Non-invertible Chiral Symmetry and Axions under Electromagnetic Duality

Work in collaboration with T. Gherghetta and J. Terning
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Outline

- What's been overlooked in U(1) gauge theory with electrons
  → Non-invertible (NI) chiral symmetry

- Is there NI chiral symmetry in a dual frame?
  → Aspects of QED, ABJ anomaly in a dual frame

- Applications: 1. Puzzle at a fixed point in a dual frame
  2. Axion mass from non-zero pairwise helicity

What's been overlooked

- In U(1) gauge theory w/ m=0 electrons,
  ∃ ABJ anomaly for $U(1)_{\text{chiral}}$

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F$$

so, under $U(1)_{\text{chiral}}$, functional integral measure varies as

$$\prod_{j=1}^{N_f} \mathcal{D}\Psi_j \, \mathcal{D}\bar{\Psi}_j \;\; \to \prod_{j=1}^{N_f} \mathcal{D}\Psi_j \, \mathcal{D}\bar{\Psi}_j \; e^{-i\alpha \mathcal{A} Q_T} \qquad\qquad Q_T = \frac{1}{8\pi^2} \int_M F \wedge F$$

but, Q_T is always an integer → real symmetry is $Z_{\mathcal{A}}$

But then, how to justify using selection rules of the full $U(1)_{\text{chiral}}$?

What's been overlooked

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  ∃ ABJ anomaly for $U(1)_{\text{chiral}}$

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$$\prod_{j=1}^{N_f} \mathcal{D}\Psi_j \, \mathcal{D}\bar{\Psi}_j \;\rightarrow \prod_{j=1}^{N_f} \mathcal{D}\Psi_j \, \mathcal{D}\bar{\Psi}_j \; e^{-i\alpha \mathcal{A} Q_T} \qquad Q_T = \frac{1}{8\pi^2} \int_M F \wedge F$$

but, Q_T is always an integer → real symmetry is $Z_{\mathcal{A}}$

Are you thinking ∃ no Abelian instanton?

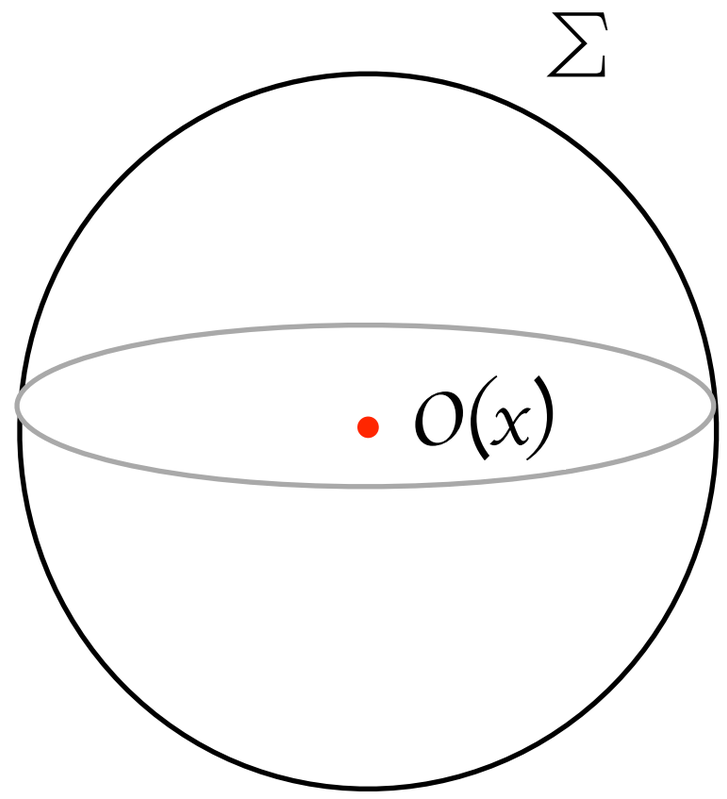
→ and thus $\mathrm{d} \star j_{\text{chiral}} \approx 0$,…… and thus ∃ $U(1)_{\text{chiral}}$??

