Radio emission from air showers. Comparison of theoretical approaches.

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Goals

- Derive the analytical solution for the emission by a charged particle moving in the matter
- Check the validity of the Endpoint formalism and ZHS by deriving both formulas from basic principles
- Find out the rage of applicability of the simulations based on the Endpoint and ZHS to the:
  - Air showers
  - SLAC

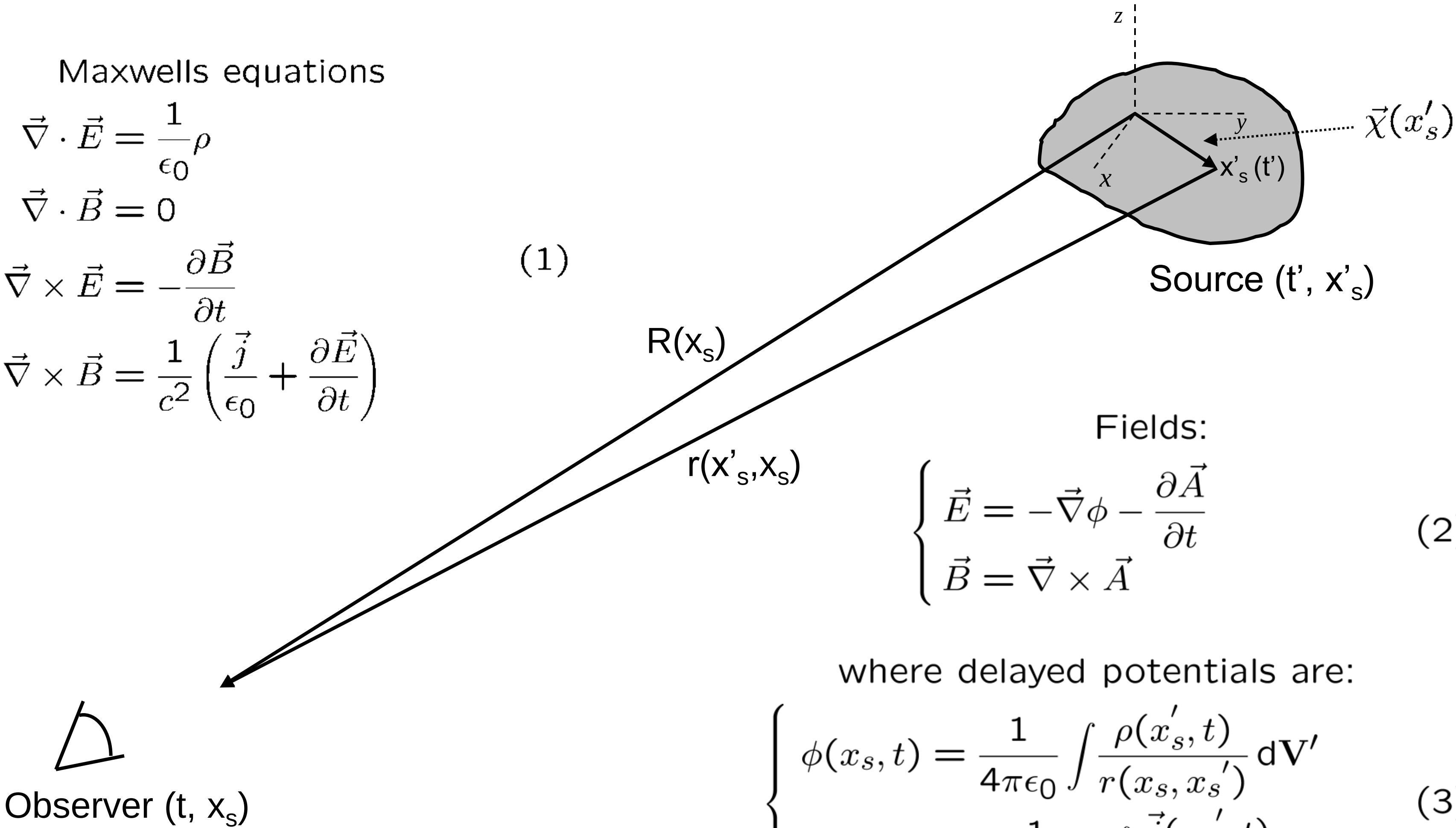
- Develop technique to avoid singularities in MC simulations at critical values for the parameters:
  - At the critical angle
  - At low frequency
  - In near zone (important for SLAC)

- Improve current MC codes.

