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ift Instituto de Física Teórica UAM-CSIC

EWBG with BARYONET

Theory and Applications

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Outline

1. EWBG: motivations
2. WKB approach
3. Boltzmann equations
   - Phenomenological model
   - BAU
4. `BARYONET`: end-to-end workflow
5. Results in `xSM`

EWBG: motivations

EWBG: motivations

- **Observed baryon asymmetry:**
$$\eta_B^{\rm obs} \simeq 8.7 \times 10^{-11}$$
far beyond what the Standard Model can generate.

- **Why the SM fails (Sakharov):** requires (i) $\Delta B \neq 0$, (ii) C and CP violation, (iii) departure from equilibrium. The SM has *too-small CPV* and only a *crossover* EW transition.

- **Electroweak baryogenesis:** a weak-scale, experimentally testable framework that ties particle physics to early-Universe dynamics (see Cohen, Kaplan, and Nelson 1990).

- **This work:** `BARYONET` provides a
$$\textbf{model} \quad \rightarrow \quad \textbf{wall profile} \quad \rightarrow \quad \textbf{transport equations} \quad \rightarrow \quad \eta_B$$
pipeline that **unifies and modernises the diffusion formalism**; **validated against the literature**; and **enables systematic scans of BSM models for baryogenesis**.

WKB approach

WKB approach - complex mass Dirac field

well known from Joyce, Prokopec, and Turok 1995; Cline, Joyce, and Kainulainen 2000; Fromme and Huber 2007; Cline and Kainulainen 2020; Kainulainen and Venkatesan 2024

We consider a single Dirac fermion with a spatially varying complex mass

$$\big(i\gamma^\mu\partial_\mu - m\,P_R - m^*P_L\big)\psi = 0, \qquad m(z) = |m(z)|\,e^{i\theta(z)}, \qquad P_{R,L} = \tfrac{1}{2}(1 \pm \gamma^5). \tag{1}$$

Looking for positive–energy solutions with plane–wave time dependence, we write

$$\Psi_s(t,z) = e^{-iEt}\begin{pmatrix} L_s(z) \\ R_s(z)\end{pmatrix} \otimes \chi_s, \quad \Rightarrow \quad L_s(z) = w(z)\,\exp\Big\{ i\int^z p_{cz}(z')\,dz'\Big\}, \tag{2}$$

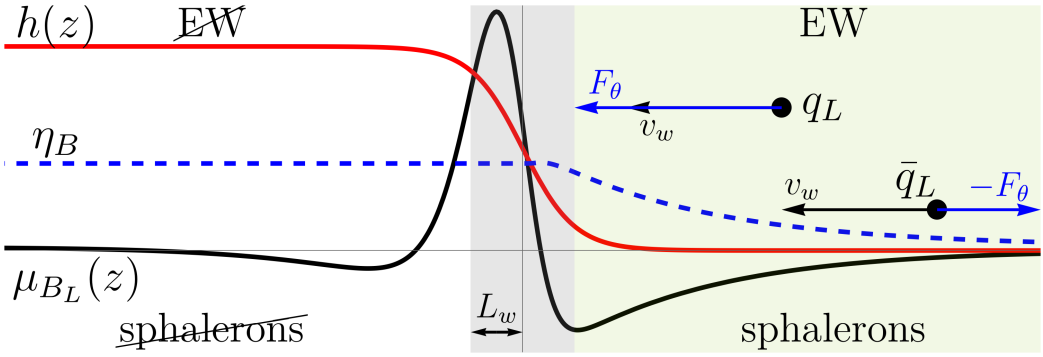
Different dispersion relation between q and $\bar{q}$

$$E = E_0 \pm \Delta E = E_0 - s_{CP}\,s\,\frac{\theta' m^2}{2E_0E_{0z}}, \tag{3}$$

where $s_{CP} = \pm 1$ and so different group velocity and semiclassical force

$$v_{gz} \equiv \frac{\partial E}{\partial p_{cz}} \simeq \frac{p_z}{E_0}\left(1 + s_{CP}\,s\,\frac{\theta' m^2}{2E_0^2E_{0z}}\right), \qquad F_z \equiv \dot{p}_z \simeq -\frac{(m^2)'}{2E_0} + s_{CP}\,s\,\frac{(m^2\theta')'}{2E_0E_{0z}} - s_{CP}\,s\,\frac{m^2(m^2)'\theta'}{4E_0^3E_{0z}},$$

WKB approach: sketch



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Boltzmann equations

Boltzmann equations

The transport problem starts from the Boltzmann equation

$$L[f] \equiv (\dot{z}\,\partial_z + \dot{p}_z\,\partial_{p_z})\,f = C[f]\,, \tag{4}$$

where we consider the **boosted distribution function**

$$f(z,\mathbf{p}) = \frac{1}{\exp\big[\beta\gamma_w\big(E_i(z,\mathbf{p}) + v_w p_z\big) - \beta\mu(z)\big] \pm 1} + \delta f(z,\mathbf{p}),$$

and we decompose the Liouville perator $L[f]$ in CP-odd/even part

$$L[f]\Big|_{\text{CP-odd}} \equiv \tfrac{1}{2}\Big(L[f] - \overline{L[\bar{f}\,]}\Big)\,, \qquad L[f]\Big|_{\text{CP-even}} \equiv \tfrac{1}{2}\Big(L[f] + \overline{L[\bar{f}\,]}\Big)\,,$$

from which we can define the CP-odd semiclassical source to be

$$S_o \equiv -\gamma_w v_w\,\frac{s\,(m^2\theta')'}{2E_0E_{0z}}f'_{0w} - s\,v_w\gamma_w\,\frac{m^2(m^2)'\theta'}{4E_0^2E_{0z}}\Big(\gamma_w f''_{0w} - \frac{f'_{0w}}{E_0}\Big). \tag{5}$$

