On Thrust resummation in Conjugate and Direct space

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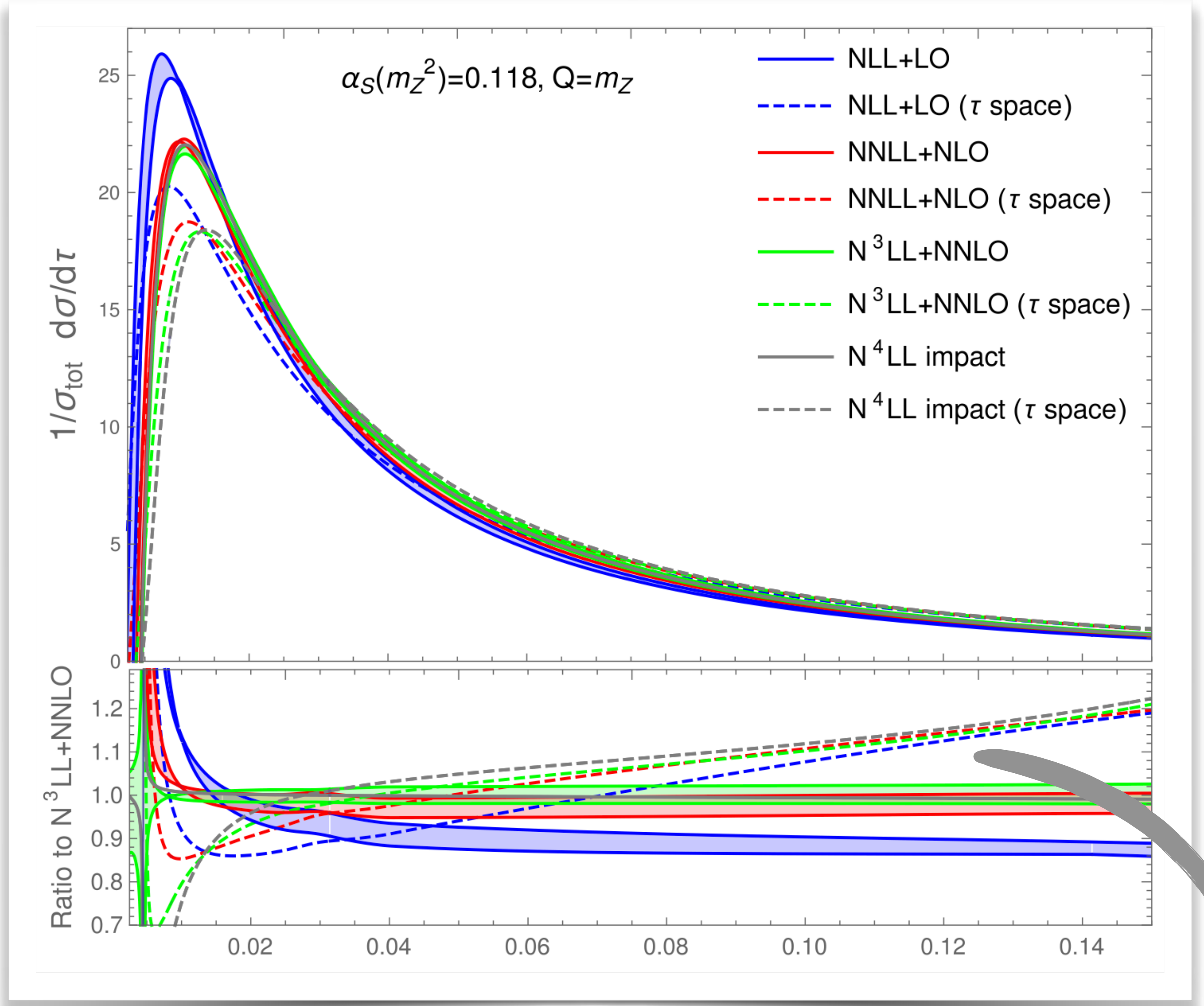
Introduction

α_s determinations from thrust based on highly accurate resummation usually lead to **smaller values** than world average

Latest addition to the **"thrust puzzle"**: impact of the **choice of the resummation space**

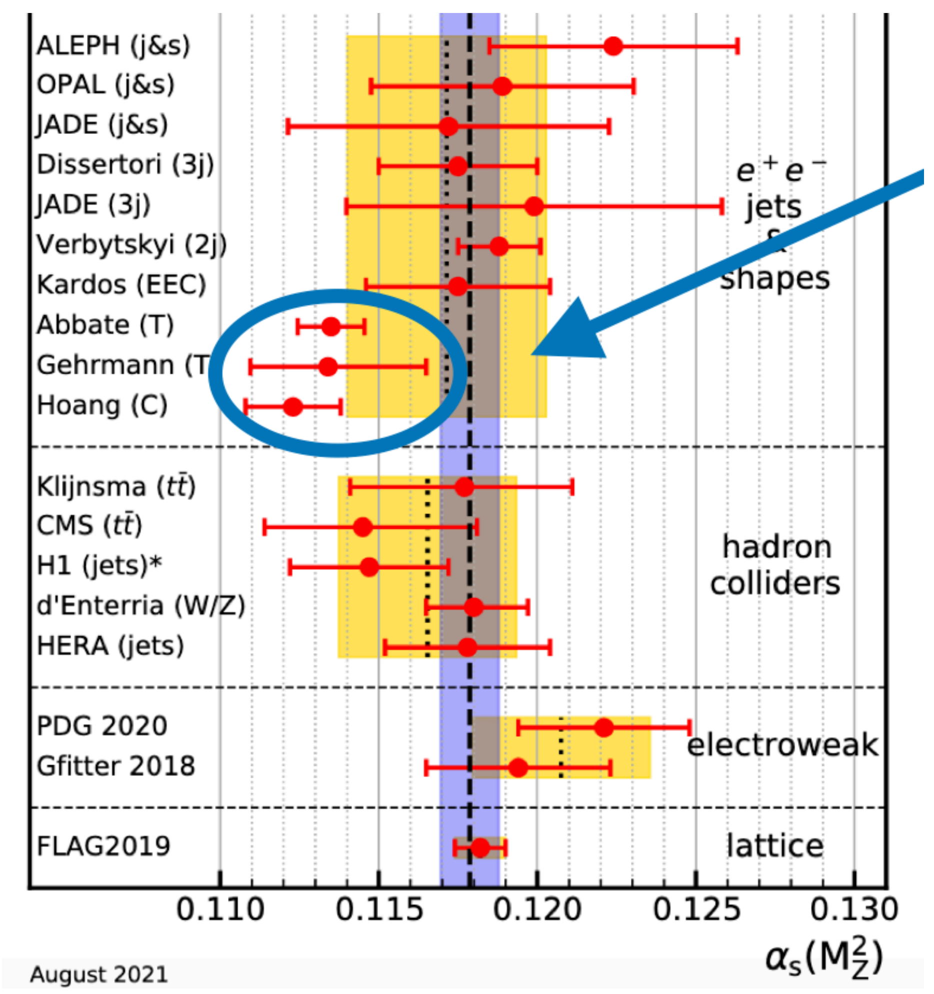


[Aglietti,Ferrera,Ju,Miao '25]

Resummation performed in **conjugate Laplace space** and **exact inversion** to τ space

Significant differences, before applying non-perturbative (NP) corrections, both in "peak" and "fit" regions