What's been overlooked

- The modern perspective on symmetry requires
  → Topological + Gauge Invariant symmetry operator
  + charged object

- For 0-form symmetry,

$$e^{i\alpha \int_\Sigma \star j^{[1]}}$$



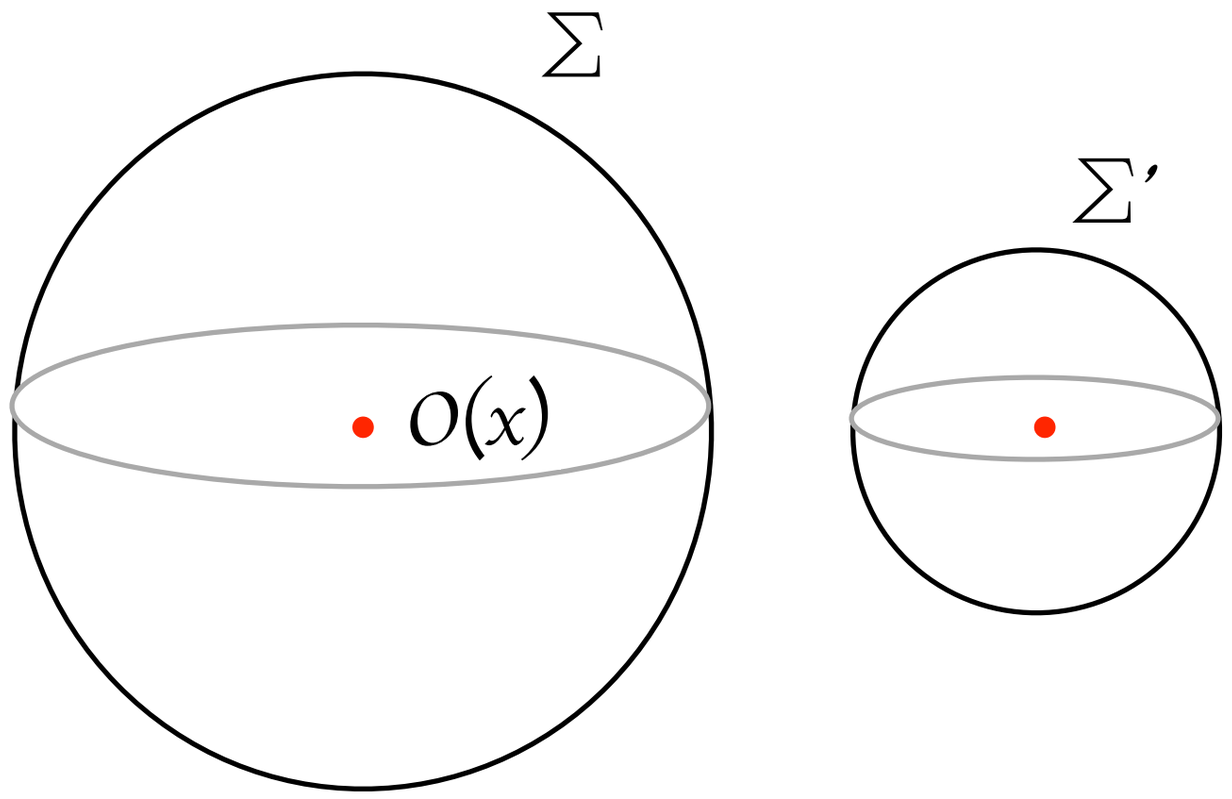
What's been overlooked

- The modern perspective on symmetry requires
  → Topological + Gauge Invariant symmetry operator
  + charged object

- For 0-form symmetry,

$$\mathrm{d} \star j^{[1]} = 0$$

$$\Rightarrow \quad e^{i\alpha \int_\Sigma \star j^{[1]}} = e^{i\alpha \int_{\Sigma'} \star j^{[1]}}$$



What's been overlooked

- The modern perspective on symmetry requires
  → Topological + Gauge Invariant symmetry operator
  + charged object

- Maybe try to redefine current?

$$\star j'_{\text{chiral}} = \star j_{\text{chiral}} - \frac{\mathcal{A}}{8\pi^2} A \wedge \mathrm{d}A$$