Start with Maxell's equations … again

Maxwells equations

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \dfrac{1}{\epsilon_0}\rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \dfrac{1}{c^2}\left(\dfrac{\vec{j}}{\epsilon_0} + \dfrac{\partial \vec{E}}{\partial t}\right) \end{cases} \qquad (1)$$



Fields:

$$\begin{cases} \vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases} \qquad (2)$$

where delayed potentials are:

$$\begin{cases} \phi(x_s, t) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \dfrac{\rho(x_s^{'}, t)}{r(x_s, x_s{'})} \mathrm{d}\mathbf{V}' \\ \vec{A}(x_s, t) = \dfrac{1}{4\pi\epsilon_0 c^2} \displaystyle\int \dfrac{\vec{j}(x_s{'}, t)}{r(x_s, x_s{'})} \mathrm{d}\mathbf{V}' \end{cases} \qquad (3)$$

$$t^{'} = t - \frac{r(x_s, x_s{'})}{c} \qquad (4)$$

Spectral components

Fourier:

$$\vec{E}(x_s,t) = Re\int \vec{E_\omega}(x_s)e^{-i\omega t}\,\mathrm{d}\omega; \quad \vec{B}(x_s,t) = Re\int \vec{B_\omega}(x_s)e^{-i\omega t}\,\mathrm{d}\omega$$

$$\vec{E}_\omega(x_s) = \frac{1}{4\pi\epsilon_0}\int \rho_\omega \nabla' \left(\frac{e^{i\vec{k}\vec{r}}}{r}\right)\mathrm{d}\mathbf{V}' + \frac{ik}{4\pi\epsilon_0 c}\int \vec{j}_\omega \frac{e^{i\vec{k}\vec{r}}}{r}\mathrm{d}\mathbf{V}'$$

$$\vec{B}_\omega(x_s) = \frac{1}{4\pi\epsilon_0 c^2}\int \vec{j}_\omega \times \nabla' \left(\frac{e^{i\vec{k}\vec{r}}}{r}\right)\mathrm{d}\mathbf{V}'$$

where: $\vec{k} = \frac{\omega}{c}\hat{r}$; $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

charge density: $\rho_\omega(x'_s) = \frac{1}{2\pi}\int \rho(x'_s,\tau)e^{i\omega\tau}\mathrm{d}\tau$;

current density: $\vec{j}_\omega(x'_s) = \frac{1}{2\pi}\int \vec{j}(x'_s,\tau)e^{i\omega\tau}\mathrm{d}\tau$

$\nabla'\left(\frac{e^{i\vec{k}\vec{r}}}{r}\right) = i\vec{k}\frac{e^{i\vec{k}\vec{r}}}{r} - \frac{e^{i\vec{k}\vec{r}}}{r^2}$; if r is large, we can drop $\frac{1}{r^2}$ term

But we will use more strict limitation later

E-field

$$E_\omega(x_s) = \frac{i}{4\pi\epsilon_0}\int \vec{k}\rho_\omega \frac{e^{i\vec{k}\vec{r}}}{r}\mathrm{d}\vec{V}' + \frac{ik}{4\pi\epsilon_0 c}\int \vec{j}_\omega \frac{e^{i\vec{k}\vec{r}}}{r}\mathrm{d}\vec{V}'$$

$$B_\omega(x_s) = \frac{i}{4\pi\epsilon_0 c^2}\int \left[\vec{j}_\omega \times \vec{k}\right]\frac{e^{i\vec{k}\vec{r}}}{r}\mathrm{d}\vec{V}'$$