Boltzmann equations

Solving directly for δf is a hard task, (see effort for the CP-even part and v_w predictions), so ...
We define a weighted phase–space average of any function $X(z, \mathbf{p})$ by

$$\langle X \rangle \equiv \frac{1}{N_1} \int d^3p\, X(z, \mathbf{p})\,, \qquad N_1 \equiv \int d^3p\, f'_{0w,\mathrm{FD}}(m=0) = -\gamma_w \frac{2\pi^3}{3} T^2\,, \tag{6}$$

We parametrise our ignorance about δf in the velocity introduced as

$$u_\ell(z) \equiv \Big\langle \Big(\frac{p_z}{E_0}\Big)^\ell \delta f(z, \mathbf{p}) \Big\rangle. \tag{7}$$

Projecting the stationary Boltzmann equation $L[f] = C[f]$ with the weight $(p_z/E_0)^\ell$ produces the ℓ-th moment equation

$$\Big\langle \Big(\frac{p_z}{E_0}\Big)^\ell L \Big\rangle = \Big\langle \Big(\frac{p_z}{E_0}\Big)^\ell \delta C \Big\rangle, \tag{8}$$

where δC is the linearised collision term.

Boltzmann equations

In a compact block–matrix form with the state vector $w(z) \equiv (\mu(z),\, u(z))^{\mathsf{T}}$ the system then reads

$$A\, w'(z) + (m^2)'(z)\, B\, w(z) = \mathcal{S}(z) + \delta\mathcal{C}(z),$$

with

$$A = \begin{pmatrix} -D_1 & 1 \\ -D_2 & R \end{pmatrix}, \qquad B = \begin{pmatrix} v_w\gamma_w\, Q_1 & 0 \\ v_w\gamma_w\, Q_2 & \bar{R} \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix}, \qquad \delta\mathcal{C} = \begin{pmatrix} \delta\mathcal{C}_1 \\ \delta\mathcal{C}_2 \end{pmatrix}.$$

Can be generalised to arbitrary ℓ. In the linearised averaged collision operator, appears inelastic + elastic + Strong Sphaleron contributions

$$\delta\mathcal{C}^a_{\ell+1} = -K^{(\ell)}_a \kappa^X_a \sum_{i,j} \widehat{\Gamma}^{\rm inel}_{i\to j} s_{ij}\, \mu_j - K^{(\ell)}_a \delta_{\ell 0} \Gamma_{ss} \mu_{ss} - \kappa^X_a\, \widehat{\Gamma}^a_{\rm tot}\, u_{\ell,a}, \qquad \widehat{\Gamma}^a_{\rm tot} \equiv \sum_{i\in a} \widehat{\Gamma}^{\rm el}_i + \widehat{\Gamma}^{\rm inel}_i\,. \tag{9}$$

and the averaged CP–odd source becomes

$$\mathcal{S}^o_\ell(z) = -v_w\gamma_w \Big[(m^2\theta')' Q^{8o}_\ell - m^2 (m^2)'\theta' Q^{9o}_\ell \Big], \tag{10}$$

with $D_\ell,\ Q_\ell,\ \bar{R},\ Q^{8o/9o}_\ell, \ldots$ the transport functions only dependent on $x = m/T$ and v_w.

Phenomenological model

Phenomenological model

EWBG: The collision sector includes precisely those processes that efficiently couple $\{t_L, b_L, t_R, h\}$:

1. charged–current W scatterings, which drive $\mu_{t_L} \simeq \mu_{b_L}$
2. QCD strong sphalerons, which rapidly equilibrate axial quark charge across flavours;
3. top helicity flips ("mass insertions"), which damp $\mu_{t_L} - \mu_{t_R}$ and feed the scalar channel
4. Higgs–number damping in the scalar sector
5. Yukawa exchange among (t_L, b_L, t_R, h)