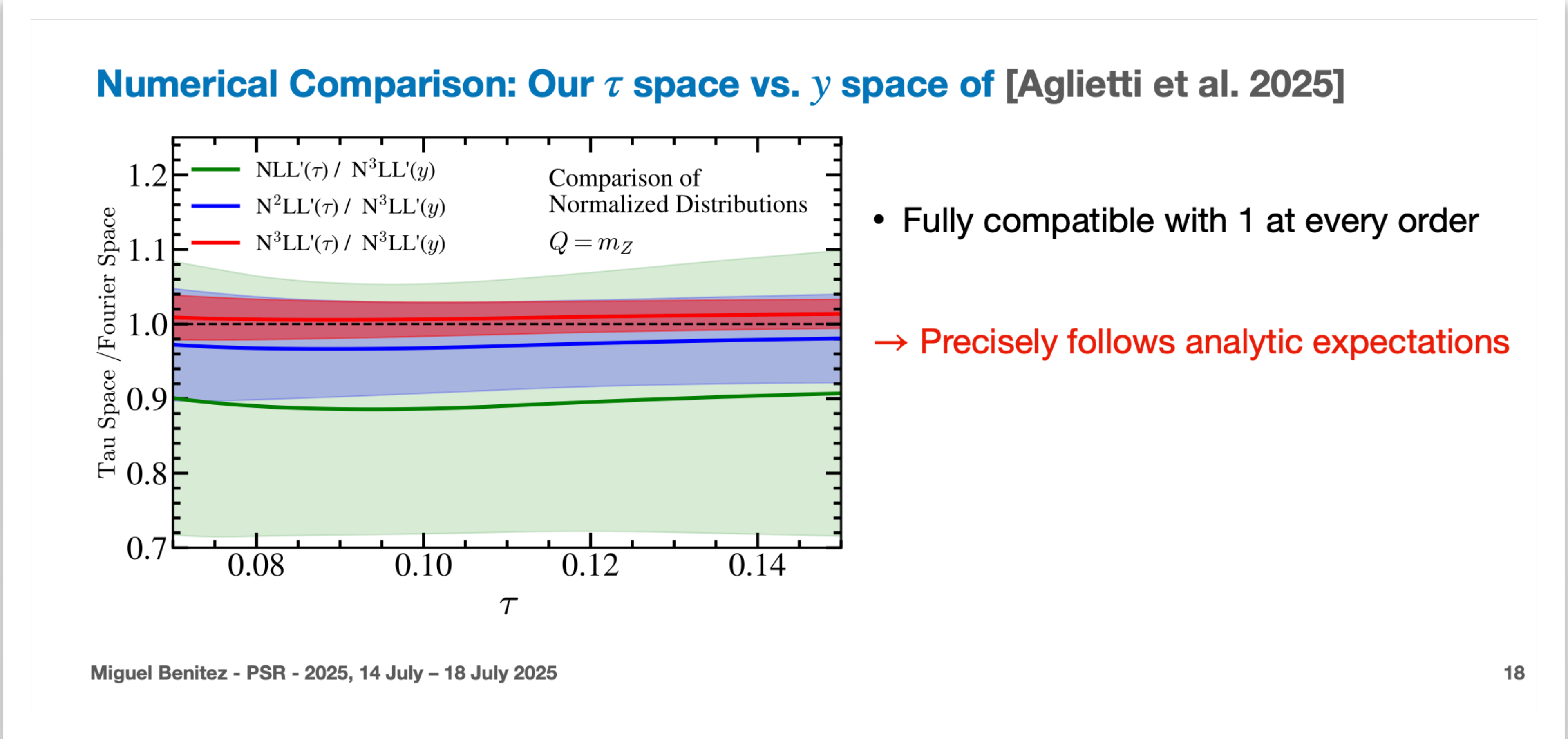
The conjugate Laplace space formulation + two-parameter Gaussian shape for NP (<u>peak region included</u>) leads to an α_s extraction **compatible with the world average**



PDG 2021 (removed from PDG 2023)

slide from Benitez PSR25

Significant differences in the **fit region** where a perturbative approach should be justified



Not confirmed in a similar exercise performed with EFT methods

Introduction

Historical considerations

There can be cases in which the choice of the resummation space, though formally subleading, **leads to significant differences**

The soft-gluon resummation for the production near **threshold** of a heavy system in hadron collisions provides an example

The argument in a nutshell: the Double Log (DL) Model

Threshold limit: $\tau \to 1, N \to \infty$

[Catani,Mangano,Nason,Trentadue 1996]

Mellin space

$$\sigma^{\rm res}(\tau) = \frac{1}{2\pi i}\int_C {\rm d}N f_N^2\, e^{a \ln^2 N} \tau^{-N}$$

$$a = 2C_R \frac{\alpha_s}{\pi}$$

Convergent series

going to momentum space

$$\longrightarrow$$

$$\sigma^{\rm res}(\tau) = \int {\rm d}x_1 {\rm d}x_2 {\rm d}z\, f(x_1) f(x_2) \delta(x_1 x_2 z - \tau) \Delta(z)$$

$$= \int_\tau^1 {\rm d}z\, e^{a \ln^2(1-z)} \frac{\rm d}{{\rm d}z} \mathscr{L}\left(\frac{\tau}{z}\right)$$

parton luminosity

Asymptotic behavior driven by

$$\int_0^1 {\rm d}z e^{a \ln^2(1-z)} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \int_0^1 {\rm d}z \ln^{2k} z = \sum_{k=0}^{\infty} \frac{a^k}{k!}(2k)!$$

Factorial growth of the coefficients!

$$\longrightarrow$$

$$\boxed{\text{Power-like ambiguities} \quad \sim \left(\frac{\Lambda}{Q}\right)^{\frac{\pi b_0}{4 C_R}}}$$

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Main considerations

- The original series in Mellin space **does not contain** factorially growing terms
- Neglecting subleading terms can lead to **large "spurious" power-like ambiguities**, unrelated to infrared renomalons
- The physical origin of these effects is understood in terms of **the violation of momentum sum rules**

Similar conclusions can be drawn at Leading Log (LL) with the caveat of the prescription of the Laundau pole

Objectives & Strategy

GOAL(s): Investigate **(semi)-analytically** the impact of the **choice of the resummation space** and the underlying **physical interpretation** in a **controlled setup** (KISS principle) and **without prejudices** (for example, fitted values of α_s)

Start from the Laplace-space formulation
τ-space resummation derived through the **analytic inversion** of the Laplace-space formula **neglecting subleading terms**

Focus on the "inversion problem": **convergence properties** of the series associated with the exact inversion of the Laplace-space formula at increasing logarithm orders