in the far zone $(\vec{k}\vec{r} >> 1)$ $\vec{E} \perp \vec{B} => \vec{E}_\omega = \left[\vec{B}_\omega \times \hat{r}\right]$

or $$E_\omega(x_s) = \frac{i\omega}{4\pi\epsilon_0 c^2}\int \left[\hat{r} \times \left[\hat{r} \times \vec{j}_\omega\right]\right]\frac{e^{i\vec{k}\vec{r}}}{r}\mathrm{d}\vec{V}'$$

if $R >> \left|\vec{\chi}(t')\right|$ **during the whole observation time**

$=> r(t') \approx R - \hat{R}\cdot\vec{\chi}(t')$ (Fraunhofer zone)

Replacing $|\vec{r}|$ with R and $\hat{r}$ with $\hat{R}$:

$$\vec{E}_\omega(x_s) = -\frac{i\omega}{4\pi\epsilon_0 c^2}\frac{e^{ikR}}{R}\int \left[\hat{R} \times \left[\hat{R} \times \vec{j}_\omega\right]\right] e^{-i\vec{k}\vec{\chi}}\mathrm{d}\vec{V}'$$

Current density

Let a charge particle move on a trajectory $\vec{\chi}(t)$

Current desity can be written as: $\vec{j}(x'_s, t') = q \int \frac{d\vec{\chi}(\tau)}{d\tau} \delta(\vec{r}(t') - \vec{\chi}(\tau)) \mathrm{d}\tau$

Recall that: $\vec{j}_\omega(x_s) = \frac{1}{2\pi} \int \vec{j}(x'_s, \tau) e^{i\omega\tau} \mathrm{d}\tau$

Substituing into equation for $\vec{E}_\omega$ will find:

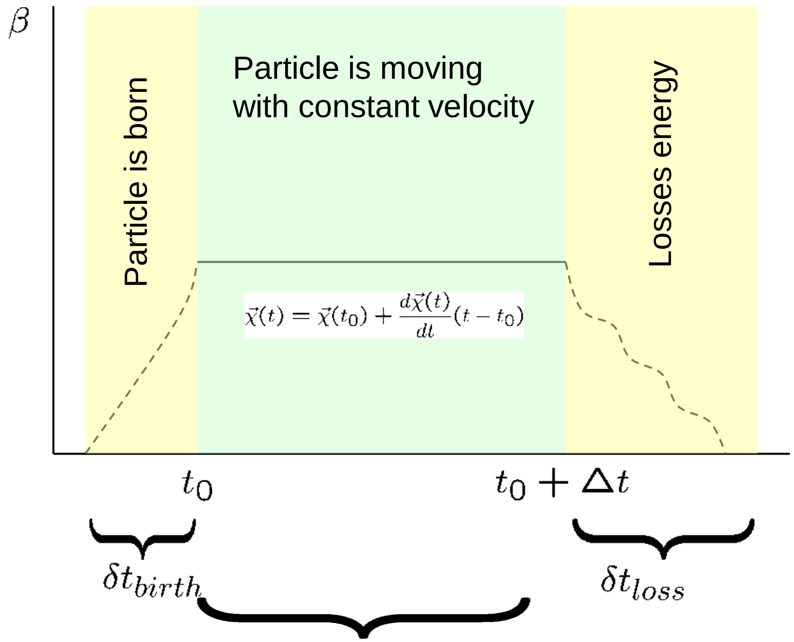
$$\vec{E}_\omega(x_s) = -\frac{i\omega q}{8\pi^2 \epsilon_0 c} \frac{e^{ikR}}{R} \int \frac{1}{c} \left[\hat{r} \times \left[\hat{r} \times \frac{d\vec{\chi}(t)}{dt}\right]\right] e^{i(\omega t - \vec{k}\vec{\chi}(t))} \mathrm{d}t$$

$\vec{\beta} = \frac{1}{c}\frac{d\vec{\chi}(t)}{dt}$; $\quad \vec{k} = n\frac{\omega}{c}\hat{r}$; $\quad \hat{R} = \hat{r}$ in Fraunthofer zone