These ingredients enter only through the components of $\delta\mathcal{C}$. Explicitly for the four-species set

$$
\begin{aligned}
\delta\mathcal{C}_1^{t_L} &\propto \widehat{\Gamma}_y^t\left(\mu_{t_L} - \mu_{t_R} + \mu_h\right) + 2\widehat{\Gamma}_m^t\left(\mu_{t_L} - \mu_{t_R}\right) + \widehat{\Gamma}_W\left(\mu_{t_L} - \mu_{b_L}\right) + \widetilde{\Gamma}_{ss}[\mu], \\
\delta\mathcal{C}_1^{b_L} &\propto \widehat{\Gamma}_y^t\left(\mu_{b_L} - \mu_{t_R} + \mu_h\right) + \widehat{\Gamma}_W\left(\mu_{b_L} - \mu_{t_L}\right) + \widetilde{\Gamma}_{ss}[\mu], \\
\delta\mathcal{C}_1^{t_R} &\propto -\widehat{\Gamma}_y^t\left(\mu_{t_L} + \mu_{b_L} - 2\mu_{t_R} + 2\mu_h\right) + 2\widehat{\Gamma}_m^t\left(\mu_{t_R} - \mu_{t_L}\right) - \widetilde{\Gamma}_{ss}[\mu], \\
\delta\mathcal{C}_1^{h} &\propto \frac{3}{2}\widehat{\Gamma}_y\left(\mu_{t_L} + \mu_{b_L} - 2\mu_{t_R} + 2\mu_h\right) + \widehat{\Gamma}_m^h\,\mu_h,
\end{aligned}
$$

Diffusion system as a boundary–value problem

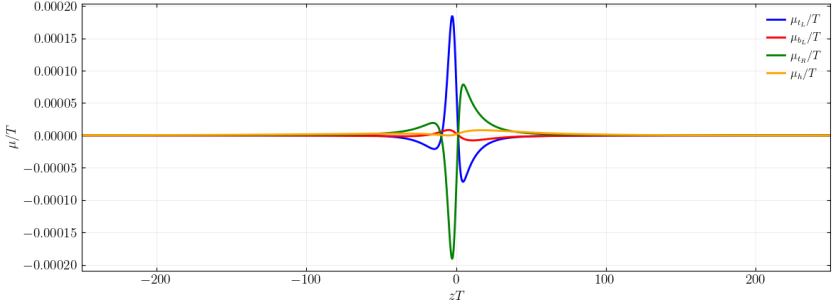
The Boltzmann system is a **boundary–value problem (BVP)**:

$$\mu_i(z),\, u_i(z) \xrightarrow[z\to\pm\infty]{} 0$$

i.e. perturbations vanish far in front of and behind the wall. Numerically we work on a finite domain

$$z \in [-\tilde{z}_{\max} L_w,\, +\tilde{z}_{\max} L_w]\,, \qquad \tilde{z}_{\max} \sim 10^4.$$

We **double the system** by introducing Lagrange multipliers $\lambda_{\mu_i}, \lambda_{u_i}$ that couple linearly to μ_i, u_i: this allows us to enforce simultaneously all BCs $\rightarrow$ `scipy.integrate.solve_bvp`



Baryon-Left chemical potential

EW sphalerons couple to the *left–handed* quark number carried by the $SU(2)_L$ doublets. The *baryon–left* chemical potential $\mu_{B_L}(z)$ that sources baryon number in the weak–sphaleron sector appear as

$$\frac{d\eta_B}{dz} \propto \Gamma_{ws}(z)\Big(\text{const}\times\mu_{B_L}(z) - A\,\eta_B(z)\Big), \qquad \rightarrow \qquad \mu_{B_L}(z) \equiv \frac{1}{2}\sum_{\text{all quarks } q}\mu_{q_L}(z)\,. \tag{11}$$

Using the light-quarks to be massless + absence of Yukawa mixing ($\mu_{q_R} = -\mu_{q_L}$)

Baryon-left chemical potential

$$\mu_{B_L}(z) = \frac{1}{2}\Big(1 + 4\,D_0^t\Big)\mu_{t_L}(z) + \frac{1}{2}\Big(1 + 4\,D_0^b\Big)\mu_{b_L}(z) + 2\,D_0^t\,\mu_{t_R}(z)\,.$$

BAU

In order to compute the BAU η_B we then need to solve

$$\frac{dn_B}{dt} = -\frac{n_f}{2}\Gamma_{ws}\big(N_c\,\mu_{B_L}\,T^2 - A\,n_B\big), \tag{12}$$

with $N_c = 3$ and in the SM $A = 15/2$. Passing to the wall frame and seeking a steady profile, $\partial_t \to -v_w\gamma_w\partial_z$, gives

$$\frac{dn_B}{dz} = \frac{n_f}{2v_w\gamma_w}\Gamma_{ws}(z)\Big(N_c\,\mu_{B_L}(z)\,T^2 - A\,n_B(z)\Big). \tag{13}$$