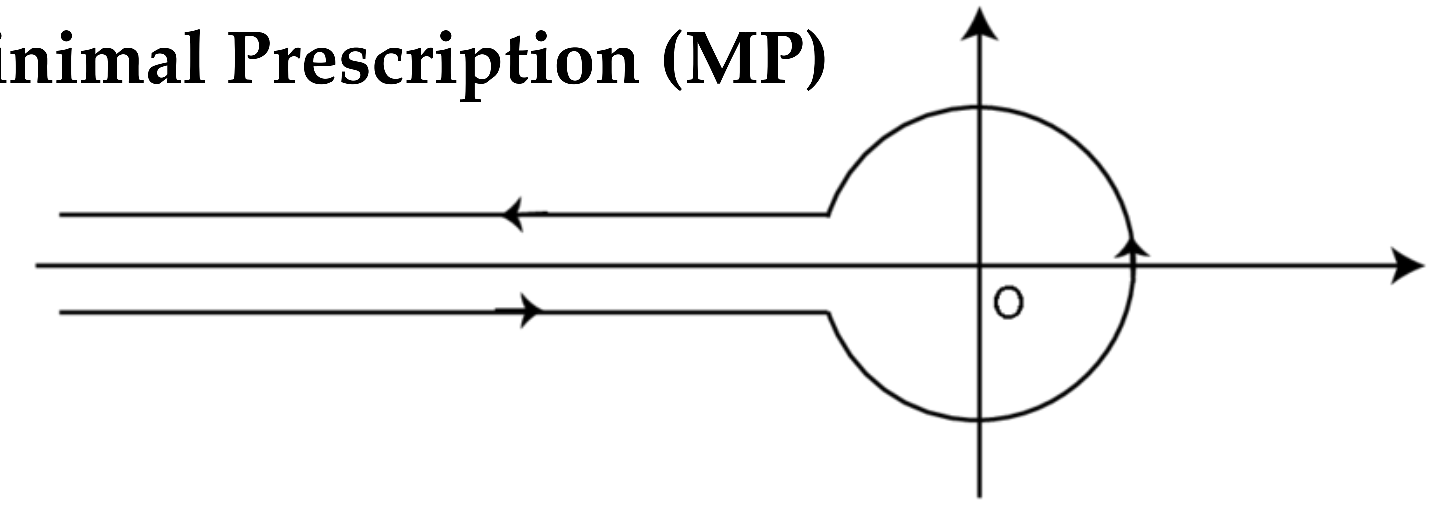
<u>A fully analytic case: the DL Model</u>

Start in Laplace space

$$R_T^{\mathrm{res}}(\tau) = \frac{1}{2\pi i}\int_C \frac{\mathrm{d}N}{N}\, e^{N\tau}\, \boxed{e^{-a\ln^2 N}} \qquad a = C_F\frac{\alpha_s(Q)}{\pi}$$

Normalised cumulant DL model

$C \equiv$ Hankel contour,
Minimal Prescription (MP)



To ease the logarithmic power counting

$x = N\tau$ Integral dominated at $x \sim 1$

$$R_T^{\mathrm{res}}(\tau) = \frac{e^{-a\ln^2\frac{1}{\tau}}}{2\pi i}\int_C \mathrm{d}x\, e^x\, e^{-(1+2\lambda)\ln x - a\ln^2 x}$$

$$= \boxed{e^{-a\ln^2\frac{1}{\tau}}}\ \boxed{\exp\left[-a\frac{\mathrm{d}^2}{\mathrm{d}(1+2\lambda)^2}\right]\frac{1}{\Gamma(1+2\lambda)}}$$

DL results

Subleading terms
Can be neglected?

At fixed $\lambda = a\ln 1/\tau$, the expansion in a in the exponent provides the tower of increasingly subleading logarithmics terms

In this case, the series has an **infinite radius of convergence**!

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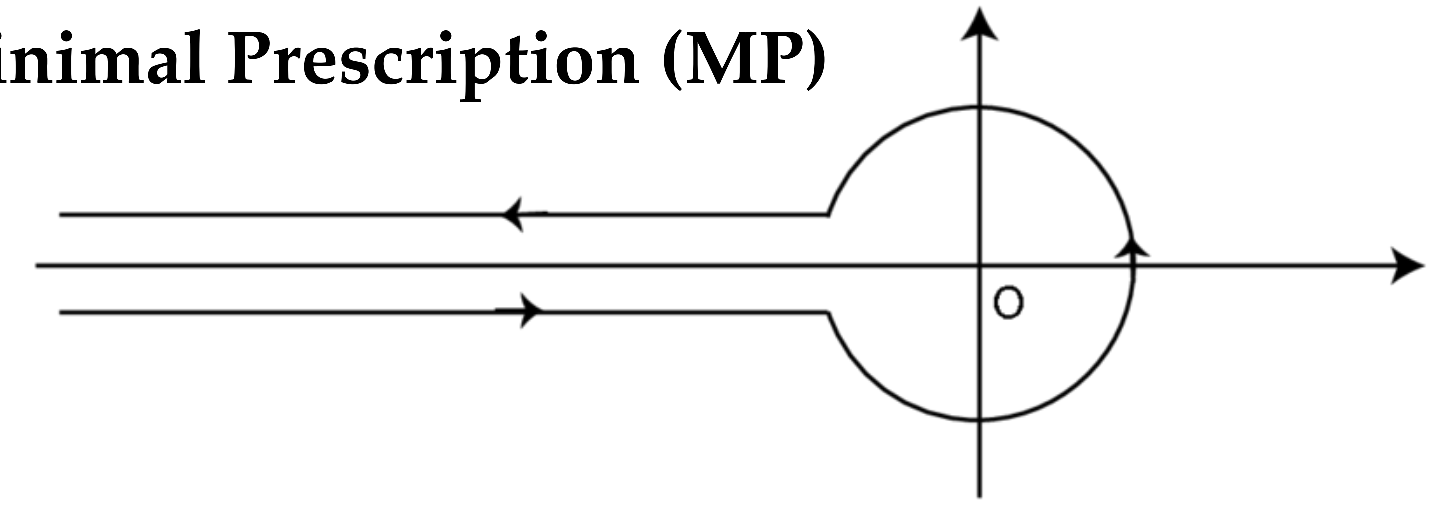

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The Leading Log case

Start in Laplace space

$$R_T^{\text{res}}(\tau) = \frac{1}{2\pi i}\int_C \frac{\mathrm{dN}}{N}\, e^{N\tau}\, e^{\frac{g_1(\lambda_N)}{a_s\beta_0}} \qquad a_s = \frac{\alpha_s(Q)}{\pi} \qquad \lambda_N = a_s\beta_0 \ln N$$

$$g_1(\lambda) = -\frac{A^{(1)}}{\beta_0}[(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda)]$$

MP is employed to deal with the **Landau Pole** associated with the **running coupling**

$$N_{\text{LP}} = e^{\frac{1}{2a_s\beta_0}} \gg 1$$

Analytic inversion $\quad x = N\tau$

$$\lambda_N = \lambda_x + \lambda_\tau \qquad \lambda_x = a_s\beta_0 \ln x \qquad \lambda_\tau = a_s\beta_0 \ln\frac{1}{\tau}$$

$$R_T^{\text{res}}(\tau) = \frac{e^{\frac{g_1(\lambda_\tau)}{a_s\beta_0}}}{2\pi i}\int_C \mathrm{d}x\, e^x \exp\left[\frac{1}{a_s\beta_0}\Big(\underbrace{\lambda_x g_1'(\lambda_\tau)}_{\text{NLL}} + \underbrace{H(\lambda_x;\lambda_\tau)}_{\text{Subleading}}\Big)\right]$$

$$= e^{\frac{g_1(\lambda_\tau)}{a_s\beta_0}} \exp\left[\frac{1}{a_s\beta_0} H\left(a_s\beta_0\frac{\mathrm{d}}{\mathrm{d}g'(\lambda_\tau)};\lambda_\tau\right)\right]\frac{1}{\Gamma[1-g_1'(\lambda_\tau)]}$$

$$H(\lambda_x;\lambda_\tau) = g_1(\lambda_x+\lambda_\tau) - g_1(\lambda_\tau) - \lambda_x g_1'(\lambda_\tau)$$

$$= \sum_{j=2}^{\infty}\frac{g_1^{(j)}(\lambda_\tau)}{j!}\lambda_x^j$$

The **logarithmic counting** corresponds to expanding in powers of λ_x

Remarks

- τ-space LL result (reproduces known literature) $\qquad R_T^{\text{res}}(\tau) = e^{\frac{g_1(\lambda_\tau)}{a_s\beta_0}}$
- Full structure including subleading terms

$$R_T^{\text{res}}(\tau) = e^{\frac{g_1(\lambda_\tau)}{a_s\beta_0}}\sum_{n=0}^{\infty} a_s^n D_n$$

What is the behavior of the series?