No, doesn't work; not gauge invariant

What's been overlooked

- Dress naive operator with the FQH action

Choi, Lam, Shao 22'
Cordova, Ohmori 22'

$$\mathcal{D}_{\frac{1}{N}}(M) = \int [Da]_M \exp\left[\oint_M \left(\frac{2\pi i}{N} \star j^{\mathrm{A}} + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)\right]$$

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$$-\frac{1}{4\pi N} A \wedge \mathrm{d}A, \quad a = -\frac{A}{N}$$

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- Possible to have symmetry operator for $\alpha = 2\pi/k$ w/ a rational k

- $U(1)_{\text{chiral}}$ is not broken to discrete invertible subgroup
  → but changed its nature
  → this justifies selection rule of $U(1)_{\text{chiral}}$ which we've used

Question

- Belief: Dual theories enjoy the same physics

- Sometimes dual theories are useful…

- Is there NI-$U(1)_{chiral}$ in a dual theory?
  Is there NI-$U(1)_{chiral}$ in a U(1) gauge theory with m=0 dyons?

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$$\mathcal{D}_{\frac{1}{N}}(M) = \int [Da]_M \exp\left[\oint_M \left(\frac{2\pi i}{N} \star j^{\mathrm{A}} + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)\right]$$

$$B^{[2]} = \frac{\mathrm{d}A}{N}$$

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F \qquad \mathrm{d} \star j_m^{[2]} = \mathrm{d}\left(\frac{F}{2\pi}\right) = 0$$

Aspects of QED

- Action

$$S = \int -\frac{1}{2e^2} F \wedge \star F + \frac{\theta}{8\pi^2} F \wedge F - A \wedge \star J$$
$$+ \int d^4x \sqrt{-g} \sum_{j=1}^{N_f} i \bar{\Psi}_j \partial_\mu \gamma^\mu \Psi_j \,,$$

- Away from electric charge, define dual gauge field $\tilde{A}$

From e.o.m of A

$$\mathrm{d}\left(\frac{1}{e^2} \star F - \frac{\theta}{2\pi} \frac{F}{2\pi} \right) = 0 \quad \rightarrow \quad \frac{\mathrm{d}\tilde{A}}{2\pi} = -\frac{1}{e^2} \star F + \frac{\theta}{2\pi} \frac{F}{2\pi}$$

Aspects of QED

- Action

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$$+ \int d^4x \sqrt{-g} \sum_{j=1}^{N_f} i \bar{\Psi}_j \partial_\mu \gamma^\mu \Psi_j \,,$$

- When $\star J \neq 0$,

$$\mathrm{d}\left(\frac{1}{e^2} \star F - \frac{\theta}{2\pi} \frac{F}{2\pi} \right) = \star J$$

$$\rightarrow \qquad Q_{\mathrm{e}} = \int_\Sigma \star J = \int_{\partial\Sigma} \left(\frac{1}{e^2} \star F - \frac{\theta}{2\pi} \frac{F}{2\pi} \right) \in \mathbb{Z}$$

Aspects of QED

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- Not possible to have
  Lorentz invariant Lagrangian with a single A
  that couples to both electric J and magnetic current K,

Aspects of QED

- Action

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- Not possible to have
  Lorentz invariant Lagrangian with a single A
  that couples to both electric J and magnetic current K,
  BUT, can formally write

$$\mathrm{d}\left(\frac{F}{2\pi}\right) = \star K \quad \rightarrow \quad Q_{\mathrm{m}} = \int_\Sigma \star K = \int_{\partial\Sigma} \frac{F}{2\pi} \in \mathbb{Z}$$

Aspects of QED

- Action

$$S = \int -\frac{1}{2e^2} F \wedge \star F + \frac{\theta}{8\pi^2} F \wedge F - A \wedge \star J$$
$$+ \int d^4x\sqrt{-g} \sum_{j=1}^{N_f} i\bar{\Psi}_j \partial_\mu \gamma^\mu \Psi_j \,,$$

- So, we have well defined

  Gauge field and its dual $(A, \tilde{A})$

  Electric current and magnetic current $(\star J, \star K)$

Dual frame and SL(2,Z)