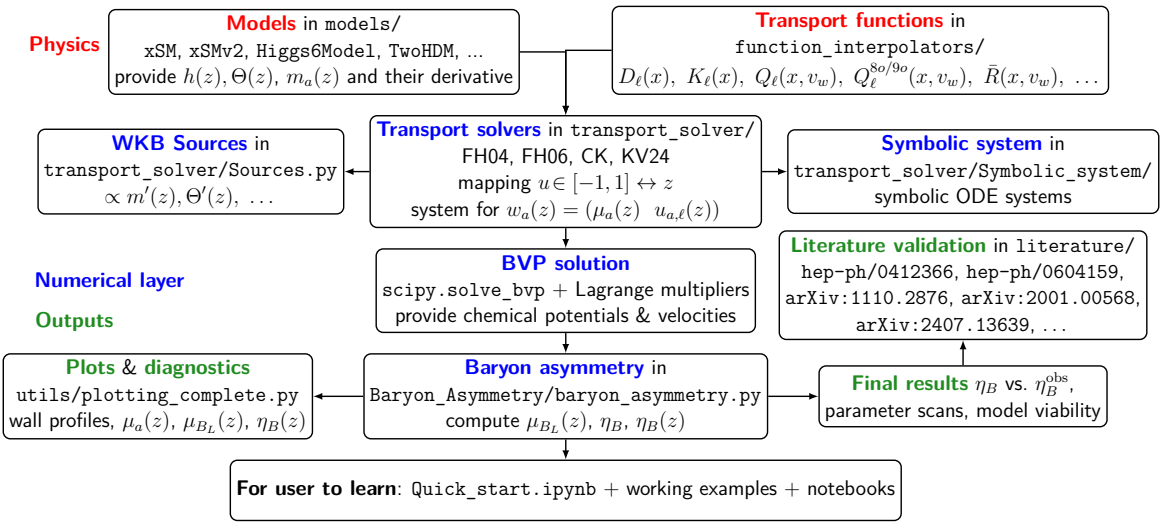
Assuming $n_B(+\infty) = 0$ then

Baryon asymmetry generated

$$\eta_B \equiv \frac{n_B(-\infty)}{s} = -\frac{45 n_f N_c}{4\pi^2 v_w\gamma_w g_* T}\int_{-\infty}^{+\infty} dz\,\Gamma_{ws}(z)\mu_{B_L}(z)\exp\left[-\frac{A n_f}{2v_w\gamma_w}\int_{-\infty}^{z}\Gamma_{ws}(u)du\right], \tag{14}$$

BARYONET
end-to-end workflow

BARYONET: end-to-end workflow (see 2510.21915)



Results in xSM

xSM - phenomenological application (see Cline and Kainulainen 2020)

We consider xSM + the dimension–five operator $i\,(s/\Lambda)\,\bar{Q}_3 H t_R$, which **induces a phase wherever the singlet s acquires a wall profile**. In the wall background the effective top mass term reads

$$y_t\, h(z)\, \bar{t}_L \left(1 + i\kappa_s \frac{s(z)}{\Lambda}\right) t_R + \text{h.c.}, \tag{15}$$

implying

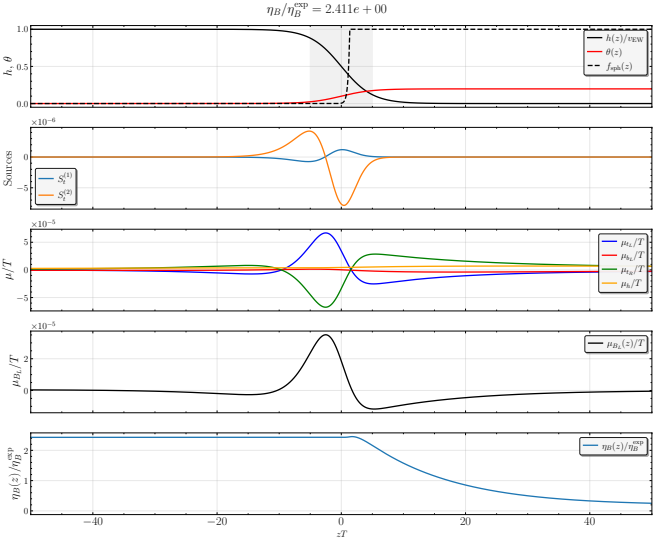
$$m_t(z) = y_t\, h(z) \sqrt{1 + \frac{s^2(z)}{\Lambda^2}}, \qquad \Theta_t(z) = \arctan\Big(\frac{s(z)}{\Lambda}\Big). \tag{16}$$

We model the background with opposite–kink profiles for Higgs and singlet

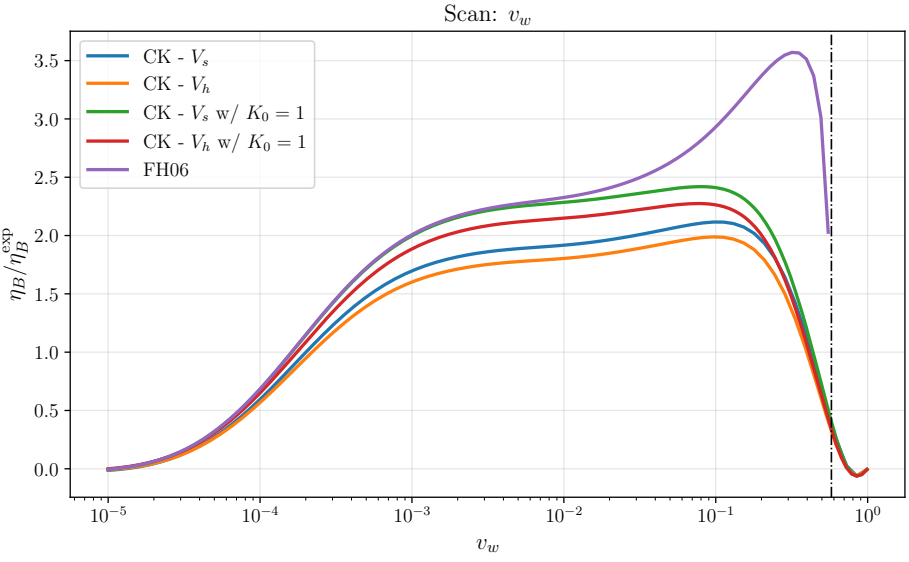
$$h(z) = \frac{v_n}{2}\Big(1 - \tanh\frac{z}{L_w}\Big), \qquad s(z) = \frac{w_n}{2}\Big(1 + \tanh\frac{z}{L_w}\Big), \tag{17}$$