$\left[\hat{r} \times \vec{\beta}\right] \times \hat{r}$ – perpendicular component of the charge velocity directed along $\hat{R}$

$$\vec{E}_\omega(x_s) = \frac{i\omega q}{8\pi^2 \epsilon_0 c} \frac{e^{ikR}}{R} \int \vec{\beta}_\perp e^{i\omega(t - \frac{n\hat{R}\vec{\chi}(t)}{c})} \mathrm{d}t$$

Straight segment



$$\int \vec{\beta}_\perp e^{i\omega(t - \frac{n\hat{R}\vec{\chi}(t)}{c})} dt = e^{i\omega(t_0 - \frac{n\hat{R}\chi(t_0)}{c})} \frac{\beta_\perp}{i\omega(1 - n\hat{R}\vec{\beta})} \left[e^{i\omega\Delta t(1 - n\hat{R}\vec{\beta})} - 1 \right] =>$$

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon_0 c} \frac{e^{ikR}}{R} e^{i\omega(t_0 - \frac{n\hat{R}\chi(t_0)}{c})} \frac{\beta_\perp}{(1 - n\hat{R}\vec{\beta})} \left[e^{i\omega\Delta t(1 - n\hat{R}\vec{\beta})} - 1 \right]$$

Let's consider small $\omega\Delta t(1 - n\hat{R}\vec{\beta})$ (has dimention of distance) (*) =>

$$e^{i\omega\Delta t(1 - n\hat{R}\vec{\beta})} \approx i\omega\Delta t(1 - n\hat{R}\vec{\beta}) + \frac{i\omega\Delta t(1 - n\hat{R}\vec{\beta})}{2} + O((i\omega\Delta t(1 - n\hat{R}\vec{\beta}))^2) =>$$

$$\vec{E}_\omega = i\omega\Delta t\vec{\beta} \perp \frac{q}{8\pi^2\epsilon_0 c} \frac{e^{ikR} e^{i\omega(t_0 - \frac{n\hat{R}\vec{\chi}(t_0)}{c})}}{R} \left[1 - \frac{1}{2} i\omega\Delta t(1 - n\hat{R}\vec{\beta}) \right]$$

Coherence length

Let's split the trajectory into Fresnel zones

so that the phase difference between nearby zones 1 and 2 is equal to π

$$=> e^{i\omega t(1-\hat{R}_2\vec{\beta}_2)+i\pi} = e^{i\omega t(1-\hat{R}_1\vec{\beta}_1)}$$

if $R_1 \approx R_2 \approx R$ and $\beta_1 \approx \beta_2 \approx \beta, \Delta t = t_2 - t_1 =>$

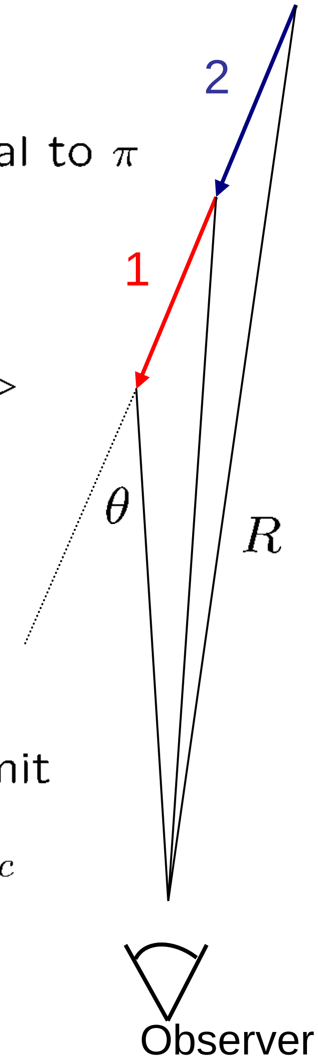
$$\Delta t\omega(1-\hat{R}\vec{\beta}) = \pi \text{ or } c\beta\Delta t = \frac{c\beta\pi}{\omega(1-\hat{R}\vec{\beta})}, \quad \frac{c\pi}{\omega} = \frac{\lambda}{2}, \quad \hat{R}\vec{\beta} = \beta cos\theta =>$$

$$L_{coh} = \frac{\beta\lambda}{2(1-n\beta cos\theta)} \text{ where } n \text{ refraction index and } \theta \neq \theta_c$$

$$L_{coh} = \frac{\sqrt{R\lambda}}{sin\theta_c} \text{ where } \theta_c \text{ Cherenkov angle}$$