- Complexified gauge coupling

$$\tau \equiv \frac{\theta}{2\pi} + \frac{2\pi}{e^2} i$$

- Under the transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

- $(A, \tilde{A})$ and $(\star J, \star K)$ transform as

$$\begin{bmatrix} d & c \\ b & a \end{bmatrix} \begin{bmatrix} A \\ \tilde{A} \end{bmatrix} = \begin{bmatrix} A' \\ \tilde{A}' \end{bmatrix} \qquad \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} \star J \\ \star K \end{bmatrix} = \begin{bmatrix} \star J' \\ \star K' \end{bmatrix}$$

Dual frame and SL(2,Z)

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Dual frame and SL(2,Z)

- From now on, we define the QED frame with

$$\mathrm{d} \star j_{\mathrm{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F$$

  and everything will be “primed” in QED frame.

- Everything in a dual frame will be “unprimed”.

ABJ (chiral) anomaly in Dual frame

- ABJ anomaly in U(1) gauge theory w/ m=0 dyonic fermions?
  → don't know how dyon measure changes under $U(1)_{\text{chiral}}$
  → But, know how electron measure changes under $U(1)_{\text{chiral}}$

QED frame

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F$$

Dual frame

?

τ

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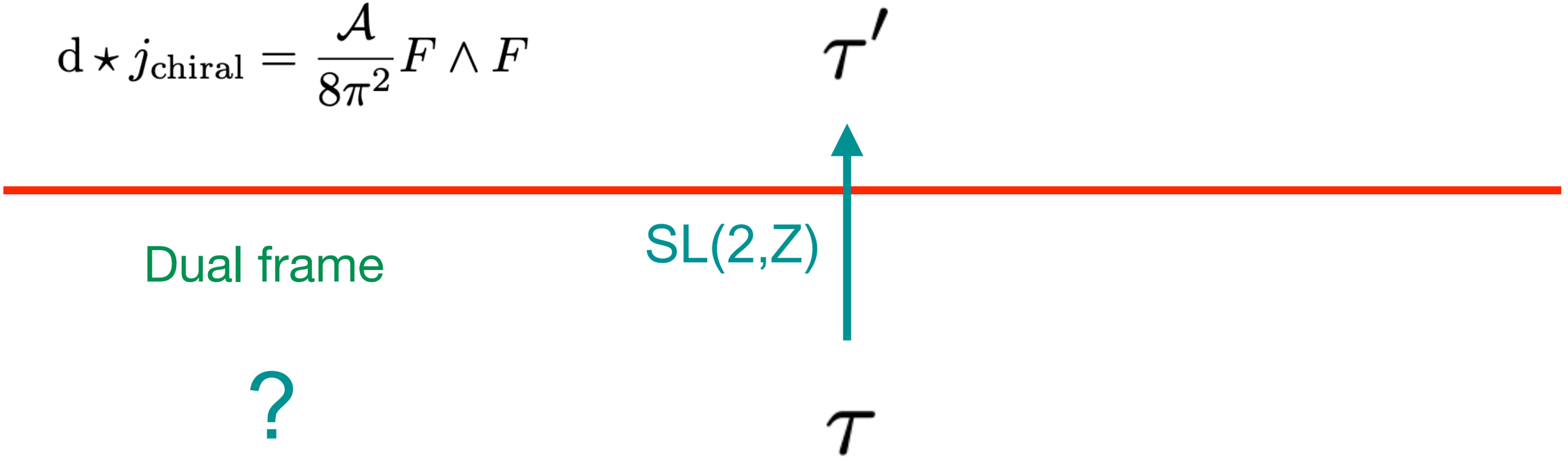

Dual frame

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QED frame

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F$$

τ' ⟶ $U(1)_{\text{chiral}}$

Dual frame

?

τ

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QED frame

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F$$

$$\tau' \xrightarrow[U(1)_{\text{chiral}}]{} \tau' - \frac{\mathcal{A}\alpha}{2\pi}$$

Dual frame

?

τ

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QED frame

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τ'

$$\tau' - \frac{\mathcal{A}\alpha}{2\pi}$$

SL(2,Z)$^{-1}$

Dual frame

?