xSM - complete analysis

BM1 : $v_w = 0.1, \quad L_w = 5, \quad \xi = v/T = 1.0, \quad y_t = 0.7, \quad y_b = \lambda_h = 0.0, \quad s_n/\Lambda = 0.2$



xSM - wall velocity v_w dependence (reproducing Cline and Kainulainen 2020)



At mean–field level, and **around the critical temperature**, the scalar potential can be written as

$$V(h,s,T) = \frac{\lambda_h}{4}\left[h^2 - v_c^2 + \frac{v_c^2}{w_c^2}s^2\right]^2 + \frac{\kappa}{4}\,s^2h^2 + \frac{1}{2}\big(T^2 - T_c^2\big)\big(c_h h^2 + c_s s^2\big), \tag{18}$$

with $\kappa \equiv \lambda_m - 2\lambda_h\, v_c^2/w_c^2$. For $\kappa > 0$ the **potential exhibits a tree–level barrier** and, at $T = T_c$, degenerate minima at $(h,s) = (v_c, 0)$ and $(0, \pm w_c)$. The thermal coefficients are

$$c_h = \frac{1}{48}\big(9g^2 + 3g'^2 + 12y_t^2 + \lambda_h\big(24 + 4\,v_c^2/w_c^2\big) + 2\kappa\big)\,, \tag{19}$$

$$c_s = \frac{1}{12}\big(\lambda_h\big(3(v_c^2/w_c^2)^2 + 4\,v_c^2/w_c^2\big) + 2\kappa\big)\,, \tag{20}$$

We can define

$$T_c^2 = \frac{\lambda_h}{c_h}\,(v_0^2 - v_c^2), \qquad m_h^2 = 2\lambda_h v_0^2, \qquad m_s^2 = \frac{1}{2}\kappa v_0^2 + \lambda_h\,(v_0^2 - v_c^2)\left(\frac{v_c^2}{w_c^2} - \frac{c_s}{c_h}\right). \tag{21}$$

A **thin–wall estimate** obtained by minimising the Euclidean action gives

$$\frac{dS_E}{dL_w} = 0 \qquad \Rightarrow \qquad L_w^2 \simeq \frac{2.7}{\kappa}\,\frac{v_c^2 + w_c^2}{v_c^2 w_c^2}. \tag{22}$$

We rewrite everything in terms of

$$\xi = \frac{v_c}{T_c}, \quad m_s, \quad \Lambda, \quad \Delta\Theta = \frac{w_c \Lambda}{\kappa_s}$$

so to have

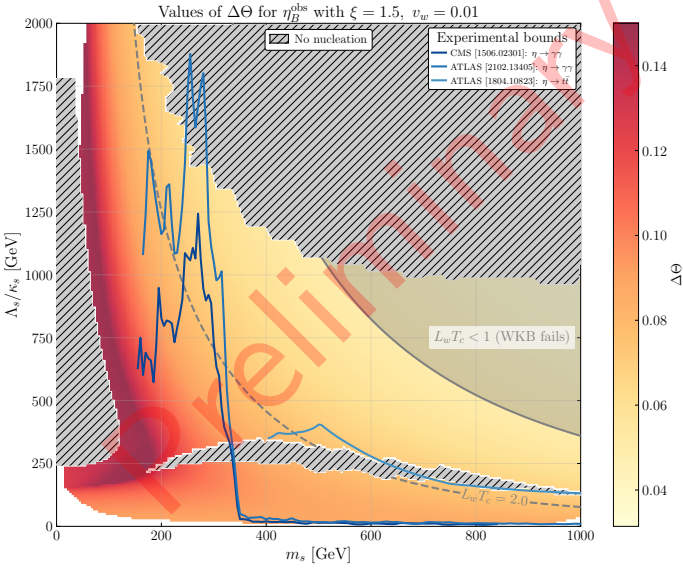
$$L_w T_c \equiv L_w T_c(\xi, \Delta\Theta, m_s, \Lambda)$$
$$S_E \equiv S_E(\xi, \Delta\Theta, m_s, \Lambda)$$

and from $S_E = 140$, we define

$$T_n/T_c \equiv T_n/T_c(\xi, \Delta\Theta, m_s, \Lambda)$$

Note

Each point in the plane with the chosen $\Delta\Theta$ reproduces the observed η_B.



preliminary

Conclusions and outlook

- We provided a **self-contained and unified derivation** of semiclassical EWBG, organising FH04, FH06, CK and KV24 into a single framework.
- We released `BARYONET`, an **end-to-end, reproducible pipeline** from wall profiles to the baryon asymmetry η_B, with symbolic equation generation and automated transport solvers.
- We performed **extensive validation** across the literature (xSM, 2HDM, Higgs–ϕ^6), establishing correctness, portability, and a consistent numerical standard.
- All key transport ingredients—**diffusion constants**, **relaxation/helicity-flip rates**, **sphaleron rates**—(not in this talk but ...) were clarified, updated, and implemented consistently in `BARYONET` (see backups).