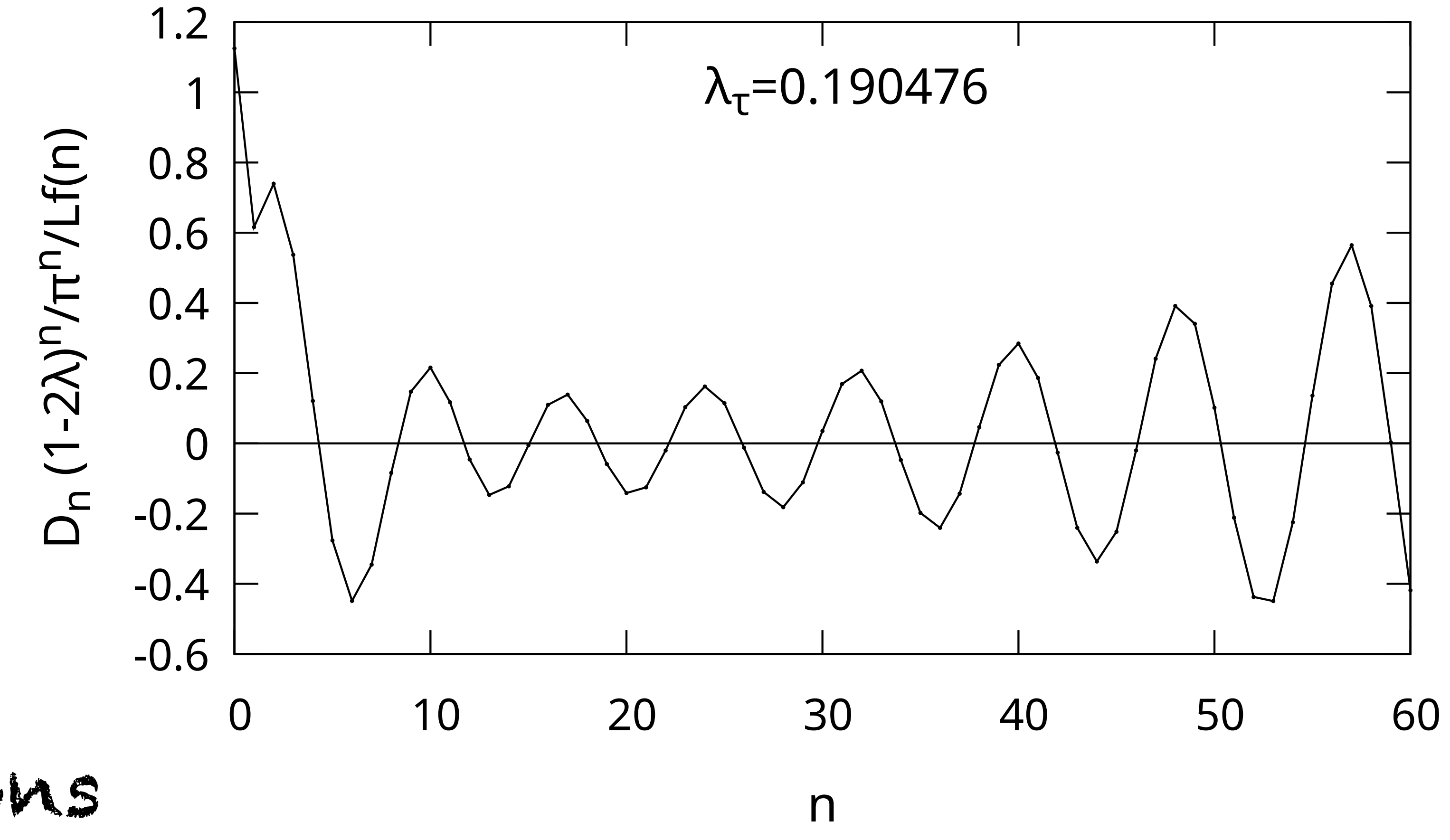
In this case, we could not solve the series analytically. We turned to **numerics**!

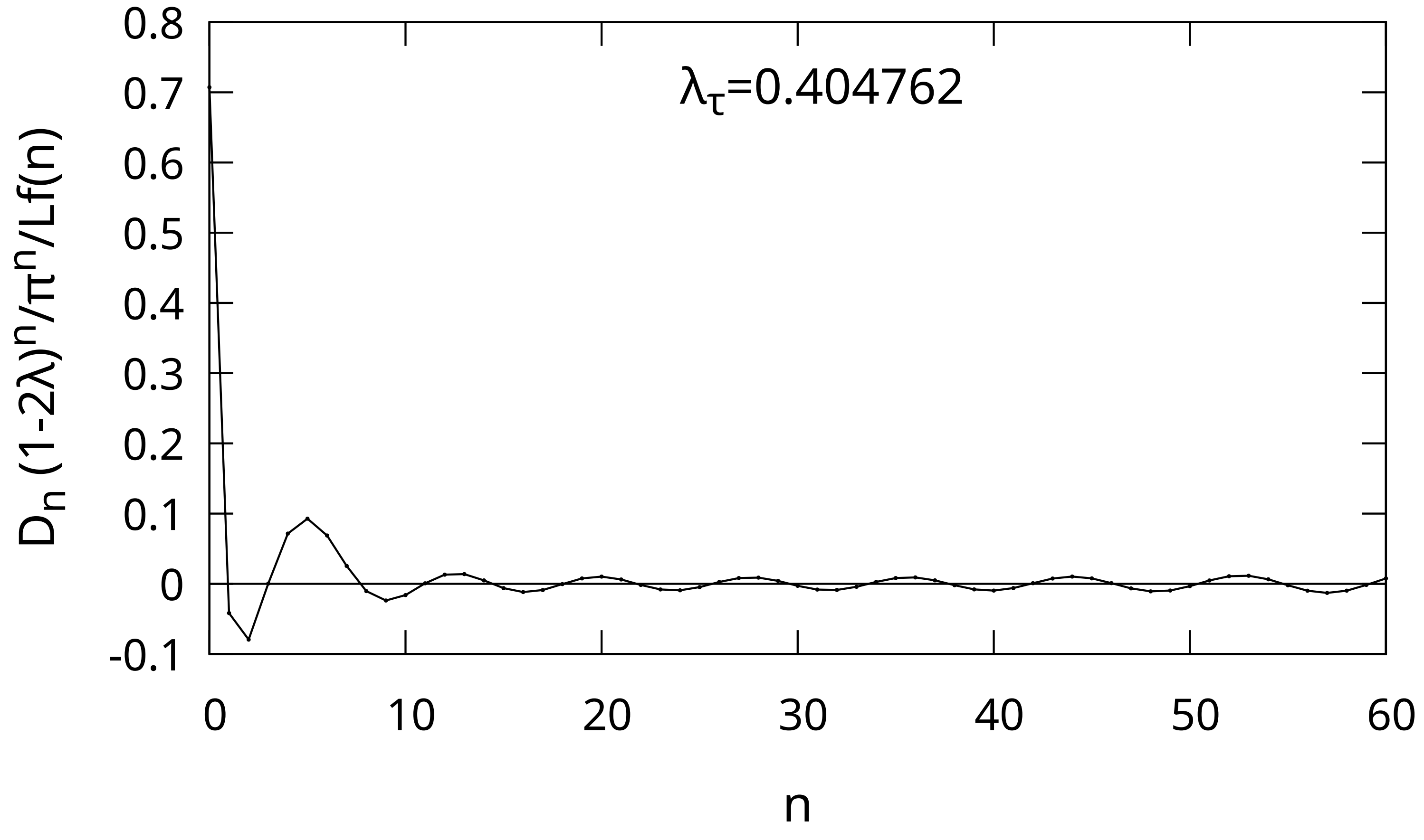
The Leading Log case: observations & discussion

