Resummation of thrust distribution in e^+e^- annihilation in QCD

Giancarlo Ferrera

Milan University & INFN, Milan

Based on:

U. Aglietti, G.F., W.-L. Ju, J. Miao, 2502.01570, PRL 134 (2025) 25

U. Aglietti, G.F., W.-L. Ju, 2506.18707

α_S from thrust at lepton colliders – CERN – 28/11/2025

Fixed-order QCD expansion

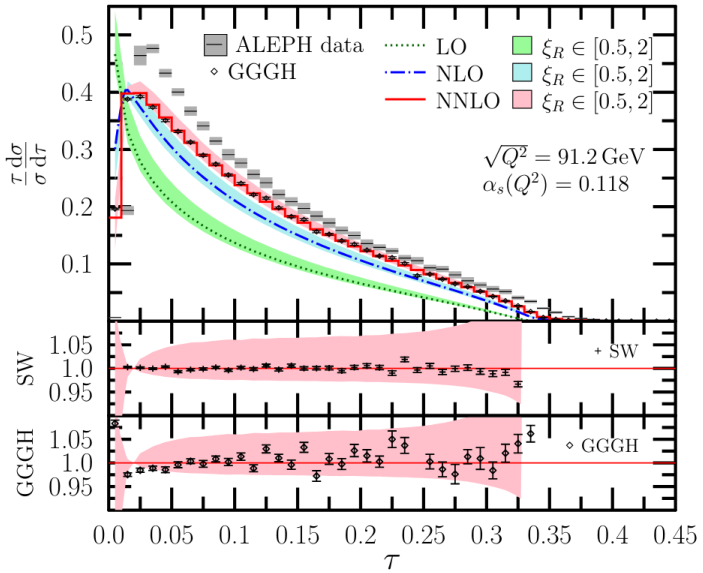
Thrust distribution can be calculated in at fixed-order in $\alpha_S = \alpha_S(\mu^2)$

$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma}{d\tau} = \delta(\tau) + \frac{\alpha_S}{\pi}\frac{d\mathcal{A}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^2\frac{d\mathcal{B}}{d\tau} + \left(\frac{\alpha_S}{\pi}\right)^3\frac{d\mathcal{C}}{d\tau} + \mathcal{O}(\alpha_S^4)\,,$$

The LO function ($\tau > 0$) is

$$\frac{d\mathcal{A}}{d\tau} = 4 + 6\tau - \frac{2}{\tau} + \left(-4 + \frac{8}{3(1-\tau)\tau}\right)\ln\left(\frac{1-2\tau}{\tau}\right) \xrightarrow{\tau\to 0} -\frac{8}{3}\frac{\ln\tau}{\tau} - \frac{2}{\tau}\,.$$

QCD corrections up to NNLO known [Gehrmann-De Ridder et al. ('07)], [Weinzierl ('09)],[Del Duca et al. ('16)]. Calculations based on a numerical integration of the matrix elements. NNLO parton-level event generator public available EERAD3 [Gehrmann-De Ridder et al. ('14)].



Sudakov resummation

- Bulk of events in the two-jet limit $\tau \to 0$ (semi-inclusive region).
- In the fixed-order expansion large Sudakov logarithms appear due to incomplete cancellation between real radiation (constrained by kinematics) and virtual (unconstrained) emissions:

$$\frac{1}{\sigma_{tot}}\frac{d\sigma}{d\tau} \overset{\tau\to 0}{\sim} \sum_{n=1}^{\infty}\sum_{k=1}^{2n-1} \alpha_S^n \frac{1}{\tau} \ln^k \frac{1}{\tau}\,.$$

- *Cumulative* cross section (probability of having a *thrust* $< \tau$):

$$R_T(\tau) \equiv \frac{1}{\sigma_{\rm tot}} \int_0^\tau d\tau' \frac{d\sigma}{d\tau'} \qquad \text{thus} \qquad \frac{1}{\sigma_{\rm tot}}\frac{d\sigma}{d\tau} = \frac{dR_T(\tau)}{d\tau}\,.$$

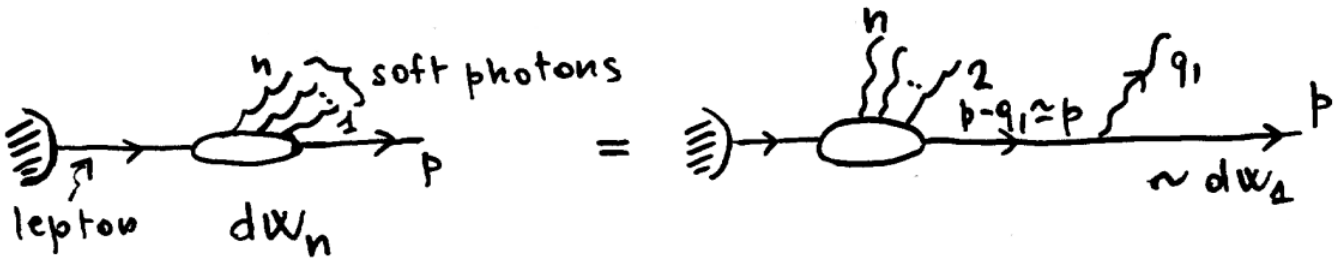
$$R_T(\tau) \overset{\tau\to 0}{\sim} \sum_{n=1}^{\infty}\sum_{k=1}^{2n} \alpha_S^n \ln^k \frac{1}{\tau}\,.$$

