Modulations of Gravitational Waves due to Non-Static Gravitational Lenses

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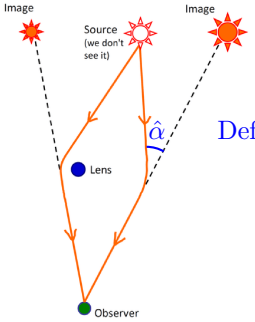
Gravitational Lensing

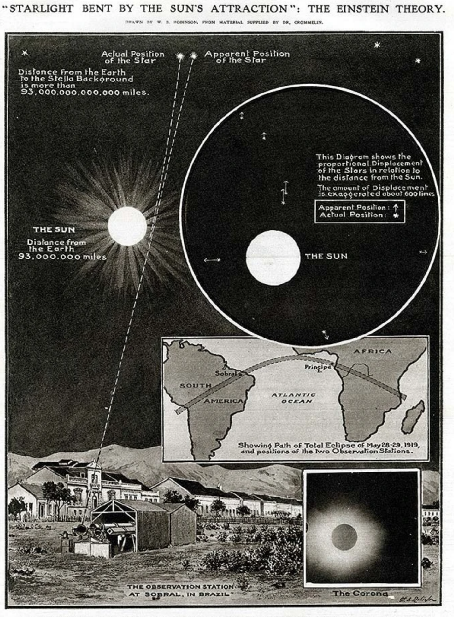


Light bent by gravity

- **1919** Frank Watson Dyson and Arthur Stanley Eddington organized two expeditions to observe the total solar eclipse of 29 May.

- Light ray that grazes the surface of the Sun should be deflected by:
  - $\sim$0.87 arcseconds
    assuming Newtonian gravity
  - $\sim$1.75 arcseconds
    assuming General Relativity

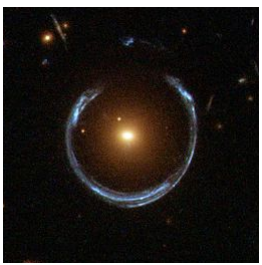
- Widespread newspaper coverage of the results led to worldwide fame for Einstein and his theories.



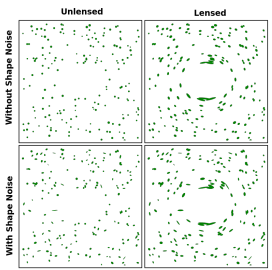
Page of the 22 November 1919 edition of *The Illustrated London News*

Classes of Gravitational Lensing

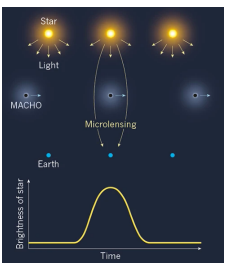
	Strong	Weak	Micro
Lens Mass Scale	Galaxies, Clusters	LSS	Stars, PBHs, Compact DM
Observable Effect	Multiple images, Arcs, Rings	Shape distortions	Brightness, Waveform modulation
Usage	Time-delay cosmography, Dark matter maps	Structure growth, Dark energy constraints	Compact DM searches, Exoplanets



Strong lensing



Weak lensing



Micro lensing

Gravitational Lensing of Gravitational Waves

- Like light, GWs can be gravitationally lensed
  - Both are deflected by gravitational potentials (spacetime curvature).
  - Governed by the same null geodesics in the geometric optics limit.
  - Can produce strong lensing (multiple images), weak lensing (distortion), and microlensing (amplification).

Key Differences

- EM: nm to mm ⇒ almost always in geometric optics.
  GW: km to millions of km ⇒ often in wave optics regime.
- EM: affected by scattering and absorption.
  GW: very weakly interact with intervening matter ⇒ ideal cosmic messengers
- EM: image distortion, magnification, time delays.
  GW: same effects *plus* interference fringes, frequency-dependent modulation.
- EM: visible in images/spectra.
  GW: detectable through waveform analysis (amplitude, phase, timing).

GW lensing provides access to lens mass scales and wave phenomena inaccessible to EM lensing.

