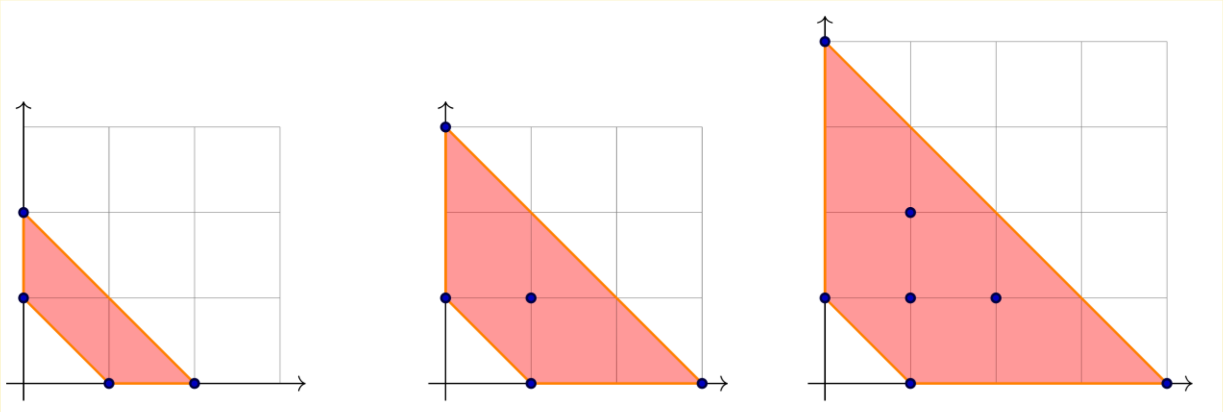
Fix $\alpha_3 = 1$

Feynman Polytope

- All possible exponent vectors:

(0,0); (1,0); (0,1)

$$P_3 = (r + D - 3)\mathrm{Newt}(\mathcal{U}) + \left(3 - \frac{D}{2}\right)\mathrm{Newt}(\mathcal{F})$$



(a) $r = 0$ (b) $r = 1$ (c) $r = 2$

$\Downarrow$ $\Downarrow$

divergent $\ell \cdot k_2$

Comparison

- Do the results of the two approaches agree?
- Yes…

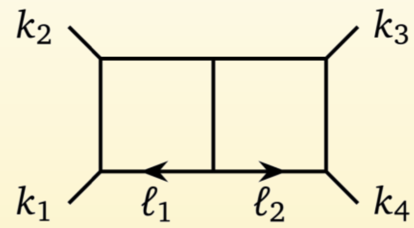
…but the comparison is subtle

  - Strongly UV convergent (by strict power counting) vs [simply] weakly UV convergent (coefficient vanishes)
  - Nontrivial numerators can vanish in parameters (special total derivs)
  - General total derivatives caught in (not locally finite but scooped up)

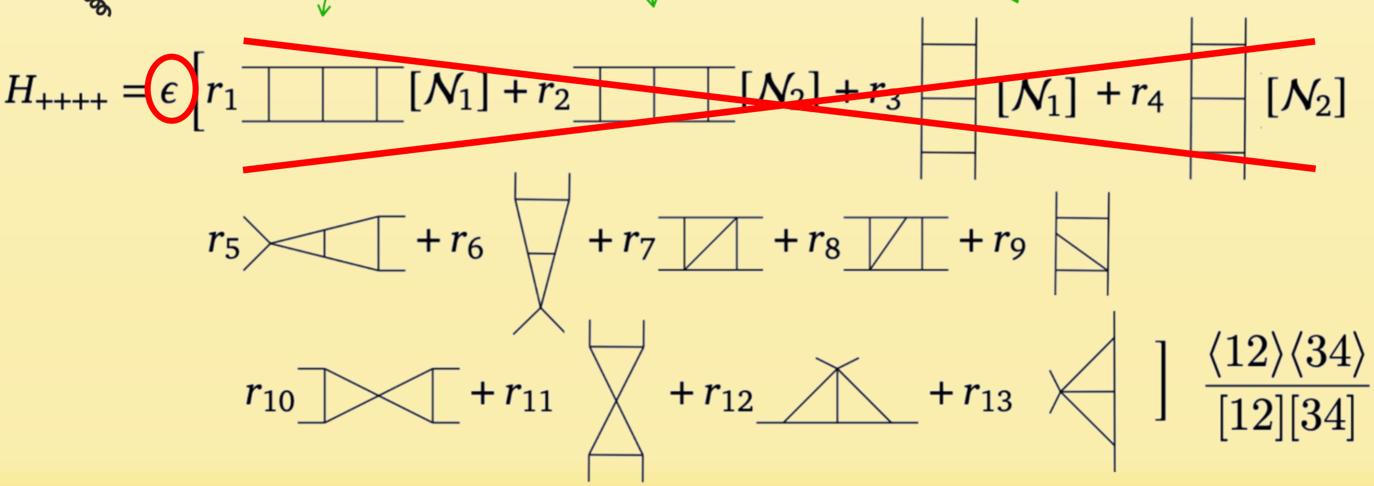
- Compared
  - Planar & nonplanar double box
  - Beetle
  - Three-loop ladder

Use of New Integrals

Look at two-loop $A_4(++++)$



$$\mathcal{N}_1 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} G\begin{pmatrix} \ell_2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}$$



Coefficients simpler too

Singularity Free Bases

- Reductions of integrals to a basis will generically have coefficients singular as $\epsilon \to 0$
- Want to avoid this
- Want to include finite integrals
- Three algorithms

Algorithm I

- Coefficients in $\mathrm{IBP}_i = C_{ij} I_j$
- Row-reduce ϵ-free part $\tilde{C}_{ij} = C_{ij}|_{\epsilon=0} \rightarrow$ (overcomplete) candidate masters
- Define a permutation so that these are ordered last
- Set aside solved integrals
- Look at full IBPs on candidate integrals; it contains singular coefficients
- Clear singular coefficients by multiplying rows by power of ϵ
- Repeat for remaining IBPs; terminate when not overcomplete

Planar Double Box

- Successfully includes two finite integrals
- Basis similar to improved one used for leading-color $A(+++\,+)$

Nonplanar Double Box

- Finite integrals are ejected from the basis

- Yet something special happens for the subleading-color contributions to $A(+++\,+)$: linear combinations of integrals can be traded for combinations of finite integrals

- Can trade five $1/\epsilon^4$ divergent integrals for finite ones

$$i\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\left(\epsilon\, c_1 B_1 + \epsilon^2 c_2 B_2 + \cdots \epsilon^2 c_7 B_7 + c_8 F_1 + \cdots c_{12} F_5\right)$$

Summary

- Finite integrals are a first step to exploring a new organization of scattering amplitudes
- Two approaches
  - Analytic approach: Landau equations + Anastasiou–Sterman scaling
  - Geometric approach: Newton polytopes + BFP theorem
- Include them in basis and avoid singular coefficients
  - All singularities are inside integrals
- Applications
  - Amplitude structure
  - Integrating