- Fixed-order expansion unreliable in the low τ region where $\alpha_S(Q)\ln^2 1/\tau \gtrsim 1$ ($\tau \lesssim 0.05$ for $Q = m_Z$).
- To obtain reliable predictions in the two-jet region, resummation of Sudakov logarithms is mandatory.

Soft gluon exponentiation

Analytic resummation feasible if:
dynamics AND kinematics factorize $\Rightarrow$ exponentiation.

- Dynamics factorization: general propriety of QCD for soft emissions [Gatheral('83)],[Frenkel,Taylor('84)], [Catani,Ciafaloni('84,'85)] analogous of eikonal approximation in QED [Yennie,Frautschi,Suura('61)]



$$dw_n(q_1,\ldots,q_n) \simeq \frac{1}{n!}\prod_{i=1}^{n} dw_1(q_i)$$

- Thrust kinematics factorize in Laplace space [Catani,Trentadue,Turnock,Webber('91)]

$$\Theta(\tau - \sum_{j=1}^{n}\frac{k_i^2}{Q^2}) = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau}\prod_{j=1}^{n} e^{-Nk^2/Q^2}$$

- Exponentiation holds in Laplace space (results then transformed into physical space):

$$\tau \ll 1 \quad \Leftrightarrow \quad N \gg 1\,, \qquad \ln 1/\tau \gg 1 \quad \Leftrightarrow \quad \ln N \gg 1\,.$$

Sudakov resummation for Thrust

- We follow the CTTW formalism [Catani,Trentadue,Turnock,Webber('91,'93)]
- Resummation applied also by [Gardi et al.('99)],[Banfi et al.('01)], [Davison,Webber('09)], [Gehrmann et al.('08)], [Dissertori et al.('09)]
- Resummation also reformulated in the framework of SCET [Schwartz('08)], [Becher,Schwartz('08)],[Hornig et al.('09)],[Almeida et al.('14)],[Abbate et al.('11,'12)],[Benitez et al.('24)]

Cumulative cross section can be written as:

$$R_T(\tau) = C(\alpha_S(Q^2))\,\Sigma(\tau, \alpha_S(Q^2)) + D(\tau, \alpha_S(Q^2))\,;$$

$C(\alpha_S)$ is a hard-virtual factor and $D(\alpha_S)$ is a *remainder* function vanishing at small τ:

$$C(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n C_n\,, \qquad D(\tau, \alpha_S) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n D_n(\tau)\,.$$

$\Sigma\,(\tau, \alpha_s)$ is a long-distance form factor (resums the large Sudakov logarithms).

In the Laplace-conjugated space:

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} = \frac{1}{2\pi i}\int_C \frac{dN}{N} e^{N\tau} e^{L\, f_1(\lambda) + f_2(\lambda) + \sum_{n=3}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^{n-2} f_n(\lambda)}$$

where the contour C runs parallel to the imaginary axis and lies to the right of all singularities.

Inversion of Laplace transform

Formal inversion from N space to τ space is:

$$\Sigma(\tau,\alpha_S) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{dN}{N}e^{N\tau}e^{\mathcal{F}(\alpha_S,L)}\,,$$

- This formula involves (formally non-integrable) Landau singularity of α_S
- Exact analytic Laplace inversion cannot be computed.

CTTW['93] : Taylor expand $\mathcal{F}(\alpha_S, L)$ around the point (not the saddle point)

$$\ln N = \ln(1/\tau) \equiv \ell\,,$$

$$\Sigma(\tau,\alpha_S) = \frac{1}{2\pi i}\int_{C}\frac{dN}{N}e^{N\tau}\exp\left[\sum_{k=0}^{\infty}\frac{\partial^k\mathcal{F}(\alpha_S,\ell)}{\partial\ell^k}\frac{\ln^k(\tau N)}{k!}\right]\,,$$

not possible to evaluate the series exactly: a new hierarchy is defined in $\tau-$space.
N^{n}LL in τ space defined by keeping the terms $\alpha_S^{n-1}(\alpha_S\ell)^k$, (for all k).
Crucial point: the correspondence $\ln N$ to $\ln(1/\tau)$ is not exact. Kinematics factorization and exponentiation are valid only in N space.