- Background metric

$$ds^2 = -[1 + 2U(t, \boldsymbol{r})]dt^2 + [1 - 2U(t, \boldsymbol{r})]d\boldsymbol{r}^2 \equiv g^{(\mathrm{B})}_{ab} dx^a dx^b$$

$|U(t, \boldsymbol{r})| \ll 1$: gravitational potential of the lensing objects

– Applicable to static and non-static gravitational lenses

- GW propagation equation

$$h_{ab;c}{}^{;c} + 2R^{(\mathrm{B})}_{cadb} h^{cd} = 0$$

Transverse Traceless Lorentz gauge condition

$$h_{ab}{}^{;b} = 0 = h^{c}{}_{c}$$

Semicolon: covariant derivative with respect to background metric $g^{(\mathrm{B})}_{ab}$

$R^{(\mathrm{B})}_{cadb}$: background Riemann tensor

Scalar Waves

- Typical wavelength of GWs $\ll$ Typical radius of spacetime curvature
  $\Rightarrow$ Eikonal expansion

$$h_{ab} = \Re\left\{e^{iS/\epsilon}\left[A_{ab} + \epsilon B_{ab} + \mathcal{O}(\epsilon^2)\right]\right\}$$

$\Re$: real part
ϵ: a formal expansion parameter

- Define the scalar amplitude A, wave vector k_a, and polarization tensor e_{ab} as

$$A \equiv \left(\frac{1}{2}A^*_{ab}A^{ab}\right)^{1/2}$$
$$k_a \equiv S_{,a}$$
$$e_{ab} \equiv A_{ab}/A$$

- The polarization tensor e_{ab} is parallel-transported along the null geodesic, and its change due to gravitational lensing is of the order of U, which is very small in the observational situation.

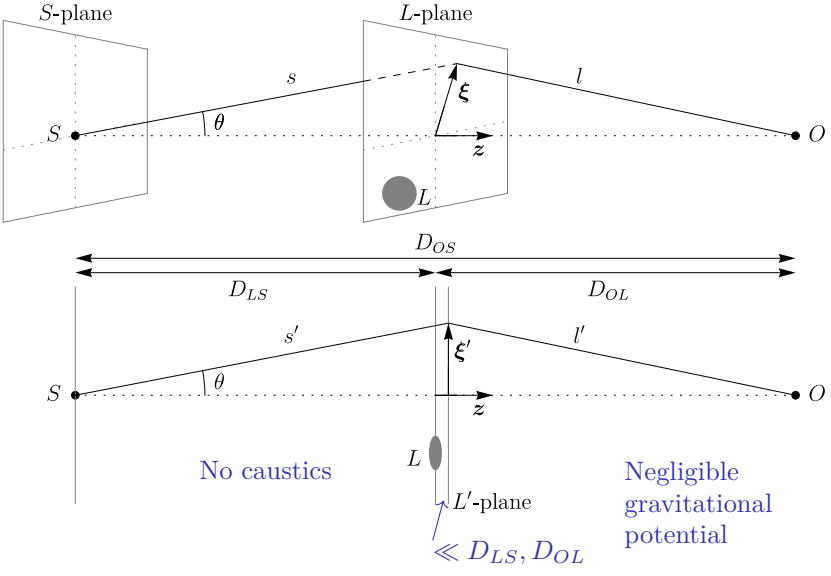
  $\Rightarrow$ Regard the polarization tensor as a constant

  $\Rightarrow$ Propagation of GWs h_{ab} can be treated as the propagation of scalar waves ϕ

$$h_{ab} = \Re\left\{\phi\ e_{ab}\right\} \simeq \Re\left\{Ae^{iS}e_{ab}\right\}$$

$h_{ab} \in \mathbb{R}$
$\phi \in \mathbb{C}$

Diagram of Gravitational Lensing System



$$\phi(t,\boldsymbol{r}|_S) \xrightarrow{U(t,\boldsymbol{r})} \phi(t,\boldsymbol{r}|_{L'}) \longrightarrow \phi(t,\boldsymbol{r}|_O)$$

Lensed Field in L'-plane

- Spherical monochromatic outgoing waves of frequency ω from the source

$$\phi(t, \boldsymbol{r}) = \frac{1}{r} e^{-i\omega(t-r)}$$

- Inserting the eikonal expansion into the propagation equation and gauge condition

$$0 = k_a k^a$$
$$0 = 2A_{,a} k^a + A k^a{}_{;a}$$

$|U| \ll 1 \Rightarrow$ Solving equations order by order of $|U|$

- Lensed field ϕ^L in the L'-plane

$$\phi^L(t, \boldsymbol{r})|_{L'} \simeq \frac{1}{s'} e^{-i\omega[t-s'+\hat{\Psi}(t,s',\boldsymbol{\xi}')]}$$

$$\hat{\Psi}(t, s', \boldsymbol{\xi}') \equiv 2 \int_0^{s'} U(t - s' + \tilde{r}, \tilde{r}, \theta, \varphi) d\tilde{r}$$

where $\boldsymbol{\xi}'$ is the coordinate in L'-plane, $\boldsymbol{\xi}' = (s' \sin\theta \cos\varphi, s' \sin\theta \sin\varphi)$.