From a numerical analysis considering up to 60 terms of the series

$$\boxed{D_n \propto \frac{\mathrm{Lf}(n)\pi^n}{(1-2\lambda_\tau)^n}\cos\left(\frac{n\pi}{4}\right)}$$

$$\mathrm{Lf}(n) = \prod_{j=2}^{n} \ln j$$ log factorial





Observations

- The series must be interpreted as **an asymptotic series, with periodic (and changing sign) coefficients.** It should be possible to resum it **without ambiguities. Very weak** log factorial growth, quite far from the factorial growth that accompanies power corrections
- The $(1-2\lambda_\tau)^n$ term can be readily related to the asymptotic behavior of the $(n+1)$ derivative of $g_1(\lambda_\tau)$, and, hence, to the **non-analiticity** introduced by the running coupling.
- Argument at the integrand level: the Taylor series of $g_1(\lambda_x+\lambda_\tau)$ around $\lambda_x = 0$ has a finite radius of convergence due to the Landau Pole. The integration over the Hankel contour will eventually encompass regions outside the domain of convergence
- This argument allows us to qualitatively interpret the observations that models like the **DL and** the **fixed-coupling model** lead to <u>convergent series</u>, while the **analytic coupling, though regularizing the Landau Pole,** leads again to <u>an asymptotic series</u>.

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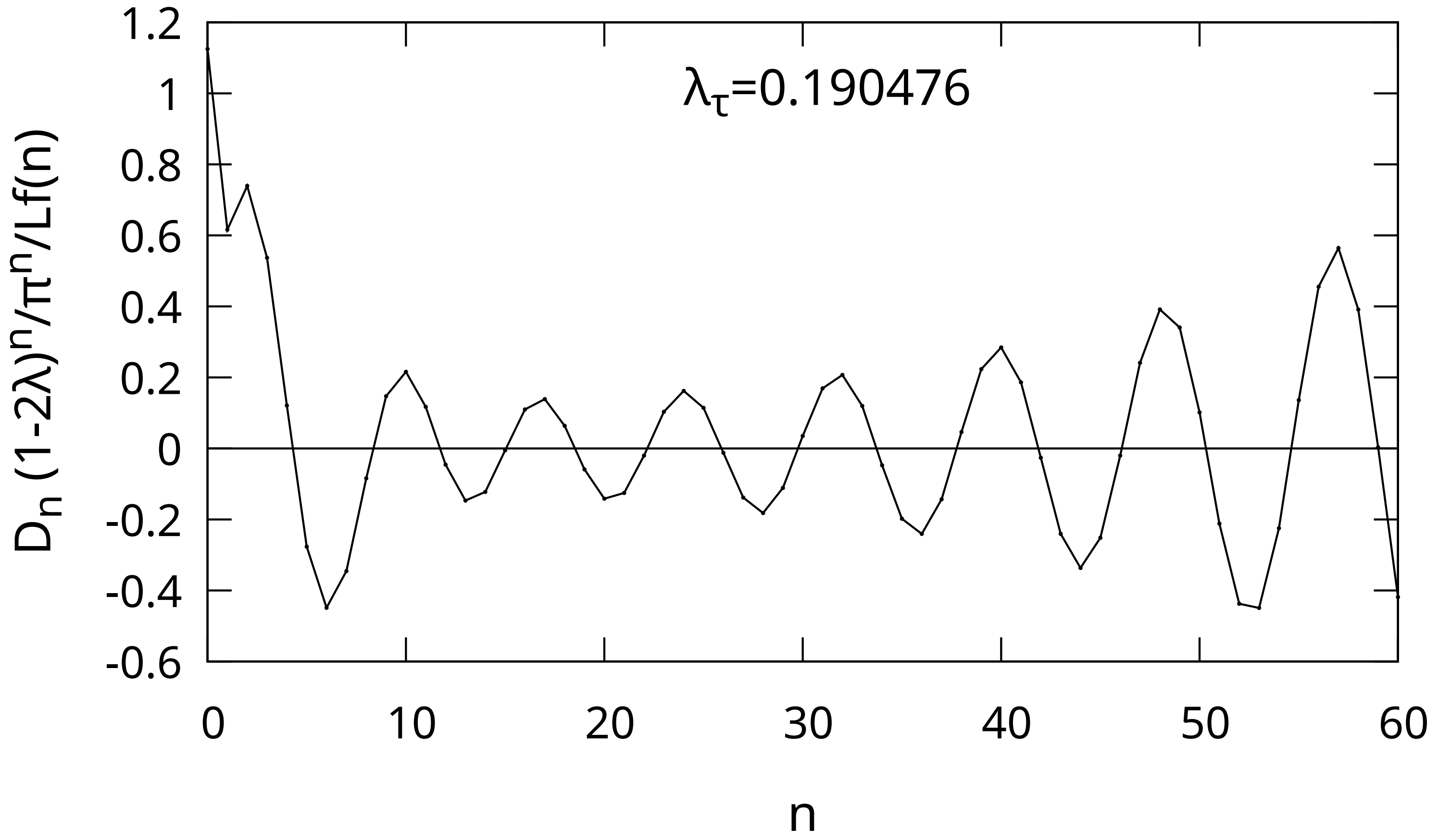
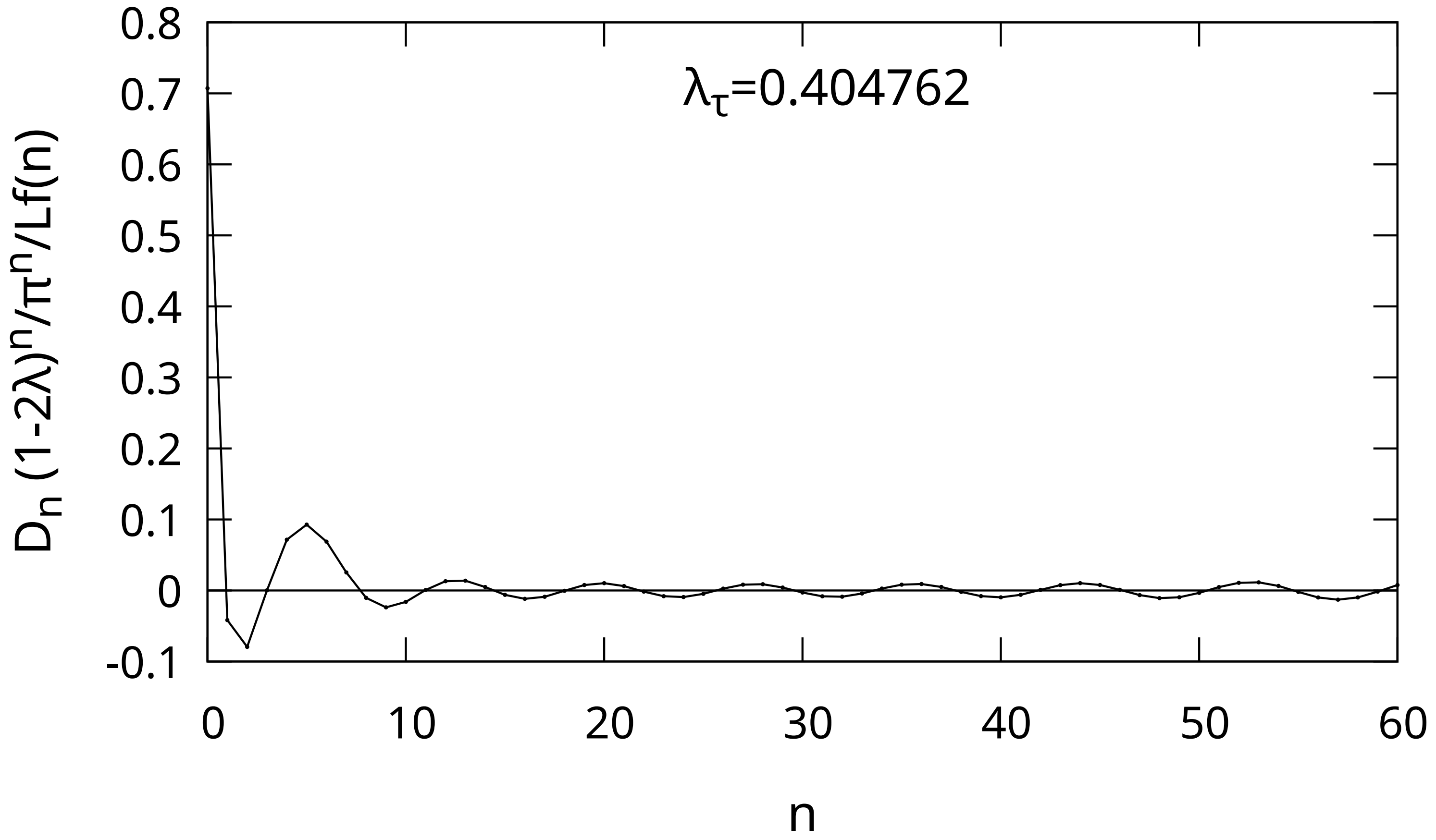