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Approximated analytic inversion

The approximated analytic form factor in τ space reads up to N^{4}LL:

$$
\begin{aligned}
\Sigma(\tau,\alpha_S) \simeq\ & \frac{1}{\Gamma[\phi]}\, e^{\ell\, f_1(\lambda_\tau)+f_2(\lambda_\tau)+\frac{\alpha_S}{\pi} f_3(\lambda_\tau)+\left(\frac{\alpha_S}{\pi}\right)^2 f_4(\lambda_\tau)+\left(\frac{\alpha_S}{\pi}\right)^3 f_5(\lambda_\tau)} \\
& \times \Bigg[1+\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)\psi_0(\phi)+\frac{1}{2}(\mathcal{F}^{(2)}(\alpha_S,\ell)+(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^2)(\psi_0^2(\phi)-\psi_1(\phi)) \\
& + \frac{1}{6}(\mathcal{F}^{(3)}(\alpha_S,\ell)+3\mathcal{F}^{(2)}(\alpha_S,\ell)\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)+(\mathcal{F}^{(1)}_{res})^3(\alpha_S,\ell)(\psi_0^3(\phi)-3\psi_0(\phi)\psi_1(\phi)+\psi_2(\phi)) \\
& + \frac{1}{24}(\mathcal{F}^{(4)}(\alpha_S,\ell)+3(\mathcal{F}^{(2)}(\alpha_S,\ell))^2+4\mathcal{F}^{(3)}(\alpha_S,\ell)\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)+6\mathcal{F}^{(2)}(\alpha_S,\ell)(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^2+(\mathcal{F}^{(1)}_{res}(\alpha_S,\ell))^4) \\
& \times (\psi_0^4(\phi)-6\psi_1(\phi)+3\psi_1^3(\phi)+4\psi_0(\phi)\psi_2(\phi)-\psi_3(\phi))\Bigg]\,,
\end{aligned}
$$

where $\Gamma(x)$ is the Euler Γ function, $\psi_n(x)\equiv\frac{d^{n+1}\ln\Gamma(x)}{dx^{n+1}}$, $\phi\equiv 1-f_1(\lambda_\tau)-\lambda f'(\lambda_\tau)$,
$\mathcal{F}^{(1)}_{res}(\alpha_S,\ell)\equiv\mathcal{F}^{(1)}(\alpha_S,\ell)-f_1(\lambda_\tau)-\lambda f'(\lambda_\tau)$, $\lambda_\tau=\alpha_S\beta_0\,\ell/\pi$.
The analytic form factor in τ space can then be re-written in an "exponentiated form"

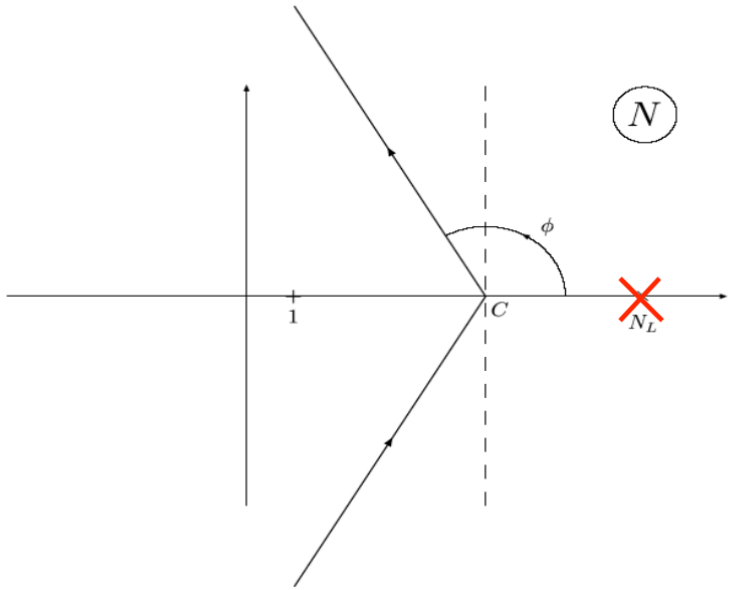
$$
\Sigma(\tau,\alpha_S)\simeq e^{\ell\, g_1(\lambda_\tau)+g_2(\lambda_\tau)+\frac{\alpha_S}{\pi} g_3(\lambda_\tau)+\sum_{n=4}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^{n-2} g_n(\lambda_\tau)}\,,
$$

Exact numerical inversion (Minimal Prescription)

$$\Sigma(\tau, \alpha_S) = \frac{1}{2\pi i} \int_{C_{MP}} \frac{dN}{N} e^{N\tau} e^{\mathcal{F}(\alpha_S, L)} ,$$

where the contour C runs parallel to the imaginary axis and lies to the right of all singularities of the integrand.

Exact numerical inversion can be performed with a prescription to avoid the Landau Pole. Minimal Prescription [Catani,Mangano,Nason,Trentadue('96)]: the contour of integration C_{MP} lies to the right of all physical singularities but to the left of the (unphysical) Landau pole. The results obtained by using this prescription converge asymptotically to the perturbative series and do not include any power correction.