Observed Lensed Field

- Beyond L'-plane

$$\Box\phi^L = 0$$

- General form of Kirchhoff's theorem
  $\Rightarrow$ observed lensed field ϕ^L

$$\phi^L(t, \boldsymbol{r}_O) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}^L_{\tilde{\omega}}(\boldsymbol{r}_O) e^{-i\tilde{\omega}t} d\tilde{\omega}$$

with $$\tilde{\phi}^L_{\tilde{\omega}}(\boldsymbol{r}_O) = \frac{1}{4\pi} \int_{L'} d^2\boldsymbol{\xi}' \left\{ \tilde{\phi}^L_{\tilde{\omega}} \frac{\partial}{\partial z} \left(\frac{e^{i\tilde{\omega}l'}}{l'} \right) - \frac{e^{i\tilde{\omega}l'}}{l'} \frac{\partial \tilde{\phi}^L_{\tilde{\omega}}}{\partial z} \right\}$$

- With small-angle approximation

$$\phi^L(t, \boldsymbol{r}_O) = \frac{1}{4\pi} \int_{L'} d^2\boldsymbol{\xi}' \frac{e^{-i\omega[t-l'-s'+\hat{\Psi}(t-l', s', \boldsymbol{\xi}')]}}{l's'} (-2i\omega - i\omega\partial_t\hat{\Psi})$$

Amplification Factor

- The amplification factor is a widely used quantity in the lensing theory

$$F(t,\omega) \equiv \frac{\phi^L(t,\boldsymbol{r}_O)}{\phi(t,\boldsymbol{r}_O)}$$

- **Non-static** gravitational lenses

$$\boxed{F(t,\omega) = (1+\frac{i}{2\omega}\partial_t)\left\{\frac{\omega}{2\pi i}\frac{D_{OS}}{D_{OL}D_{LS}}\int_L d^2\boldsymbol{\xi}\exp\left(i\omega\left[\frac{1}{2}\boldsymbol{\xi}^2\frac{D_{OS}}{D_{OL}D_{LS}}-\Psi(t-D_{OL},\boldsymbol{\xi})\right]\right)\right\}}$$

with deflection potential

$$\Psi(t,\boldsymbol{\xi}) \equiv 2\int U(t+z,\boldsymbol{\xi},z)dz$$

- **Static** gravitational lenses

$$F_0(\omega) = \frac{\omega}{2\pi i}\frac{D_{OS}}{D_{OL}D_{LS}}\int_L d^2\boldsymbol{\xi}\exp\left(i\omega\left[\frac{1}{2}\boldsymbol{\xi}^2\frac{D_{OS}}{D_{OL}D_{LS}}-\Psi_0(\boldsymbol{\xi})\right]\right)$$

with

$$\Psi_0(\boldsymbol{\xi}) \equiv 2\int U(\cdot,\boldsymbol{\xi},z)dz$$

Amplification Factor

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with deflection potential

$$\Psi(t,\boldsymbol{\xi}) \equiv 2\int U(t+z,\boldsymbol{\xi},z)dz$$

– $\{\cdots\}$ term is similar to the quasi-static approximation, in which the usual lensing solution is evaluated at each snapshot, effectively treating the lens as static at any given instant.

– $\frac{i}{2\omega}\partial_t\{\cdots\}$ term arises from the next-order expansion of the wave equation in the presence of a non-static gravitational potential. It captures first-order corrections due to the genuine time dependence of the lens.