τ

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QED frame

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{8\pi^2} F \wedge F \qquad \tau' \qquad \tau' - \frac{\mathcal{A}\alpha}{2\pi}$$

SL(2,Z)$^{-1}$

Dual frame

?

τ

$\tau_{\{\alpha\}}$

ABJ (chiral) anomaly in Dual frame

- Complexified gauge coupling after $U(1)_{chiral}$-rotating phases of dyonic fermions

$$\begin{aligned}\tau_{\{\alpha\}} &= \frac{d\tau'_{\{\alpha\}} - b}{-c\tau'_{\{\alpha\}} + a} = \frac{\tau - \mathcal{A}d(c\tau + d)\frac{\alpha}{2\pi}}{1 + \mathcal{A}c(c\tau + d)\frac{\alpha}{2\pi}} \\ &= \tau - \mathcal{A}(c\tau + d)^2\frac{\alpha}{2\pi} + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right]\end{aligned}$$

ABJ (chiral) anomaly in Dual frame

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$$= \tau - \mathcal{A}(c\tau + d)^2 \frac{\alpha}{2\pi} + \mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right]$$

ABJ (chiral) anomaly in Dual frame

- Complexified gauge coupling after $U(1)_{\text{chiral}}$-rotating phases of dyonic fermions

$$
\begin{aligned}
\mathrm{Re}[\Delta\tau] \;=\; & \mathcal{A}\left\{-\left(c\frac{\theta}{2\pi}+d\right)^2+\left(c\frac{2\pi}{e^2}\right)^2\right\}\frac{\alpha}{2\pi} \\
& +\mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right], \\
\mathrm{Im}[\Delta\tau] \;=\; & \mathcal{A}\left\{-2\left(c\frac{\theta}{2\pi}+d\right)\left(c\frac{2\pi}{e^2}\right)\right\}\frac{\alpha}{2\pi} \\
& +\mathcal{O}\left[\left(\frac{\alpha}{2\pi}\right)^2\right].
\end{aligned}
$$

ABJ (chiral) anomaly in Dual frame

- Complexified gauge coupling after $U(1)_{\text{chiral}}$-rotating phases of dyonic fermions

$$\int \prod_{i=1}^{N_f} \mathcal{D}\Psi_i \mathcal{D}\bar{\Psi}_i \mathcal{D}A \; e^{iS - i\alpha \int \mathrm{d}\star j_{\text{chiral}}}$$

$$= \int \prod_{i=1}^{N_f} \mathcal{D}\Psi_i \mathcal{D}\bar{\Psi}_i \mathcal{D}A \; e^{iS_{\{\alpha\}}} \,,$$

$$\exp\left[-i\alpha \int \mathrm{d}\star j_{\text{chiral}}\right]$$

$$= \exp\left[i \int -\frac{\mathrm{Im}[\Delta\tau]}{4\pi} F \wedge \star F + \frac{\mathrm{Re}[\Delta\tau]}{4\pi} F \wedge F\right]$$

ABJ (chiral) anomaly in Dual frame

- ABJ chiral anomaly in a dual frame

$$
\begin{aligned}
\mathrm{d} \star j_{\mathrm{chiral}} &= -\frac{\mathrm{Re}[\Delta\tau]}{4\pi\alpha} F \wedge F + \frac{\mathrm{Im}[\Delta\tau]}{4\pi\alpha} F \wedge \star F \,, \\
&= \frac{\mathcal{A}}{8\pi^2} \left\{ \left(c\frac{\theta}{2\pi} + d \right)^2 - \left(c\frac{2\pi}{e^2} \right)^2 \right\} F \wedge F \\
&\quad - \frac{\mathcal{A}}{8\pi^2} \left\{ 2 \left(c\frac{\theta}{2\pi} + d \right) \left(c\frac{2\pi}{e^2} \right) \right\} F \wedge \star F
\end{aligned}
$$

ABJ (chiral) anomaly in Dual frame

- Provided there exists non-invertible $U(1)_{\text{chiral}}$ in a dual frame

$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{2} \star \hat{j}^{[2]} \wedge \star \hat{j}^{[2]}$$