Remainder function and unitarity

Outside $\tau \ll 1$ resummation is not justified. Matching with F.O. results is necessary

$$D(\tau, \alpha_S) = R_T(\tau, \alpha_S)|_{f.o.} - [C(\alpha_S(Q^2))\, \Sigma(\tau, \alpha_S(Q^2))]_{f.o.}\,;$$

Resummation ambiguity at large τ only partially solved by matching. We impose the *physical* constraint on total probability:

$$R_T(\tau_{\max}) = 1\,, \qquad (R_T(\tau) \geq 0)$$

otherwise violated by higher-order terms. It can be fulfilled in τ space using CTTW['93]

$$\ell \equiv \ln(1/\tau) \quad \mapsto \quad \tilde{\ell} \equiv \ln(1/\tau - 1/\tau_{\max} + 1) \overset{\tau \ll 1}{=} \ell + \mathcal{O}(\tau)\,,$$

which acts as a perturbative unitarity constraint: $\Sigma(\tau = \tau_{\max}, \alpha_S)|_{\ell \to \tilde{\ell}} = 1\,.$

An analogous constraint can be imposed in N space:

$$L \equiv \ln N \quad \mapsto \quad \tilde{L} \equiv \ln(N + N_c) \overset{N \gg 1}{=} L + \mathcal{O}(1/N)\,,$$

with N_c a constant of $\mathcal{O}(1)$.

To reduce impact of resummation at large τ a smooth switching procedure can be also imposed:

$$\frac{d\sigma}{d\tau} \to w(\tau)\frac{d\sigma}{d\tau} + (1 - w(\tau))\left.\frac{d\sigma}{d\tau}\right|_{f.o.}, \qquad \textit{where} \quad \begin{cases} w(\tau) = 1 & \tau \leq \tau_{\rm sw.} \\ w(\tau) < 1 & \tau > \tau_{\rm sw.} \end{cases}$$

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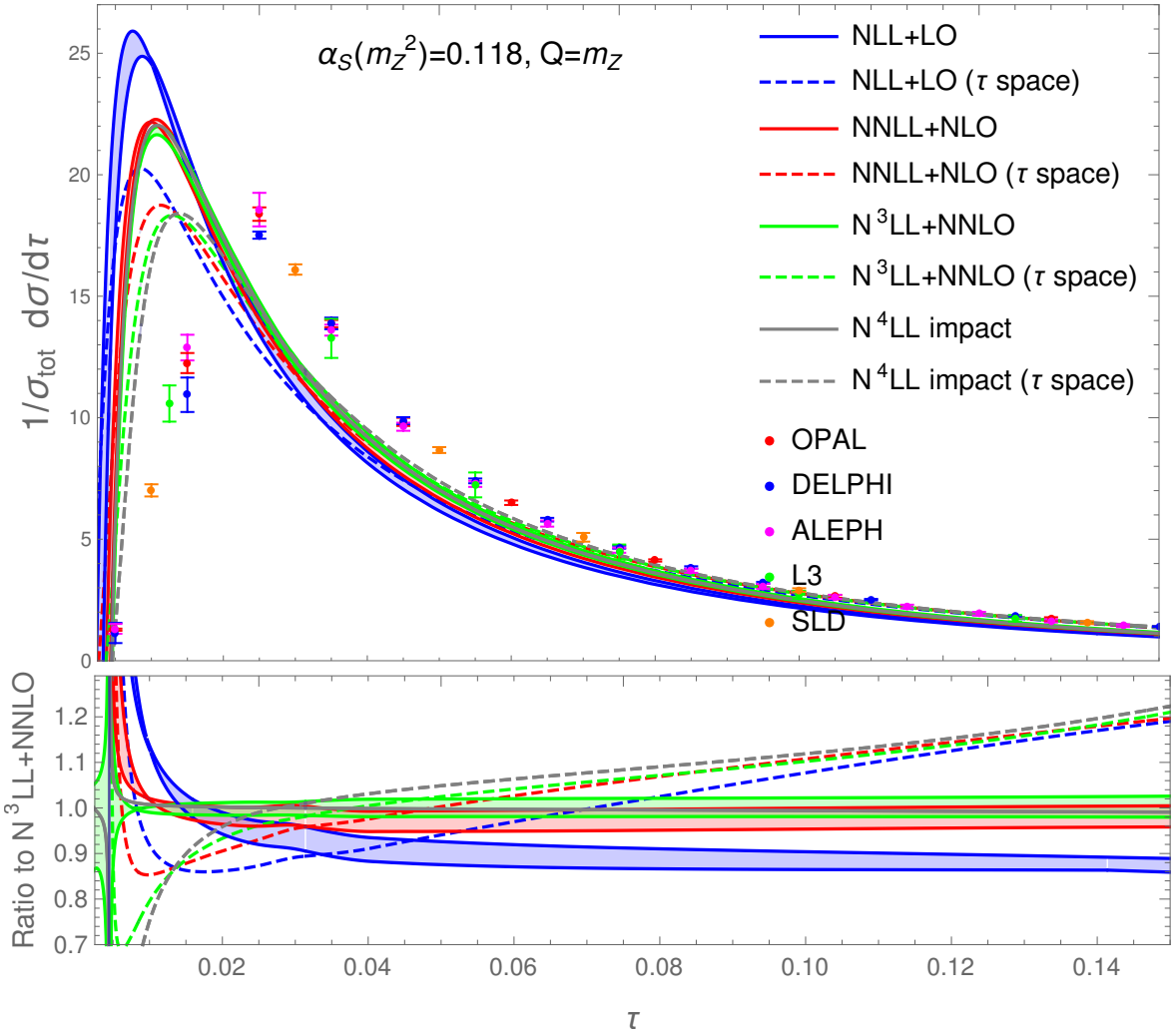
Numerical results: perturbative effects