Outlook

- **Extend physical reach:** incorporate **CP-violating flavour mixing**, **EDM constraints** and more **Collider bounds** to directly link EWBG scenarios with current and future experiments.
- **Beyond the fluid network**. Replace the moment-truncated equations with a direct solution of the full out-of-equilibrium distribution using spectral methods or kinetic-operator eigenmodes, like in CP-even approach for v_w. This eliminates closure ambiguities and yields systematically improvable accuracy.

Thanks for the attention!

Backups

Collider bounds on the NP scale Λ_s/κ_s

Assuming the pseudoscalar state s couples to gluons via an anomalous operator such that

$$\sigma(gg \to s) = \left(\frac{\kappa_s v_{\mathrm{EW}}}{\Lambda_s}\right)^2 \sigma_A(m_s),$$

and that the experimental bound σ_{exp} is given for $\sigma(gg \to s) \times \mathrm{BR}(s \to \gamma\gamma)$, we can write

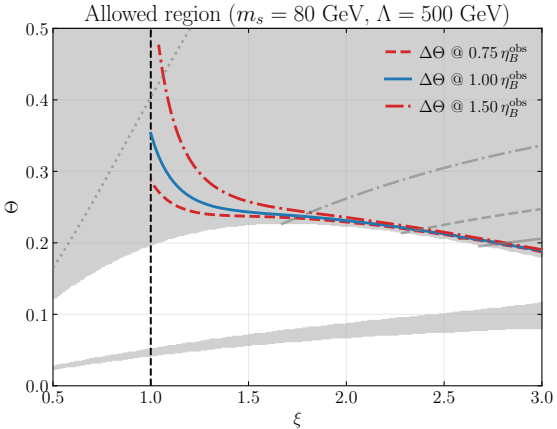
$$\left(\frac{\kappa_s v_{\mathrm{EW}}}{\Lambda_s}\right)^2 \sigma_A(m_s)\,\mathrm{BR}(s \to \gamma\gamma) \leq \sigma_{\mathrm{exp}}(m_s).$$

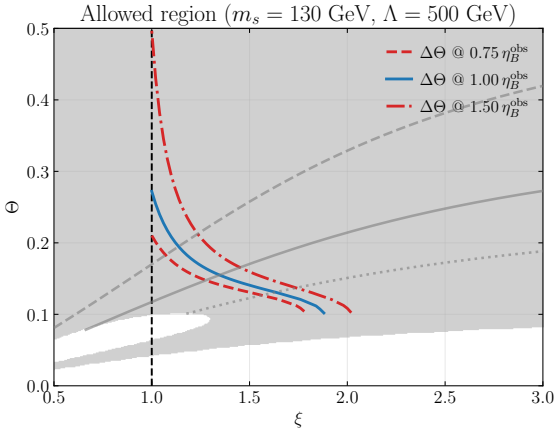
Inverting this for Λ_s/κ_s, we obtain

$$\frac{\Lambda_s}{\kappa_s} \geq v_{\mathrm{EW}} \sqrt{\frac{\sigma_A(m_s)\,\mathrm{BR}(s \to \gamma\gamma)}{\sigma_{\mathrm{exp}}(m_s)}}.$$

Where $\sigma_A(m_s)$ has been derived with `Sushi` at NNLO in QCD. Analogous for $s \to t\bar{t}$.

Reproducing arXiv:1110.2876 and arXiv:1302.6713





`Higgs6Model`: ϕ^6 Benchmark

Setup: Extend the SM Higgs potential with a stabilising dimension–six operator:

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{8L^2}\phi^6.$$

Here ϕ^6 term permits $\lambda < 0$ while keeping the potential bounded $\rightarrow$ **tree–level barrier** and strong FOEWPT.
Wall and CPV.

- Wall profile modelled as a kink: $\phi(z) = \frac{v_c}{2}(1 - \tanh(z/L_w))$. Walls typically thick: $3 \lesssim L_w T_c \lesssim 16$.
- CP violation via dimension–six top-Yukawa operator: $m_t(z) = |m_t(z)|e^{i\Theta_t(z)}$, with $\Delta\Theta_t$ controlling the CP–odd source.

Transport and computation.

- Use FH06 diffusion network with species $\{t, b, t_c, h\}$.
- Solve full z–dependent moment equations and convert $\mu_{B_L}(z) \rightarrow \eta_B$ via the weak–sphaleron equation.

Parameter viability.

- Require strong transition: $\xi \equiv v_c/T_c \gtrsim 1.1$ to avoid washout.
- Viable region: $5 \lesssim L/T_c \lesssim 8.5$ with $\lambda < 0$.