Discussion

- There are no clear theoretical arguments in favour of a particular resummation space
- We have studied the impact of a different prescription for the Landau Pole (Borel prescription), which differs by non-perturbative power corrections. We find very small effects
- The neglected higher-order terms **can be sizeable**. This should happen in regions where **perturbative resummation breaks down**
- Do "canonical" **scale variation** bands cover these effects?

We consider a conservative 7-point variation of the renormalization scale μ_R and of the resummation scale μ_L around Q, up and down by a factor of 2

$$\ln\left(\frac{1}{v}\right) \to \ln\left(\frac{\mu_L}{Qv}\right), \quad v \in \left\{\tau, \frac{1}{N}\right\}$$

Conjugate vs Direct space: pure resummed

Same reasoning can be applied to **higher logarithmic orders**

$$g(\lambda) = \frac{1}{a_s\beta_0} g_1(\lambda) + f_2(\lambda) + a_s f_3(\lambda) + a_s^2 f_4(\lambda) + a_s^3 f_5(\lambda)$$

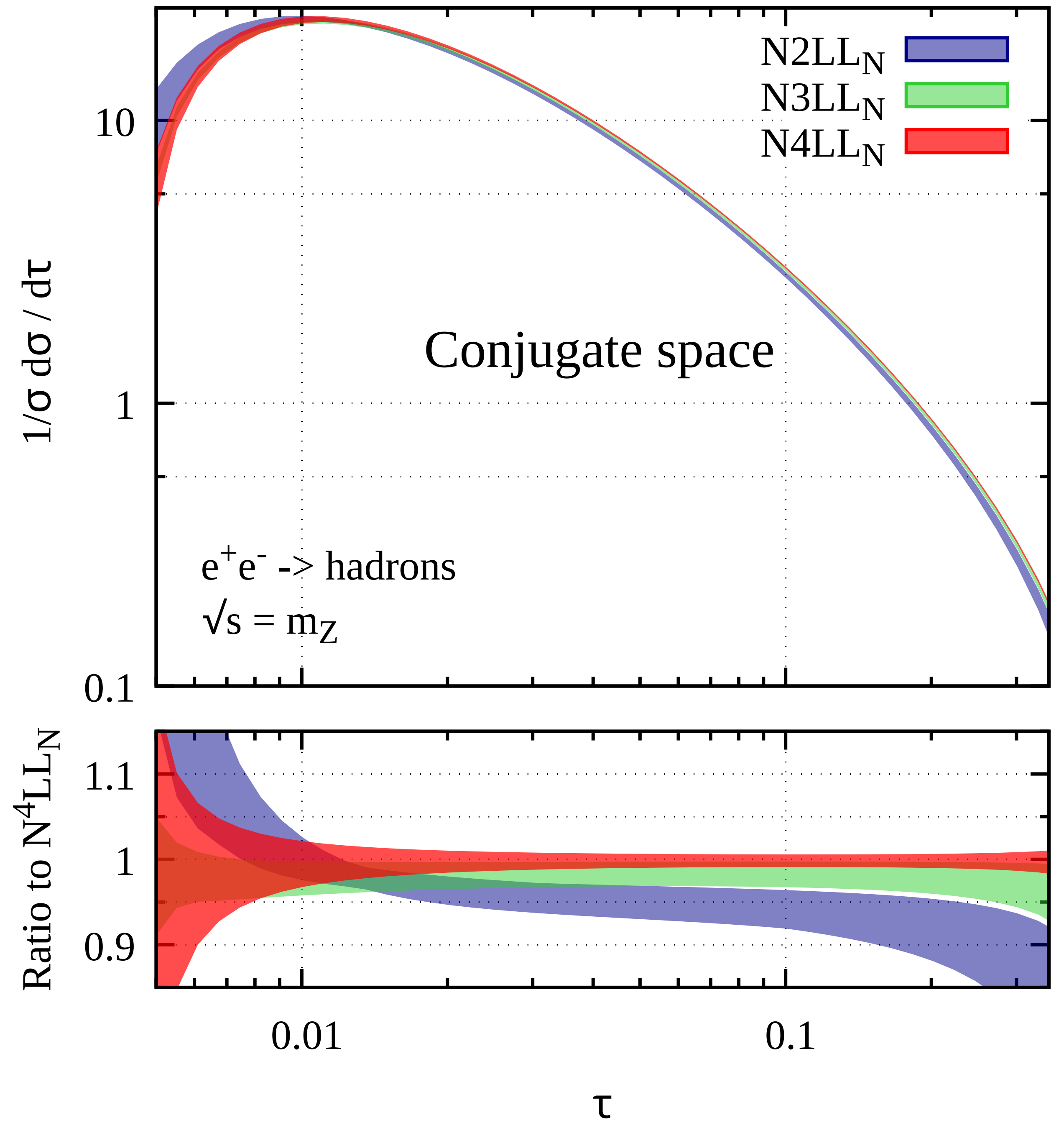
Counting at the exponent

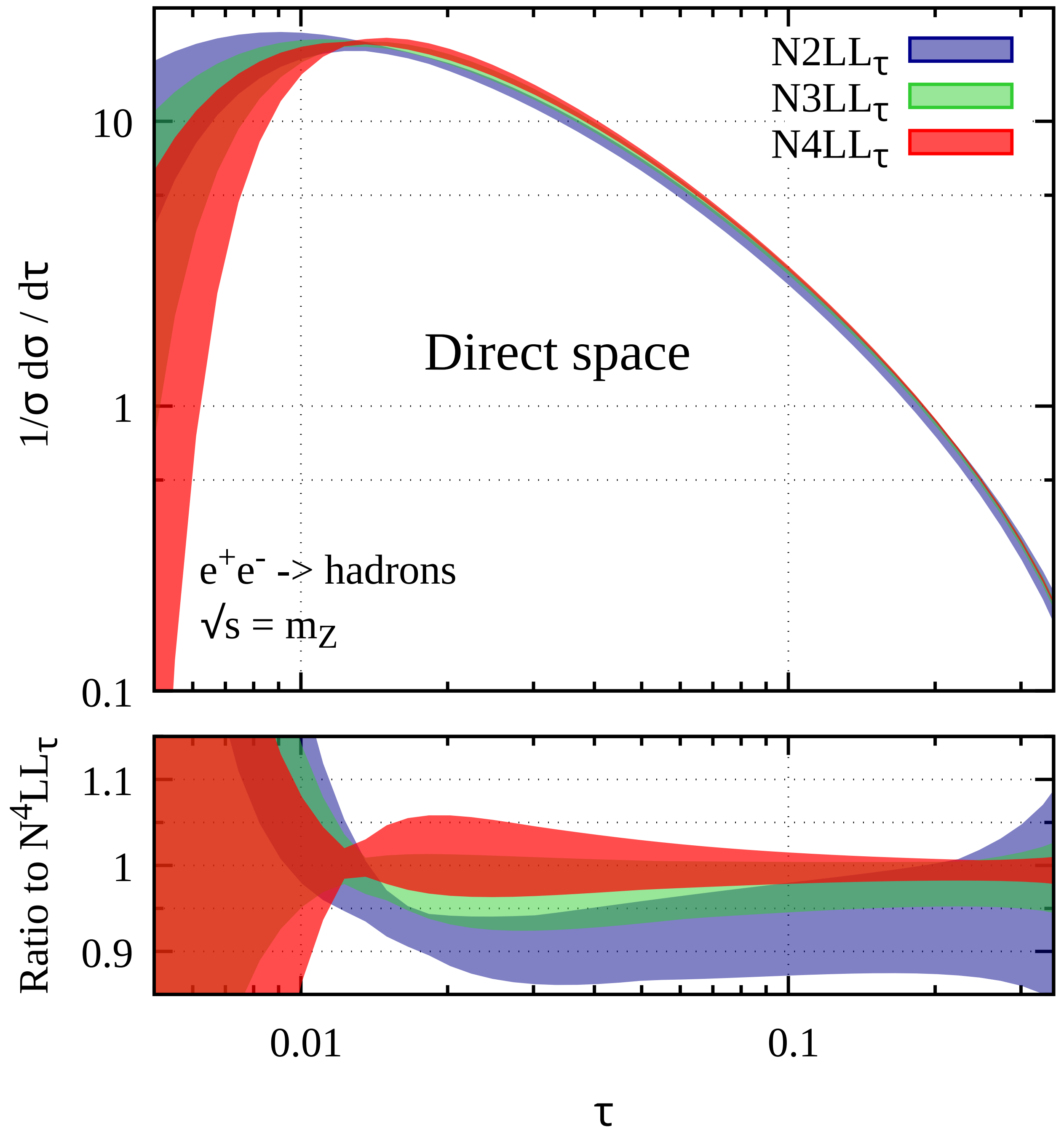
$$R_T^{\mathrm{res}}(\tau) = \frac{1}{2\pi i}\int_C \frac{\mathrm{d}N}{N}\, e^{N\tau}\, e^{g(\lambda_N)}$$

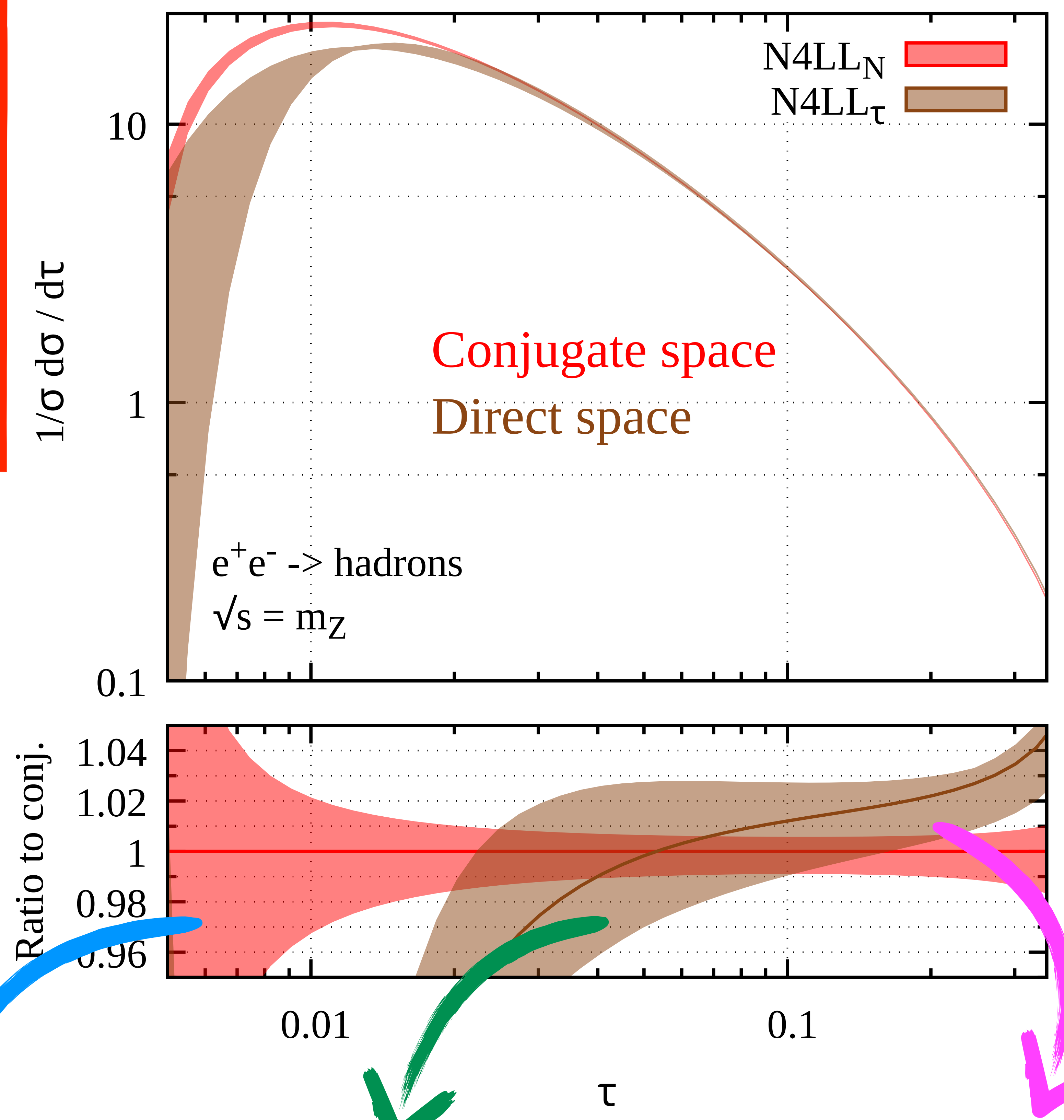
$$= e^{g(\lambda_\tau)} \exp\left[\frac{1}{a_s\beta_0} H\left(a_s\beta_0 \frac{\mathrm{d}}{\mathrm{d}g'(\lambda_\tau)};\lambda_\tau\right)\right] \frac{1}{\Gamma[1-g_1'(\lambda_\tau)]}$$

$$H(\lambda_x;\lambda_\tau) = g(\lambda_x+\lambda_\tau) - g(\lambda_\tau) - \lambda_x g_1'(\lambda_\tau)$$