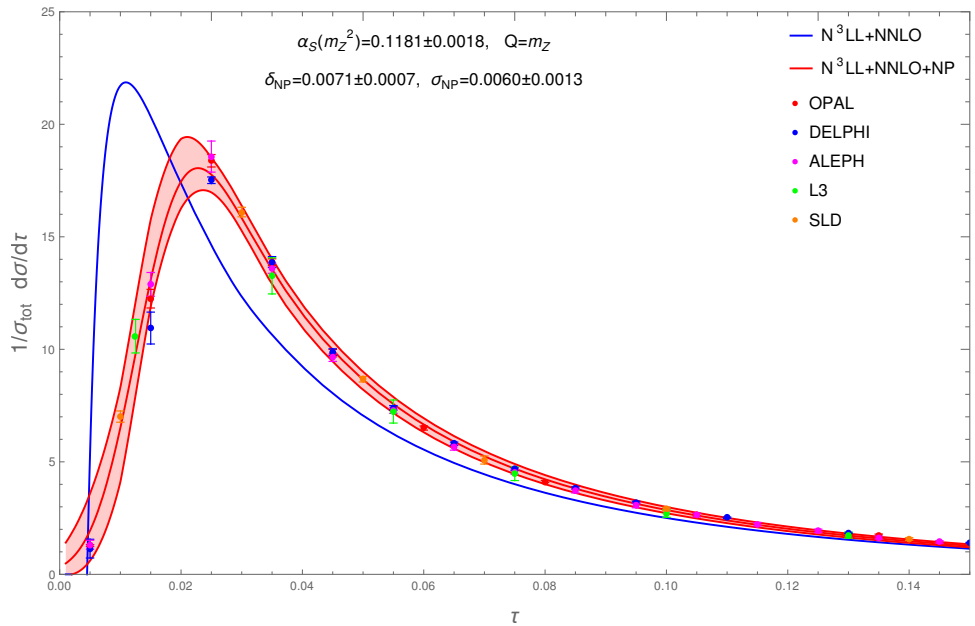
Thrust distribution at $Q = 91.1876$ GeV in pQCD. Results from resummation in Laplace-conjugated space (solid bands), including renormalization scale variations $Q/2 \leq \mu_R \leq 2Q$, compared with physical τ-space approximated results (dashed lines).

Numerical results: non perturbative effects

NP effects included using an analytic model based on a correlation [Catani at al.('91)] or shape function [Korchemsky,Sterman('99)] $f_{NP}(\tau_h, \tau)$ depending on 2 parameters:

$$\frac{d\sigma_h}{d\tau_h} = \int d\tau \frac{d\sigma}{d\tau} f_{NP}(\tau, \tau_h)\,,$$

$$f_{NP}(\tau_h, \tau) = \frac{1}{\sqrt{2\pi}\sigma_{NP}} \exp\left[-\frac{(\tau_h - \tau - \delta_{NP})^2}{2\,\sigma_{NP}^2}\right],$$