Geometric Optics Limit

- Dimensionless time delay

$$T(\boldsymbol{x};\boldsymbol{y},\tau) \equiv \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2 - \psi(\tau,\boldsymbol{x})$$

$\boldsymbol{x} \equiv (\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}_L)/\xi_*$
$\boldsymbol{y} \equiv -\bar{\boldsymbol{\xi}}_L/\xi_*$
$t_* \equiv \frac{D_{OS}}{D_{OL}D_{LS}}\xi_*^2$
$w \equiv \omega t_*$
$\tau \equiv t/t_*$
$\psi \equiv \Psi/t_*$

- Stationary points $\boldsymbol{x}_j(\boldsymbol{y},\tau)$ of $T(\boldsymbol{x};\boldsymbol{y},\tau) \Leftarrow \partial_{\boldsymbol{x}} T(\boldsymbol{x};\boldsymbol{y},\tau) = 0$
  - Corresponds to the position of j-th image in geometric optics.
  - Time dependence of $\boldsymbol{x}_j(\boldsymbol{y},\tau)$ means that the image position could vary with time.
- Geometric Optics, $w \gg 1$

$$F(t,\omega) = \sum_j F_j(t,\omega) = \sum_j (1 + \frac{i}{2w}\partial_\tau)\left\{\left|\mu\big(\boldsymbol{x}_j(\boldsymbol{y},\tau)\big)\right|^{1/2} \exp\Big(iwT\big(\boldsymbol{x}_j(\boldsymbol{y},\tau)\big) - i\pi n_j\Big)\right\}$$

where

$$\mu\big(\boldsymbol{x}_j(\boldsymbol{y},\tau)\big) = 1/\det\left[\partial_a\partial_b T\big(\boldsymbol{x}_j(\boldsymbol{y},\tau)\big)\right]$$

$n_j = 0, 1/2, 1$ when $\boldsymbol{x}_j(\boldsymbol{y},\tau)$ is a minimum, saddle, maximum point of $T(\boldsymbol{x};\boldsymbol{y},\tau)$, respectively.

$$h_{ab} = \Re\{\phi\ e_{ab}\}$$
$$h_{ab} \in \mathbb{R} \qquad \phi \in \mathbb{C}$$
$$F(t,\omega)^* = F(t,-\omega)$$

At observer's position $\boldsymbol{r}_O$

$$h(t) \overset{\text{FT}}{\sim} \int d\omega\ \tilde{h}(\omega)\ e^{-i\omega t}$$

$$\text{Gravitational Lensing} \Big\Downarrow\ e^{-i\omega t} \to e^{-i\omega t}\ F(t,\omega)$$

$$\int d\omega\ \tilde{h}^L(\omega)\ e^{-i\omega t} \overset{\text{FT}}{\sim} h^L(t) \sim \int d\omega\ \tilde{h}(\omega)\ e^{-i\omega t}\ F(t,\omega)$$

$$\Downarrow \qquad \sim \int d\omega \int dt'\ e^{i\omega(t'-t)}\ h(t')\ F(t,\omega)$$

$$\tilde{h}^L(\omega) \sim \int dt \int d\omega'\ e^{i(\omega-\omega')t}\ \tilde{h}(\omega')\ F(t,\omega')$$

$$\text{Static Lens} \Big\Downarrow\ F(t,\omega) \to F_0(\omega)$$

$$\tilde{h}^L(\omega) \sim \tilde{h}(\omega)\ F_0(\omega)$$

Static Gravitational Lens

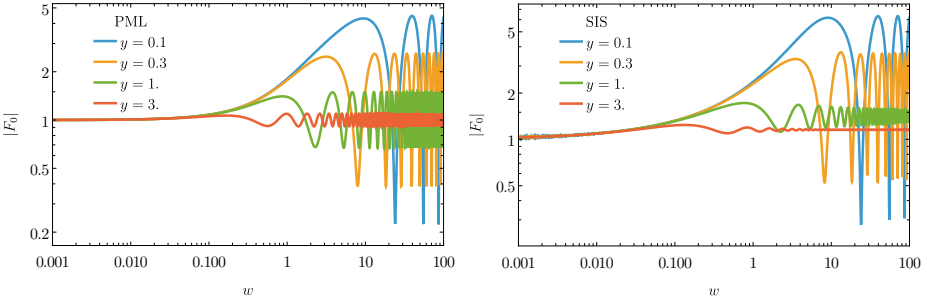
- Point-Mass Lens

$$F_0(w, \boldsymbol{y}) = \exp(\frac{\pi w}{4} + \frac{iw}{2}\ln\frac{w}{2})\ \Gamma(1 - \frac{iw}{2})\ {}_1F_1(\frac{iw}{2}, 1; \frac{iw}{2}y^2)$$

$$\boldsymbol{y} \equiv -\bar{\boldsymbol{\xi}}_L/\xi_*$$
$$t_* \equiv \frac{D_{OS}}{D_{OL}D_{LS}}\xi_*^2$$
$$w \equiv \omega t_*$$