To obtain the later, one should get away from the Fraunthofer limit

$$\text{The approximation (*) works if } c\beta\Delta t << \begin{cases} \dfrac{\beta\lambda}{2(1-n\beta cos\theta)}, & \text{if } \theta \neq \theta_c \\ \dfrac{\sqrt{R\lambda}}{sin\theta_c}, & \text{if } \theta = \theta_c \end{cases}$$



Related definitions can be found in Tamm (1939),
Collection of science publications (in Russian),
Moscow 1975

Coherence time

Corresponding coherent time is:

$$\Delta t_{coh} = \frac{\pi}{\omega(1-\hat{R}\vec{\beta})}$$

If the charge is moving from infinity toward the observer viewed at the angle θ

and stops at the moment t_0 =>

Integration from $-\infty$ to $t_0$can be replaced by

1/2 of the integral over any Fresnel zone

for example for the first (closes) Fresnel zone:

$$\int_{-\infty}^{t_0} \vec{\beta}_\perp e^{i\omega(t-\frac{n\hat{R}\vec{\chi}(t)}{c})}\mathrm{d}t = \frac{1}{2}\int_{t_0-\frac{\pi}{\omega(1-\hat{R}_1\vec{\beta})}}^{t_0} \vec{\beta}_\perp e^{i\omega(t-\frac{n\hat{R}_1\vec{\chi}(t)}{c})}\mathrm{d}t$$

where $\hat{R}_1$ radius vector to the first Fresnel zone

Calculating the integral above we will get the same expression for

$\vec{E}_\omega$ as before

Although the particle emmits from the whole track, it appears as if

all emission comes from one Fresnel zone only

The choice of the "emmiting" zone is arbitrary

- Can we use this to speed up the MC for the air showers?

Emission from the ends of the trajectory (yellow regions)

Partcle birth:

$$\vec{E}_\omega = -i\omega \frac{qe^{ikR}e^{i\omega(t_0 - \frac{1}{c}n\hat{R}\vec{\chi}(t_0))}}{8\pi\epsilon_0 cR} \int \vec{\beta}_\perp \mathrm{d}t$$

integration over the time of the particle "birth" δt_{birth}

Energy loss:

$$\vec{E}_\omega = -i\omega \frac{qe^{ikR}e^{i\omega(t_0 + \Delta t - n\hat{R}(\frac{1}{c}\vec{\chi}(t_0)) + \vec{\beta})}}{8\pi\epsilon_0 cR} \int \vec{\beta}_\perp \mathrm{d}t$$

$\int \vec{\beta}_\perp \mathrm{d}t \approx \vec{\beta} sin\theta \delta t_{birth}$, and $\beta \approx 1, sin\theta \leq 1$

– is path of the particle travel as seen by the observer

$\int \vec{\beta}_\perp \mathrm{d}t \leq \delta t_{birth}$

if $\delta t_{birth} << \Delta t$

than we can neglect the emission in yellow regions

– split the whole trajectory by the small enough segments and sum up the emission from all the segments

Sum over all straight segments

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon_0 c}\sum \frac{e^{ikR_m}}{R_m} e^{i\omega(t_0^m - \frac{n\hat{R_m}\chi^m(t_0)}{c})} \frac{\vec{\beta}_{m\perp}}{(1-n\hat{R}_m\vec{\beta}_m)}\left[e^{i\omega\Delta t_m(1-n\hat{R}_m\vec{\beta}_m)} - 1\right]$$