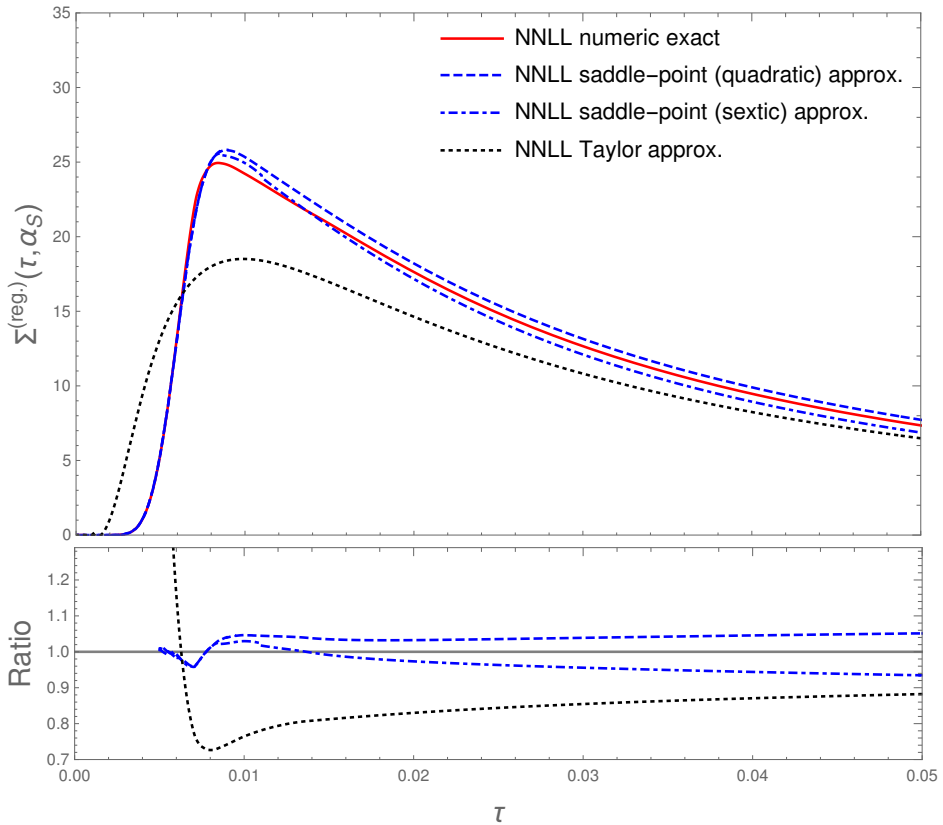
The thrust distribution at $Q = 91.1876$ GeV at N^3LL+NNLO in QCD without (blue solid line) and with (red band) the inclusion of NP effects.

Analytic Saddle-Point inversion [Aglietti,G.F.,W.-L. Ju('25)]

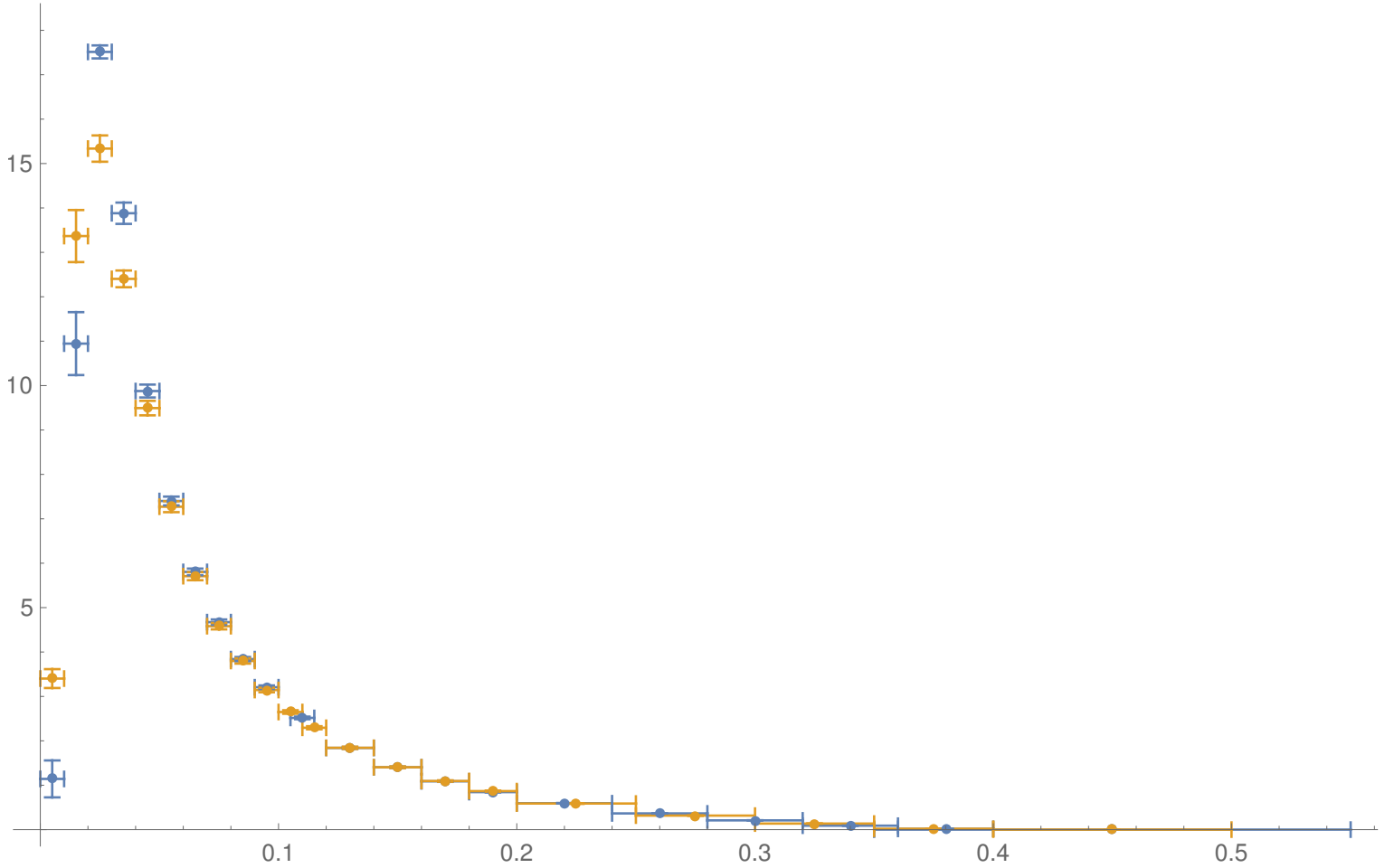
The true saddle point $\overline{N}$ is the following stationary point:

$$\overline{N}\tau = 1 - \frac{d}{d\lambda}\left[\lambda f_1(\lambda) + \beta_0 \sum_{n=1}^{\infty}\left(\frac{\alpha_S}{\pi}\right)^n f_{n+1}(\lambda)\right]_{\lambda\to\overline{\lambda}}$$

where $\overline{\lambda} = \beta_0\alpha_S \ln(\overline{N})/\pi$. The (usual) Taylor expansion is around $N = 1/\tau$: the *free* theory ($\alpha_S \to 0$) saddle point. The application of the SP method is complicated by the presence of the Landau singularity which has to be regularized.



Delphi data



D. Wicke (Wuppertal U.),Thesis PhD Wuppertal U.(1999) Report number: WUB-DIS-1999-05, WU-B-DIS-1999-05 (blue), Delphi Exp. Eur.Phys.J.C 14 (2000) 57-584, 2000 (yellow).

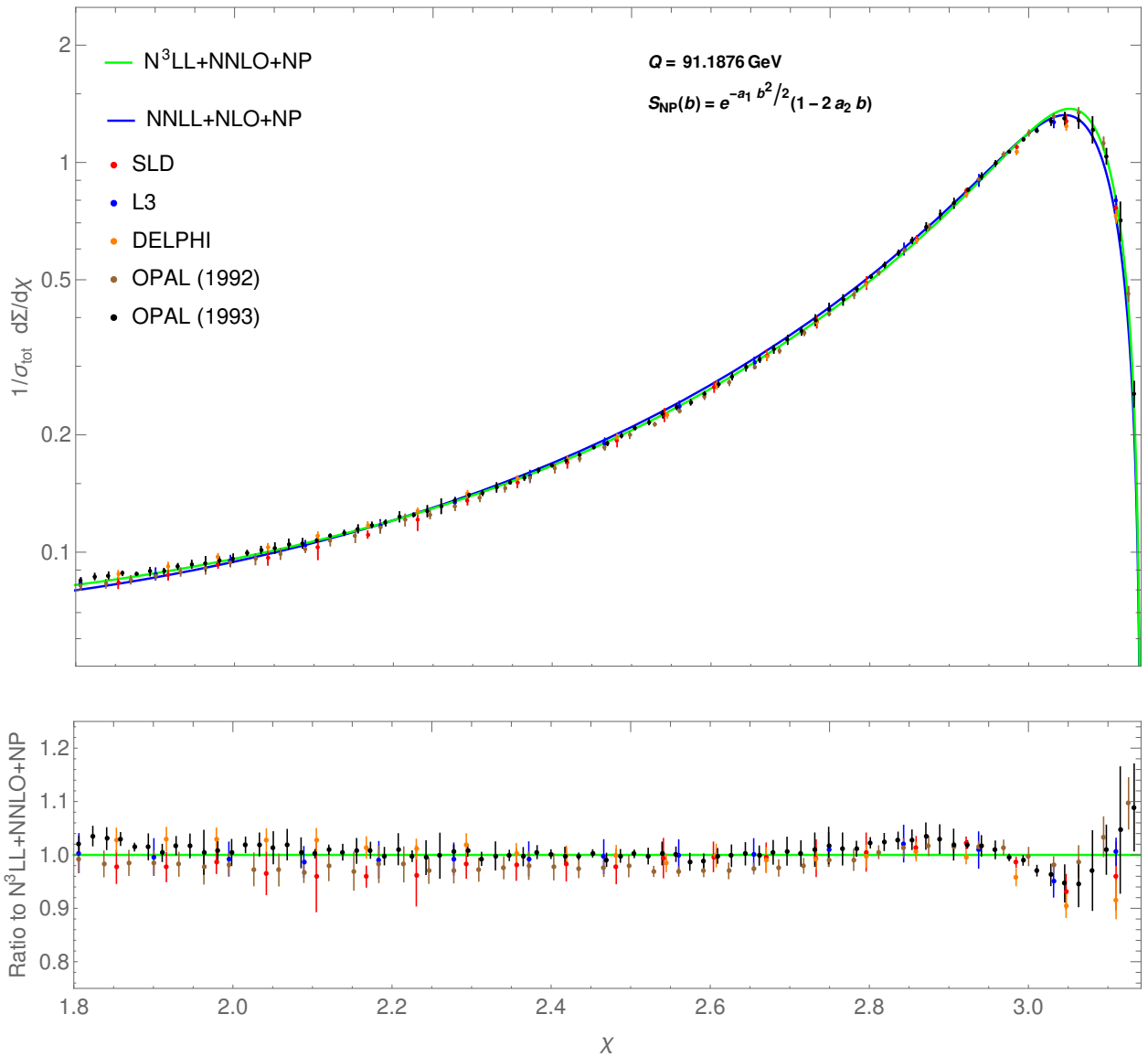
plus a data/theory based estimate of non perturbative corrections. The final quoted results are unbelievably precise: $\alpha_s(m_Z) = 0.1135 \pm 0.0011$ from the tail of the Thrust distribution [41] and $\alpha_s(m_Z) = 0.1140 \pm 0.0015$ from the first moment of the Thrust distribution [42] (note the low central value). I think that this is a good example of an underestimated error which is obtained within a given machinery without considering the limits of the method itself. Another allegedly