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$$\mathrm{d} \star j_{\text{chiral}} = \frac{\mathcal{A}}{2} \star \hat{j}^{[2]} \wedge \star \hat{j}^{[2]} \longrightarrow \star \hat{j}^{[2]} = \left(c \frac{\theta}{2\pi} + d \right) \frac{F}{2\pi} - c \frac{\star F}{e^2}$$

- 1-form symmetry generated by $\star \hat{j}^{[2]}$ corresponds to 1-form magnetic symmetry in QED frame

Chiral rotation at a fixed point

- In a dual frame,

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\ln\mu} = -\frac{4\pi}{e'^3}\frac{\mathrm{d}e'}{\mathrm{d}\ln\mu}i \xrightarrow{\mathrm{SL(2,Z)}} \frac{\mathrm{d}\tau}{\mathrm{d}\ln\mu} = -(c\tau+d)^2\frac{4\pi}{e'^3}\frac{\mathrm{d}e'}{\mathrm{d}\ln\mu}i$$

- Beta functions in the dual frame

$$\beta_\theta \equiv \frac{\mathrm{d}\theta}{\mathrm{d}\ln\mu} = \frac{16\pi^2}{e'^3}\beta_{e'}\left(c\frac{\theta}{2\pi}+d\right)\left(c\frac{2\pi}{e^2}\right)$$

$$\beta_e \equiv \frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = \frac{e^3}{e'^3}\beta_{e'}\left\{\left(c\frac{\theta}{2\pi}+d\right)^2-\left(c\frac{2\pi}{e^2}\right)^2\right\}$$

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∃ a fixed point

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Chiral rotation at a fixed point

- But then, ABJ anomaly in a dual frame

$$
\begin{aligned}
\mathrm{d} \star j_{\mathrm{chiral}} &= -\frac{\mathrm{Re}[\Delta\tau]}{4\pi\alpha} F \wedge F + \frac{\mathrm{Im}[\Delta\tau]}{4\pi\alpha} F \wedge \star F \,, \\
&= \frac{\mathcal{A}}{8\pi^2} \left\{ \left(c\frac{\theta}{2\pi} + d \right)^2 - \left(c\frac{2\pi}{e^2} \right)^2 \right\} F \wedge F \\
&\quad - \frac{\mathcal{A}}{8\pi^2} \left\{ 2 \left(c\frac{\theta}{2\pi} + d \right) \left(c\frac{2\pi}{e^2} \right) \right\} F \wedge \star F
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&\quad -\frac{\mathcal{A}}{8\pi^2}\left\{2\left(c\frac{\theta}{2\pi}+d\right)\left(c\frac{2\pi}{e^2}\right)\right\}F\wedge\star F
\end{aligned}$$

at the fixed point, this reduces to

$$\mathrm{d}\star j_{\mathrm{chiral}}=-\frac{\mathcal{A}}{8\pi^2}\left\{2\left(c\frac{\theta}{2\pi}+d\right)\left(c\frac{2\pi}{e^2}\right)\right\}F\wedge\star F$$

Chiral rotation at a fixed point

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Can rotate away kinetic term??

Chiral rotation at a fixed point

- From the “unreasonable” action without F^2,

$$S \supset \int \frac{\theta_*}{8\pi^2} F \wedge F - A \wedge \star J \quad \longrightarrow \quad \mathrm{d}\left(-\frac{\theta_*}{2\pi} \frac{F}{2\pi} \right) = \star J$$

Chiral rotation at a fixed point

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$$S \supset \int \frac{\theta_*}{8\pi^2} F \wedge F - A \wedge \star J \longrightarrow \mathrm{d}\left(-\frac{\theta_*}{2\pi}\frac{F}{2\pi}\right) = \star J$$

- However, rotating away F^2 w/ NI $U(1)_{\text{chiral}}$ means

$$\mathrm{d} \star \hat{j}^{[2]} = 0$$

- And at the fixed point, this further means

$$\mathrm{d}F = \pm \mathrm{d} \star F$$

$$\begin{aligned} \star\hat{j}^{[2]} &= \left(c\frac{\theta}{2\pi} + d\right)\frac{F}{2\pi} \pm c\frac{\star F}{e^2} \\ &= \left(c\frac{\theta}{2\pi} + d\right)\left(\frac{F}{2\pi} \pm \frac{\star F}{2\pi}\right) \end{aligned}$$