Compare to ZHS:

$$\vec{E}_\omega(\vec{x}(t),\nu) = \frac{q}{c}\frac{e^{ikR_{1.5}}}{R_{1.5}}\frac{e^{2\pi i\nu(n_{1.5}\beta_{1.5}cos\theta_{1.5})t_2} - e^{2\pi i\nu(n_{1.5}\beta_{1.5}cos\theta_{1.5})t_1}}{1-n_{1.5}\beta_{1.5}cos\theta_{1.5}}\beta_{1.5}sin\theta_{1.5}$$

where index 1.5 means values taken in the middle of the track segment

These two formilas are identical

Remember that we used approximation: $\vec{k}\vec{r} >> 1$ to derive it.

Ends of a particle track approach

From Maxwell's equations one can get:

$$\vec{E}(t,x_s) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{r}-\vec{\beta})(1-\beta^2)}{r^2(1-\hat{r}\vec{\beta})^3}\right] + \frac{q}{4\pi\epsilon_0 c}\left[\frac{\hat{r}\times\left[(\hat{r}-\vec{\beta})\times\frac{d\vec{\beta}}{dt}\right]}{r(1-\hat{r}\vec{\beta})^3}\right]$$

(see Landau for example for the derivation)

all values are taken at delayed time t'

For magnetic field: $\vec{B}(t,x_s) = \frac{1}{c}\vec{E}(t,x_s)\times\hat{r}$

Notice that: $\frac{dt}{dt'} = 1 - \hat{r}\vec{\beta}$

and in Fraunhofer zone: $\hat{R} \approx \hat{r}$ =>

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon_0 c}\int\frac{\hat{r}\times\left[(\hat{r}-\vec{\beta})\times\frac{d\vec{\beta}}{dt'}\right]}{r(1-\hat{r}\vec{\beta})^2}e^{i\omega(t'+\frac{\vec{\chi}(t')}{c})}\mathrm{d}t'$$

A funny excersise is to show that: $\frac{d}{dt'}\left[\frac{\hat{r}\times\left[\hat{r}\times\vec{\beta}\right]}{1-\hat{r}\vec{\beta}}\right] = \frac{\hat{r}\times\left[(\hat{r}-\vec{\beta})\times\frac{d\vec{\beta}}{dt'}\right]}{(1-\hat{r}\vec{\beta})^2}$ =>

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}\int\frac{d}{dt'}\left[\frac{\vec{\beta}_\perp}{1-\hat{R}\vec{\beta}}\right]e^{i\omega(t'-\frac{\hat{R}\vec{\chi}(t')}{c})}\mathrm{d}t', \quad \text{where: } \vec{\beta}_\perp = \left[\hat{r}\times\vec{\beta}\right]\times\hat{r}$$

Ends of a particle track

Since the acceleration is zero in green zone,

the emission only comes from yellow regions

For left yellow zone (particle birth and acceleration):

$$\vec{E}_\omega(x_s) = -\frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}\int\frac{d}{dt}\left[\frac{\vec{\beta}_\perp}{1-\hat{R}\vec{\beta}}\right]e^{i\omega(t-\frac{\hat{R}\vec{\chi}(t)}{c})}\mathrm{d}t = -\frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}e^{i\omega(t_0-\frac{\hat{R}\vec{\chi}(t_0)}{c})}\frac{\vec{\beta}_\perp}{1-\hat{R}\vec{\beta}}$$

For right yellow zone (particle energy loss and deceleration):

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}e^{i\omega(t_0+\Delta t-\hat{R}(\frac{\vec{\chi}(t_0)}{c}+\Delta t\vec{\beta}))}\frac{\vec{\beta}_\perp}{1-\hat{R}\vec{\beta}}$$

We do not care how the particle velocity changes in yellow regions if $\omega\delta t << 1$