- Singular Isothermal Sphere

$$F_0(w, \boldsymbol{y}) = -iwe^{iwy^2/2}\int_0^\infty dx\ xJ_0(wxy)\exp\left(iw\left[\frac{1}{2}x^2 - x\right]\right)$$



Time-independent but frequency-dependent amplification

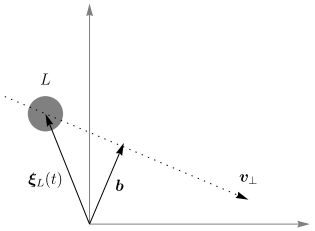
Uniformly Moving Lens

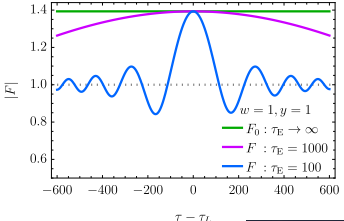
$$\boldsymbol{\xi}_L(t) = \boldsymbol{b} + \boldsymbol{v}_\perp (t - t_0)$$

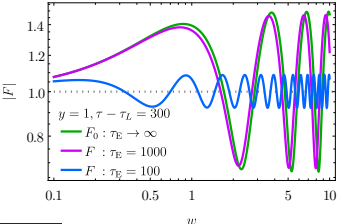
$$F(t,\omega) = K(w)\,(1+\frac{i}{2w}\partial_\tau)\;{}_1F_1(\frac{iw}{2},1;\frac{iw}{2}[y^2+(\tau-\tau_L)^2/\tau_E^2])$$

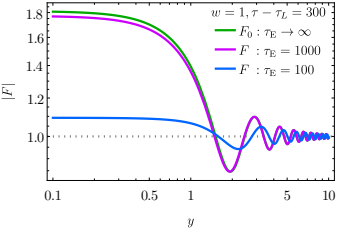
$$K(w) \equiv \exp(\frac{\pi w}{4} + \frac{iw}{2}\ln\frac{w}{2})\;\Gamma(1-\frac{iw}{2})$$

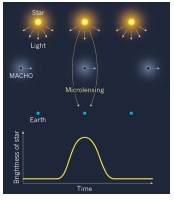
$$\tau_E \equiv t_E/t_*, \quad t_E \equiv \xi_*/|\boldsymbol{v}_\perp| = R_{\rm E}/|\boldsymbol{v}_\perp|,$$
$$\tau_L \equiv t_L/t_*, \quad t_L \equiv t_0 + D_{OL}.$$



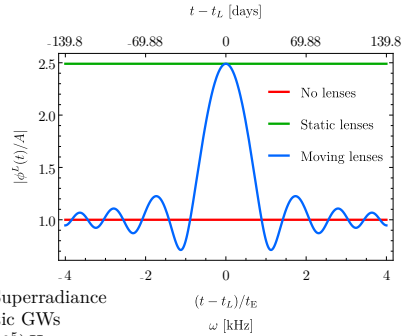


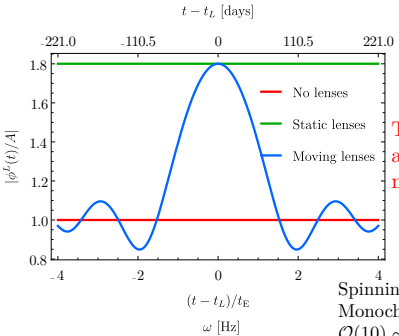






Lensed Monochromatic GWs by Uniformly Moving Lens

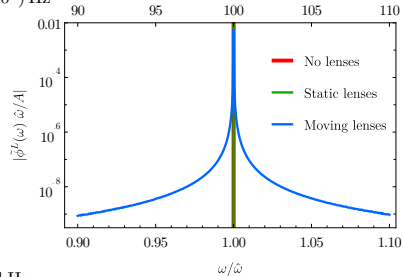


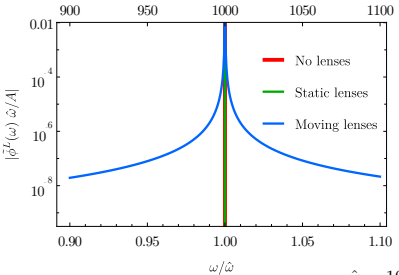


Time-varying amplitude modulations

QCD Axion Superradiance
Monochromatic GWs
$\mathcal{O}(10^4) \sim \mathcal{O}(10^5)\,\mathrm{Hz}$