TwoHDM: CPV and Two–Field Dynamics

Setup: Two scalar doublets H_1, H_2 with soft Z_2 breaking: $\mu_3^2 e^{i\phi} H_1^\dagger H_2$. Effective potential includes tree, CW, counterterms, and thermal pieces:

$$V_{\rm eff}(h_1, h_2, \theta; T) = V_0 + V_1 + V_{\rm CT} + V_1^T.$$

EWPT and wall profile: At T_c: obtain broken-phase vevs (v_1, v_2) and CP phase $\theta_{\rm brk}$. Strength and direction:

$$\xi = \frac{v_c}{T_c}, \qquad v_c = \sqrt{2(v_1^2 + v_2^2)}, \qquad \tan\beta_T = \frac{v_2}{v_1}.$$

Kink wall along valley direction: $h(z) = \dfrac{v_c}{2}\left[1 - \tanh\left(\dfrac{z}{L_w}\right)\right], \qquad L_w \simeq \sqrt{\dfrac{v_c^2}{8V_b}}.$

Masses and CPV sources.

- Field projections: $h_1(z) = h(z)\cos\beta_T,\ h_2(z) = h(z)\sin\beta_T.$
- Top/bottom masses: $m_t(z) = y_t h_2(z)/\sqrt{2},\ m_b(z) = y_b h_1(z)/\sqrt{2}.$
- CP phase kink: $\theta(z) = \theta_{\rm brk} - \dfrac{\Delta\theta}{2}[1 + \tanh(z/L_w)].$
- Relevant source phase: $\Theta(z) = \theta_t(z) = \theta(z)/(1 + \tan^2\beta_T).$

Transport: Implement FH06 transport for 2HDM (`FH06for2HDM`). Reproduced known behaviour: stronger transitions (larger ξ) and moderate v_w enhance η_B; thicker walls suppress it.

Higgs6Model and TwoHDM comparison

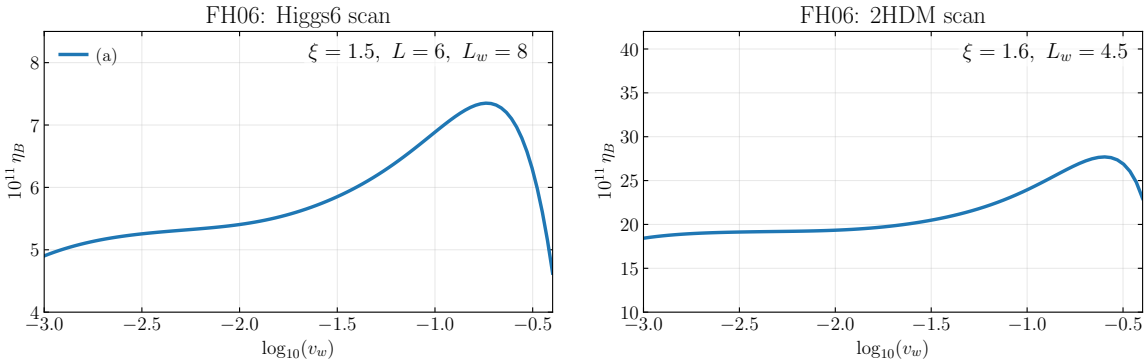


Figure: **FH06 scans.** Left—Higgs6Model and Right—TwoHDM

Interaction rates

Interaction rates			
Interaction name	**Rates used in the literature**[1]	**Previous update from** Cline and Laurent 2021	**Last update (this work)**
Top Yukawa rate	$\widehat{\Gamma}_y^t \simeq 4.2 \times 10^{-3} T$	$\widehat{\Gamma}_y^t \simeq 3 \times 10^{-2} T$	$\widehat{\Gamma}_y^t \simeq 10^{-2} T$
Top Helicity flipping	$\widehat{\Gamma}_m^t \simeq \dfrac{m_t^2}{63T}$	$\widehat{\Gamma}_m^t \simeq \dfrac{0.79 m_t^2}{T}$	$\widehat{\Gamma}_m^t \simeq \dfrac{0.26 m_t^2}{T}$
Higgs Yukawa rate	$\widehat{\Gamma}_m^h \simeq \dfrac{m_W^2}{50T}$	$\widehat{\Gamma}_m^h \simeq \dfrac{m_W^2}{25T}$	$\widehat{\Gamma}_m^h \simeq \dfrac{1.5 m_W^2}{T}$
W boson rate	$\widehat{\Gamma}_W \simeq \widehat{\Gamma}_{\rm tot}^h$	—"—	—"—
Strong sphaleron	$\Gamma_{ss} \simeq 4.9 \times 10^{-4}\, T$	$\Gamma_{ss} \simeq 8.7 \times 10^{-3}\, T$	$\Gamma_{ss} \simeq 2.7 \times 10^{-3}\, T$
Weak sphaleron	$\Gamma_{ws} \simeq 10^{-6} T$	$\Gamma_{ws} \simeq 6.3 \times 10^{-6} T$	$\Gamma_{ws} \simeq 8 \times 10^{-7} T$

[1] Bodeker et al. 2005; Fromme and Huber 2007; Konstandin 2013; Cline and Kainulainen 2020