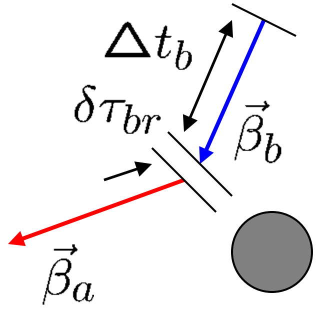
Adding the emission from both "ends" together we get:

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}e^{i\omega(t_0-n\frac{\hat{R}\vec{\chi}(t_0)}{c})}\frac{\vec{\beta}_\perp}{(1-n\hat{R}\vec{\beta})}\left[e^{i\omega\Delta t(1-n\hat{R}\vec{\beta})}-1\right]$$

Caution about simply adding the refraction index to account for the medium

- see discussion later

Bremsstrahlung



Let's:

Δt_b – time from t_0 before the scattering

$\delta\tau_{br}$ – time of the interaction

$\vec{\beta}_b$ particle velocity before the scattering

$\vec{\beta}_a$ particle velocity after the scattering

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon c}\frac{e^{ikR}}{R}e^{i\omega(t_0+\Delta t_b - n\frac{\hat{R}\vec{\chi}(t_0)}{c})}\left[\frac{\vec{\beta}_{a\perp}}{(1-n\hat{R}\vec{\beta}_b)} - \frac{\vec{\beta}_{b\perp}}{(1-n\hat{R}\vec{\beta}_a)}\right]$$

Summing up over all straight trajectory tracks:

$$\vec{E}_\omega(x_s) = -\frac{q}{8\pi^2\epsilon c}\sum\frac{e^{ikR_{m+1}}}{R_{m+1}}e^{i\omega(t_0^{m+1} - n\frac{\hat{R}_{m+1}\vec{\chi}^{m+1}(t_0)}{c})}\left[\frac{\vec{\beta}_{m+1\perp}}{(1-n\hat{R}\vec{\beta}_{m+1})} - \frac{\vec{\beta}_{m\perp}}{(1-n\hat{R}\vec{\beta}_m)}\right]$$

Compare this to what we got for ZHS:

$$\vec{E}_\omega(x_s) = \frac{q}{8\pi^2\epsilon_0 c}\sum\frac{e^{ikR_m}}{R_m}e^{i\omega(t_0^m - \frac{n\hat{R}_m\chi^m(t_0)}{c})}\frac{\vec{\beta}_{m\perp}}{(1-n\hat{R}_m\vec{\beta}_m)}\left[e^{i\omega\Delta t_m(1-n\hat{R}_m\vec{\beta}_m)} - 1\right]$$

Assuming that velocity at the beginning and the end of the whole track is zero two formulas above are the same

End points formula:

$$\vec{E} = \frac{q}{c}\frac{e^{ikR_{1.5}}}{R_{1.5}}\frac{e^{2\pi i\nu(1-n_{1.5}\beta_{1.5}cos\theta_{1.5})(t_{end}-t_{start})}}{1-n_{1.5}\beta_{1.5}cos\theta_{1.5}}\beta_{1.5}cos\theta_{1.5}$$

Conclusions

- ZHS and Endpoint Formalism are mathematically identical by derivation. More strictly, if both are derived with no medium present.
- The physics reason is the E-field cancellation from nearby Fresnel zones along the track due to the phase change by π

- ZHS and Endpoint formalism are only valid in Fraunhofer zone where dropping the static term term is possible => It is not correct to say that Endpoints -> ZHS in far zone as both are derived in this approximation.

- Low frequency limit is not valid for both approaches as the kr >> 1 condition is violated.

- In Endpoint we might have to subtract very close terms each going to infinity (at n close to 1). Should switch to different approximation.

Additional Conclusions

- Endpoint is the "full spectrum" solution of Maxwell's equations. Accounting for the medium is not that trivial. Simple addition of the phase shift works for ZHS. Not important for air showers with 10 MHz – 3 GHz range in mind.
- We should interpret the emission in the presence of medium as emission from *all* particles moving with acceleration, including the charges in the medium. We did not account for the particle interaction with the charges in the medium in both approaches. MC simulations are taking this into account.

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