Thank you

Back up slides

Energy Energy Correlation function

[Aglietti,G.F.('24)]



Comparison at N^{3}LL+NNLO and NNLL+NLO with NP effects parameterized by a form factor

$$S(Q,b) \to S(Q,b)S_{NP}(b)$$

$$S_{NP} = \exp\{-a_1 b^2\}(1 - a_2 b)$$

[Dokshitzer,Marchesini, Webber('99)]

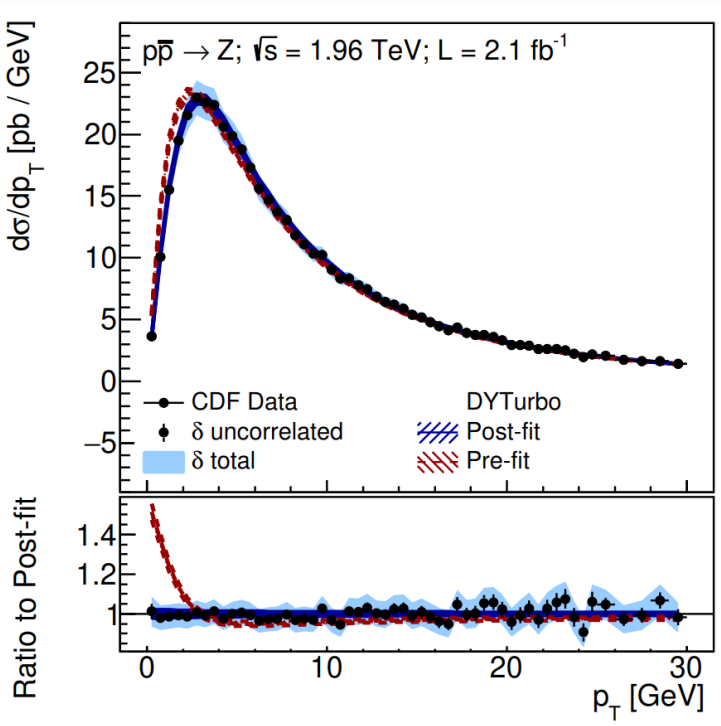
Fit results

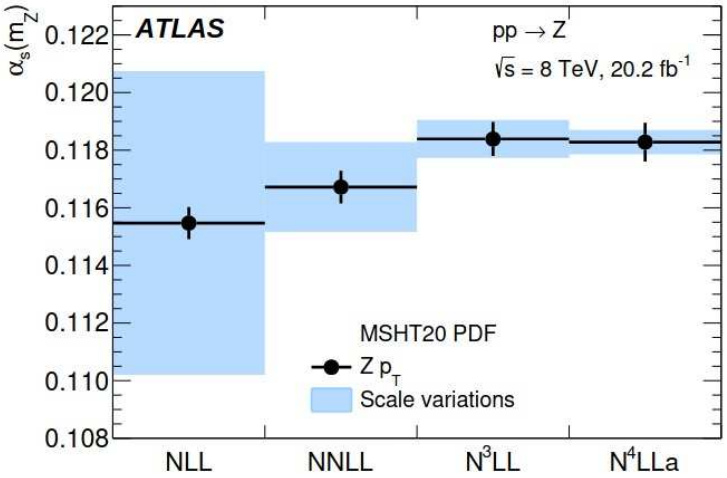
NNLL+NLO:
$\alpha_S(m_Z) = 0.121 \pm 0.002$,
$a_1 = 1.9 \pm 1.4 GeV^2$,
$a_2 = 0.4 \pm 0.1 GeV$

N^{3}LL+NNLO:
$\alpha_S(m_Z) = 0.120 \pm 0.002$,
$a_1 = 1.8 \pm 1.4 GeV^2$,
$a_2 = 0.3 \pm 0.1 GeV$

Drell–Yan transverse momentum

[Camarda,G.F.,Schött('22)],[ATLAS Coll.('23)]





Simultaneous fit of $\alpha_S(m_Z)$ and NP parameters at N^{4}LL+$\mathcal{O}(\alpha_S^3)$:

$$\alpha_S(m_Z) = 0.11828^{+0.00084}_{-0.00088}$$