- τ-space results obtained by truncating the exponential operator at the required order and defining suitable $f_i^{(\tau)}$ functions
- The spectrum is obtained as derivative of the cumulant*







Large differencies near and below the peak

Reasonable overlap in the "intermediate region"

At large τ, modified logs and fixed order matching matter

Conjugate vs Direct space: matched results

Some work towards **matched predictions** and comparison in the "fit region".
As a first step, we adopt a plain **additive matching**, and we follow Giancarlo's approach to introduce **modified logs**

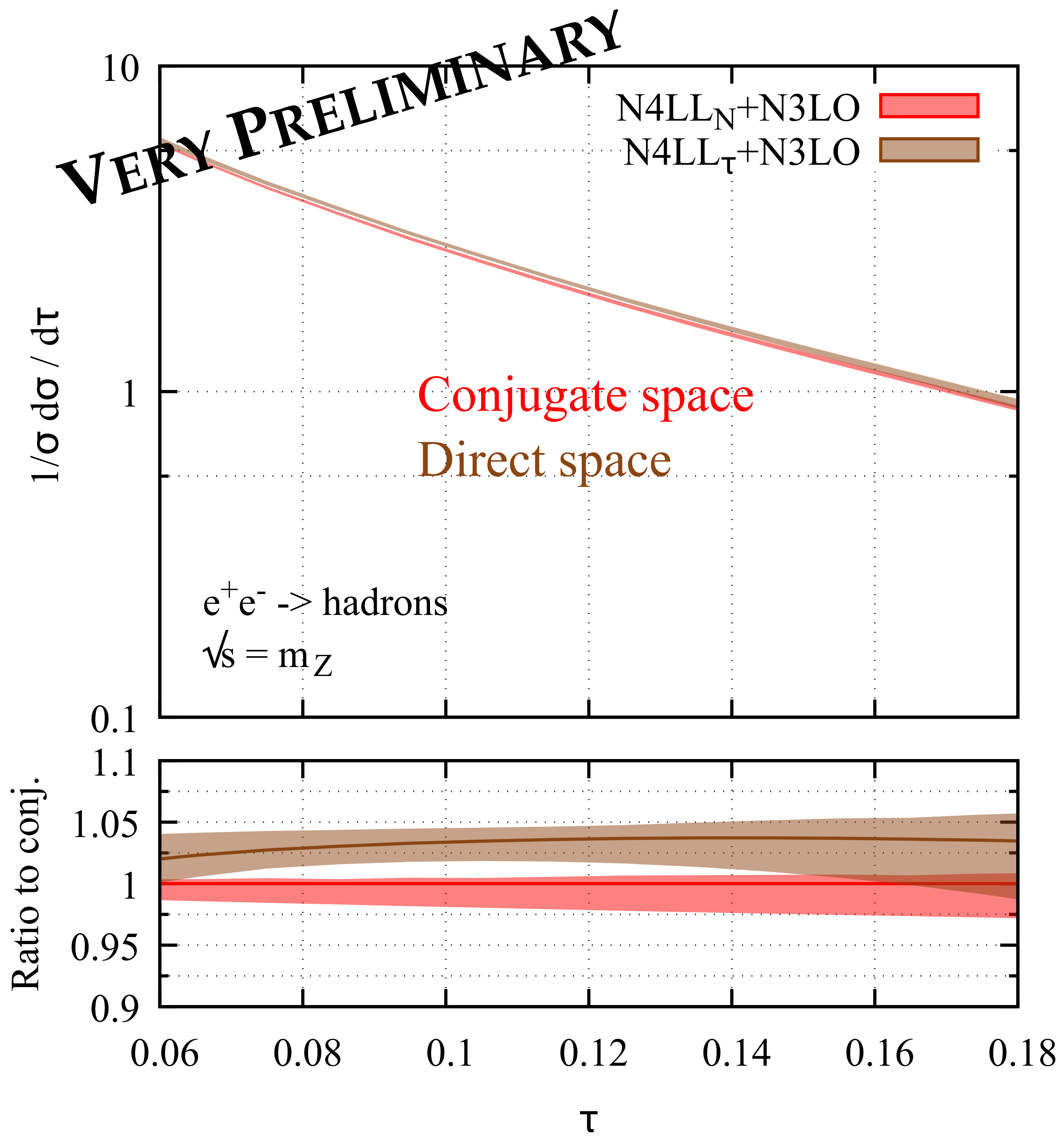
In the **preliminary** study, fixed-order predictions are taken from [Weinzierl 2009].
To be improved, but immaterial for the this comparison

$$\frac{\mathrm{d}}{\mathrm{d}\tau}R_T(\tau) = \frac{1}{2\pi i}\boxed{C(\alpha_s)}\int_C \mathrm{d}N\, e^{N\tau}\, e^{g(\lambda_N)} + \boxed{\mathrm{NS}(\alpha_s)}$$

$$= \boxed{C(\alpha_s)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ e^{g(\lambda_\tau)} \exp\left[\frac{1}{a_s\beta_0}H\left(a_s\beta_0\frac{\mathrm{d}}{\mathrm{d}g'(\lambda_\tau)};\lambda_\tau\right)\right]\frac{1}{\Gamma[1-g_1'(\lambda_\tau)]}\right\} + \boxed{\mathrm{NS}(\alpha_s)}$$

Boundary terms

Fixed order minus
resummed expanded



Linear modified logs

$$\ln\left(\frac{1}{\tau}\right) \to \ln\left(\frac{1}{\tau} - \frac{1}{\tau_{\max}} + 1\right)$$

Exact kinematic boundary $\tau_{\max}$

n	3	4	5
$\tau_{\max}$	0.3333	0.4226	0.4539

$$\ln N \to \ln\left(N - N_0\right)$$

N_0 fixed by unitary constraint: $\displaystyle\int_0^{\tau_{\max}} \frac{\mathrm{d}}{\mathrm{d}\tau}R_T(\tau;N_0) = 1$

From very preliminary results, $\mathcal{O}(2\%)$ differences, almost flat behavior
Uncertainty bands do not contain matching uncertainties

For Discussion

We tackled the problem from a different angle: we have analyzed the convergence properties of the **series associated with the inverse Laplace transform** to disclose potential ambiguities associated with asymptotic series (as in threshold resummation)

Our semi-analytical analysis shows that, in fact, the **series is an asymptotic one**, but its coefficients only has **"log-factorial" growth**, a behavior too weak to produce ambiguities usually associated with factorially growing terms.

Due to the asymptotic origin of the series, **neglected subleading terms in the inversion can be, however, sizeable** at small τ. This explains the large difference found around and below the peak, where perturbative resummation is expected to fail

On the contrary, in the "intermediate region", conjugate and direct space pure resummed results are more consistent.

Very preliminary results suggest that the situation does not change drammatically after the matching with the fixed-order

Other interesting perspectives

A potential issue related to the choice of the resummation space does not implication only in the extraction of α_s
Methods of automatic resummation, like Ceasar, Ares, parton showers in general, are usually formulated in direct space and are based on the corresponding counting of the logarithms.

We are currently exploring the connection to the ARES - RADISH - shower approaches