Spinning neutron stars
Monochromatic GWs
$\mathcal{O}(10) \sim \mathcal{O}(10^3)\,\mathrm{Hz}$





Spectral broadening

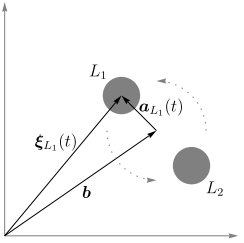
$\hat{\omega} = 100\,\mathrm{kHz}$
$M_L = 1\,M_\odot$
$D_{OL} = D_{LS} = 1\,\mathrm{kpc}$

GWs lensed by uniformly moving lens with $v_L = 100\mathrm{km/s}$ and $y = 0$.

$\hat{\omega} = 1000\,\mathrm{Hz}$
$M_L = 50\,M_\odot$
$D_{OL} = D_{LS} = 50\,\mathrm{pc}$

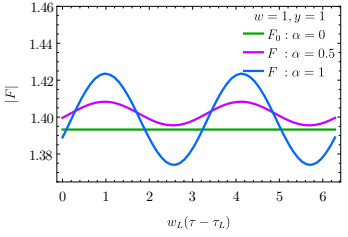
Orbiting Binary Lenses

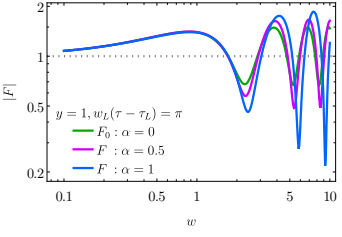
$$\boldsymbol{a}_{L_1}(t) = +\frac{q}{1+q} R\Big(\cos[\omega_L(t-t_0)],\ \sin[\omega_L(t-t_0)]\Big)$$
$$\boldsymbol{a}_{L_2}(t) = -\frac{1}{1+q} R\Big(\cos[\omega_L(t-t_0)],\ \sin[\omega_L(t-t_0)]\Big)$$

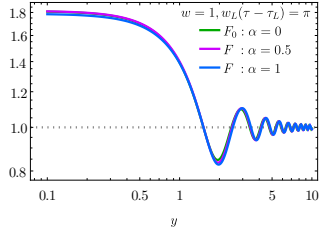


$$F(t,\omega) = (1+\frac{i}{2w}\partial_\tau)\Bigg\{\frac{w}{2\pi i}\int_0^\infty x\,dx\int_0^{2\pi} d\varphi\ \exp\Bigg(iw\Bigg[\frac{1}{2}(x^2-2xy\sin\varphi+y^2)$$
$$-\frac{1}{2(1+q)}\ln\{x^2-2x\frac{q}{1+q}\alpha\cos[\varphi-w_L(\tau-\tau_L)]+(\frac{q}{1+q}\alpha)^2\}$$
$$-\frac{q}{2(1+q)}\ln\{x^2+2x\frac{1}{1+q}\alpha\cos[\varphi-w_L(\tau-\tau_L)]+(\frac{1}{1+q}\alpha)^2\}\Bigg]\Bigg)\Bigg\}$$

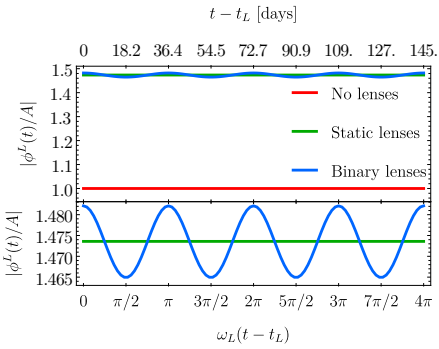
$$\alpha \equiv R/\xi_*$$
$$w_L \equiv \omega_L t_*$$
$$\tau_L \equiv t_L/t_*$$
$$t_L \equiv t_0 + D_{OL}$$