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- However, rotating away F^2 w/ NI $U(1)_{\text{chiral}}$ means

$$\mathrm{d} \star \hat{j}^{[2]} = 0$$

- And at the fixed point, this further means

$$\mathrm{d}\left(\mp\frac{\theta_*}{2\pi}\frac{\star F}{2\pi}\right) = \star J$$

Chiral rotation at a fixed point

- This is equivalent to e.o.m from the action before performing $U(1)_{\text{chiral}}$ at the fixed point

$$S \supset \int -\frac{1}{2e_*^2} F \wedge \star F + \frac{\theta_*}{8\pi^2} F \wedge F - A \wedge \star J$$

$$\begin{aligned}
\star J &= \mathrm{d}\left(\frac{1}{e_*^2} \star F - \frac{\theta_*}{2\pi}\frac{F}{2\pi}\right), \\
&= \mathrm{d}\left(\pm\frac{\theta_*}{2\pi}\frac{\star F}{2\pi} \pm \frac{Q_\mathrm{e}}{Q_\mathrm{m}}\frac{\star F}{2\pi} - \frac{\theta_*}{2\pi}\frac{F}{2\pi}\right) \\
&= \mathrm{d}\left(\mp\frac{\theta_*}{2\pi}\frac{\star F}{2\pi}\right),
\end{aligned}$$

- $U(1)_{\text{chiral}}$-rotating away F^2 doesn't remove kinetic term!

Axion mass from mutually non-local charges

- Combined conservation equation for 2-form currents

$$\frac{\mathrm{Im}[\tau]}{2\pi}(\mathrm{d}\star F' + i\mathrm{d}F') = \star J' + \tau \star K'$$

- Under SL(2,Z), one can show

$$\begin{aligned} \mathrm{d}\star \hat{j}^{[2]} &= \frac{\mathrm{Im}[(c\tau^* + d)(\star J + \tau \star K)]}{\mathrm{Im}[\tau]} \\ &= -c \star J + d \star K\,, \end{aligned}$$

- This gives the condition for breaking of NI-U(1)$_{\text{chiral}}$

$$-c\,Q_{\mathrm{e}2} + d\,Q_{\mathrm{m}2} = -\frac{1}{Q'_e}\left(Q_{\mathrm{m}}Q_{\mathrm{e}2} - Q_{\mathrm{e}}Q_{\mathrm{m}2}\right) \neq 0$$

Axion mass from mutually non-local charges

- When the pairwise helicity is not zero,

$$-c\,Q_{\mathrm{e}2} + d\,Q_{\mathrm{m}2} = -\frac{1}{Q'_e}\left(Q_{\mathrm{m}}Q_{\mathrm{e}2} - Q_{\mathrm{e}}Q_{\mathrm{m}2}\right) \neq 0$$

one can show

$$Q_T = \frac{1}{2}\int_M \star\hat{j}^{[2]} \wedge \star\hat{j}^{[2]} \neq 0 \qquad \text{(because } Q_T \propto \text{pairwise helicity)}$$

- NI-U(1)$_{\text{chiral}}$ is broken to $Z_{\mathcal{A}}$

$$\prod_{j=1}^{N_f} \mathcal{D}\Psi_j\,\mathcal{D}\bar{\Psi}_j \;\rightarrow \prod_{j=1}^{N_f} \mathcal{D}\Psi_j\,\mathcal{D}\bar{\Psi}_j\; e^{-i\alpha\mathcal{A}Q_T}$$

Summary

- ABJ anomaly in a dual frame is still quadratic in $\star\hat{j}^{[2]}$

- The expression $\star\hat{j}^{[2]} = \left(c\frac{\theta}{2\pi} + d\right)\frac{F}{2\pi} - c\frac{\star F}{e^2}$

  is already signaling kinetic term and topological term in a dual frame is mixture of $F \wedge F$ and $F \wedge \star F$

- In Axion-Maxwell, two particles with non-zero pairwise helicity leads to non-vanishing axion potential.