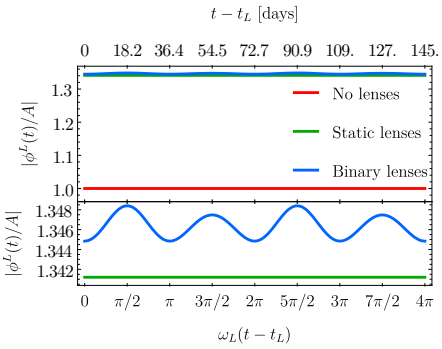






Lensed Monochromatic GWs by Orbiting Binary Lenses





Monochromatic GWs lensed by orbiting binary gravitational lenses with $\omega_L = 10^{-6}\,\mathrm{Hz}$ and $y = 1$.

frequency beyond observation scope

$\hat{\omega} = 100\,\mathrm{kHz}$
$M_{L_1} = M_{L_2} = 1\,M_\odot$
$D_{OL} = D_{LS} = 1\,\mathrm{kpc}$

frequency inside observation scope

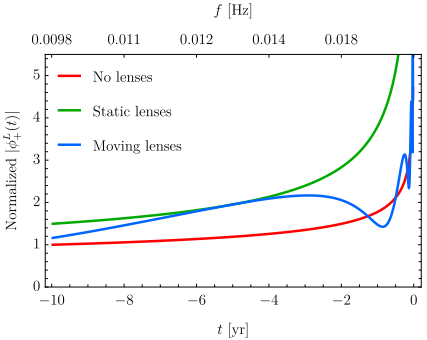
$\hat{\omega} = 1000\,\mathrm{Hz}$
$M_{L_1} = 50\,M_\odot$
$M_{L_2} = 10\,M_\odot$
$D_{OL} = D_{LS} = 50\,\mathrm{pc}$

Lensed GWs from Binary Inspiral

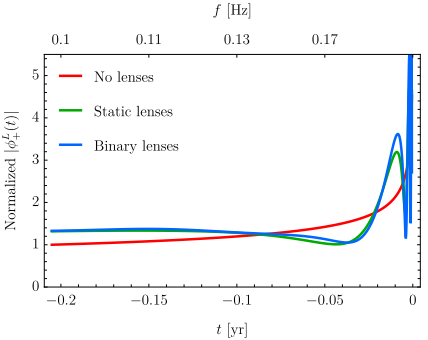
Chirp signal of GWs from the inspiral of circular orbiting binary

$$h_+(t) \equiv \Re\{\phi_+(t)\} = \frac{1}{d}\frac{GM_c}{c^2}\left(\frac{5GM_c}{c^3(t_c-t)}\right)^{1/4}\frac{1+\cos^2\iota}{2}\cos\left[-2\left(\frac{5GM_c}{c^3(t_c-t)}\right)^{-5/8}+\Phi_0\right]$$

Lensed GWs originated from inspiral of circular orbiting binary at $D_{OS} = 1\,\mathrm{Mpc}$.



GWs from source $M_{c,S} = 40\,M_\odot$
lensed by uniformly moving lens
$M_L = 5\times10^5\,M_\odot$, $D_{LS} = 1\,\mathrm{pc}$
$t_L = -5\,\mathrm{yr}$, $v_L = 100\mathrm{km/s}$, $y = 0$



GWs from source $M_{c,S} = 10\,M_\odot$
lensed by orbiting binary lenses
$M_{L_1} = M_{L_2} = 5\times10^4\,M_\odot$, $D_{LS} = 1\,\mathrm{pc}$
$\omega_L = 10^{-6}\,\mathrm{Hz}$, $y = 1$

Current Status

Observational Signatures

- Multiple GW images arriving with measurable time delays.
- Repeated signals with identical waveforms and sky positions.
- Frequency-dependent modulation and interference fringes in the waveform.
- Possible distortions in the chirp signal.

Current Status

- No *conclusive* evidence for lensed GWs yet, several candidate pairs (e.g. GW170809 & GW170814) under investigation.
- Upper limits placed on lensing optical depth and compact object density.

Challenges

- Lensing degeneracies: magnified events mimic higher-mass mergers.
- Large sky localization uncertainty hinders cross-identification.
- Need for high-SNR, well-modeled signals to resolve waveform distortions.

Summary and Outlook

- Lensing modifies GW signals through:
  - Amplification, time delays, waveform distortions
  - Interference fringes and frequency-dependent modulations in the wave optics regime
- Lensing opens a new observational window for:
  - Cosmology (e.g., time-delay H_0 measurement)
  - Astrophysics (e.g., probing compact dark matter)
  - Fundamental physics (e.g., testing gravity theories)
- Outlook
  - Current detectors (LIGO/Virgo/KAGRA) may detect first lensed GW events in the near future.
  - Future observatories (LISA, Taiji, TianQin, ET, CE) will significantly enhance sensitivity to lensing signatures.
  - Gravitational lensing of GWs is poised to become a precision tool for probing the Universe.

Backup

- Background metric
$$ds^2 = a^2(\eta)\big\{-[1+2U(\eta,\boldsymbol{r})]d\eta^2 + [1-2U(\eta,\boldsymbol{r})]d\boldsymbol{r}^2\big\}$$

- Spherical outgoing waves from the source
$$\phi(t(\eta),\boldsymbol{r}) = \frac{1}{a(\eta)r}e^{-i\omega(\eta-r)}$$

- Local coordinates in the neighborhood of lenses when GWs pass through
$$\mathfrak{t} \equiv a_L\eta, \qquad \boldsymbol{\zeta} \equiv a_L\boldsymbol{\xi}, \qquad \mathfrak{z} \equiv a_L z,$$
where $a_L \equiv a(\eta_L) \equiv 1/(1+Z_L)$, η_L is the conformal time when GWs pass through the lenses and Z_L is the corresponding redshift.

- Angular diameter distance d_- and comoving distance D_-
$$\begin{aligned} d_{OL} &= a_L D_{OL} = D_{OL}/(1+Z_L),\\ d_{OS} &= a_S D_{OS} = D_{OS}/(1+Z_S),\\ d_{LS} &= a_S D_{LS} = D_{LS}/(1+Z_S), \end{aligned}$$
where Z_S is the redshift of the source.

- Amplification factor
$$F(t,\omega) = (1+\frac{i}{2\omega_{Z_L}}\partial_{\mathfrak{t}})\left\{\frac{\omega_{Z_L}}{2\pi i}\frac{d_{OS}}{d_{OL}d_{LS}}\int_L d^2\boldsymbol{\zeta}\exp\left(i\omega_{Z_L}\left[\frac{1}{2}\boldsymbol{\zeta}^2\frac{d_{OS}}{d_{OL}d_{LS}} - \Upsilon(\mathfrak{t}-d_{OL},\boldsymbol{\zeta})\right]\right)\right\},$$
where $\omega_{Z_L} \equiv \omega(1+Z_L)$.

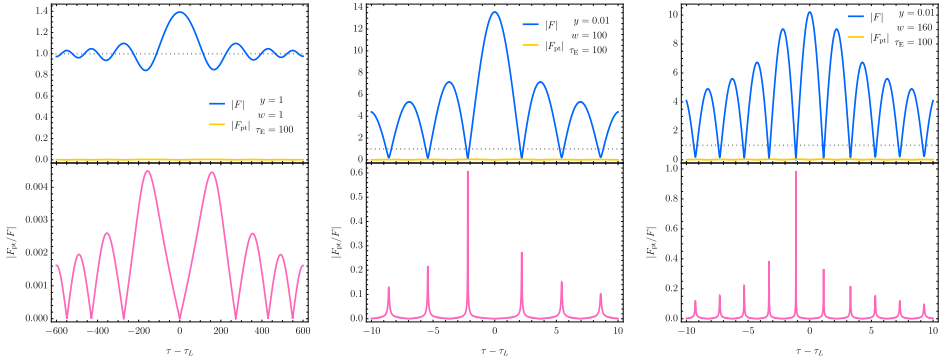
To calculate the amplification factor at the present epoch, one can simply do the calculations in the local coordinates of lenses following equation above, and then transfer the results to cosmic time by $\mathfrak{t} \simeq a_L t = t/(1+Z_L)$.

Effect of the time-derivative term

$$F(t,\omega) = F_{\rm qs}(t,\omega) + F_{\rm pt}(t,\omega)$$

$$F_{\rm qs}(t,\omega) = K(w)\ \ {}_1F_1(\frac{iw}{2},1;\frac{iw}{2}[y^2+(\tau-\tau_L)^2/\tau_E^2])$$

$$F_{\rm pt}(t,\omega) = K(w)\ \frac{i}{2w}\partial_\tau\left\{\ {}_1F_1(\frac{iw}{2},1;\frac{iw}{2}[y^2+(\tau-\tau_L)^2/\tau_E^2])\right\}$$



Examples showing higher w and smaller y can make the time-derivative term more significant at certain intervals.