Diffusion constants and numerical values

Diffusion constants			
Quarks	$D_q \simeq 6/T$	$D_q \simeq 7.4/T$	$D_{q_L} \simeq 7.1/T$, $D_{q_R} \simeq 7.6/T$
Higgs	$D_h \simeq 20/T$	—"—	$D_h \simeq 14/T$
Leptons	✗[1]	$D_{l_R} \simeq 490/T$, $D_{l_L} \simeq 90/T$	$D_{l_R} \simeq 412/T$, $D_{l_L} \simeq 100/T$
Couplings			
Strong int.	$\alpha_s \simeq 1/7$	$\alpha_s(m_z) \simeq 0.1179$	
Weak int.	$\alpha_w \simeq 1/30$	$\alpha_w(m_z) \simeq 0.0338$	
Weinberg angle	$\tan\theta_W \simeq 0.29$	$\tan\theta_W \simeq 0.302$	

[1] Joyce, Prokopec, and Turok 1996a; Joyce, Prokopec, and Turok 1996b

Higher–moment analysis: Formulation

Moment hierarchy.

- For each species, assemble $w(z) = \big(\mu,\, u_1, \dots, u_{\ell-1}\big)^{\top}$ and stack the ℓ CP–odd equations.
- System retains the structure $A\,w' + B\,w = \mathcal{S} + \delta\mathcal{C}$, with $A, B \in \mathbb{R}^{\ell\times\ell}$.

$$A \;=\; \begin{pmatrix} -D_1 & 1 & 0 & \cdots & 0 \\ -D_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -D_{\ell-1} & 0 & \cdots & 0 & 1 \\ -D_\ell & R_1 & \cdots & R_{\ell-2} & R_{\ell-1} \end{pmatrix}, \qquad B = (\partial_z m^2) \begin{pmatrix} Q_1\, v_w\gamma_w & 0 & 0 & \cdots & 0 \\ Q_2\, v_w\gamma_w & \bar{R} & 0 & \cdots & 0 \\ Q_3\, v_w\gamma_w & 0 & 2\,\bar{R} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Q_\ell\, v_w\gamma_w & 0 & 0 & \cdots & (\ell-1)\,\bar{R} \end{pmatrix},$$

Inversion. Closed analytic expression for A^{-1} (triangular + rank-1 piece) allows rewriting the system as

$$w' = A^{-1}(\mathcal{S} + \delta\mathcal{C} - Bw)\,.$$

Higher–moment analysis: Convergence and stability

Convergence study.

- Solve hierarchy up to $N_{\mathrm{PERT}} = 30$ using: **(i)** constant–R closure and **(ii)** variance truncation.
- Increasing N_{PERT} suppresses $\mu_{B_L}(z)$ and stabilises η_B.
- Variance closure converges earlier and with smaller spread.

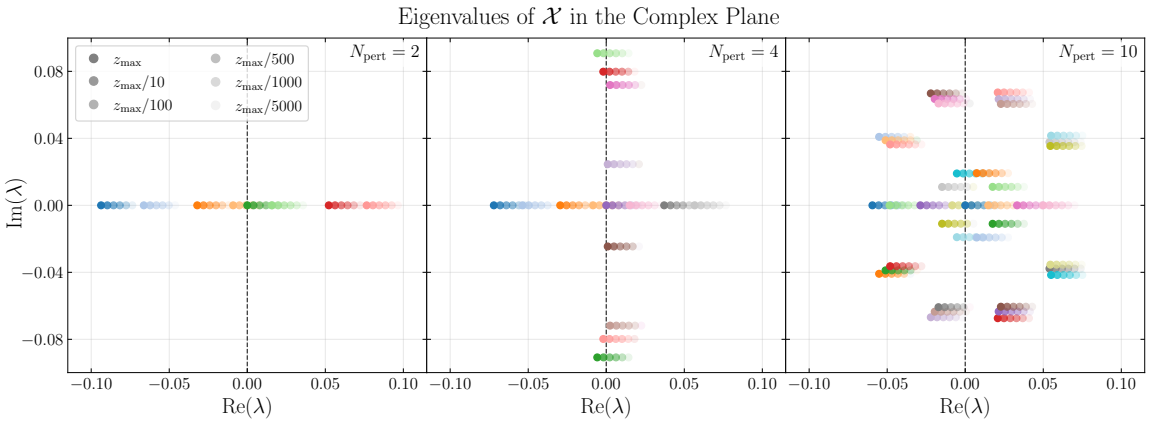
Stable truncation sequence.

- Analyse far–from–wall limit $w' = \mathcal{X}w$; eigenvalues of $\mathcal{X}$ determine decay.
- Stable sequence: $N_{\mathrm{PERT}} = 2 + 4k$ (eigenvalues remain well inside left half–plane).
- Other sequences show small $|\operatorname{Re}\lambda|$ and oscillatory/unstable modes.

Numerical issues.

- Lagrange multipliers double the dynamical dimension and add slow modes.
- At large N_{PERT}: amplified stiffness and near–singular blocks.
- Robust fix: eliminate multipliers analytically or impose BCs without enlarging the state (planned for `BARYONETv2`).

Higher–moment analysis: eigenvalues



Higher–moment analysis: analysis

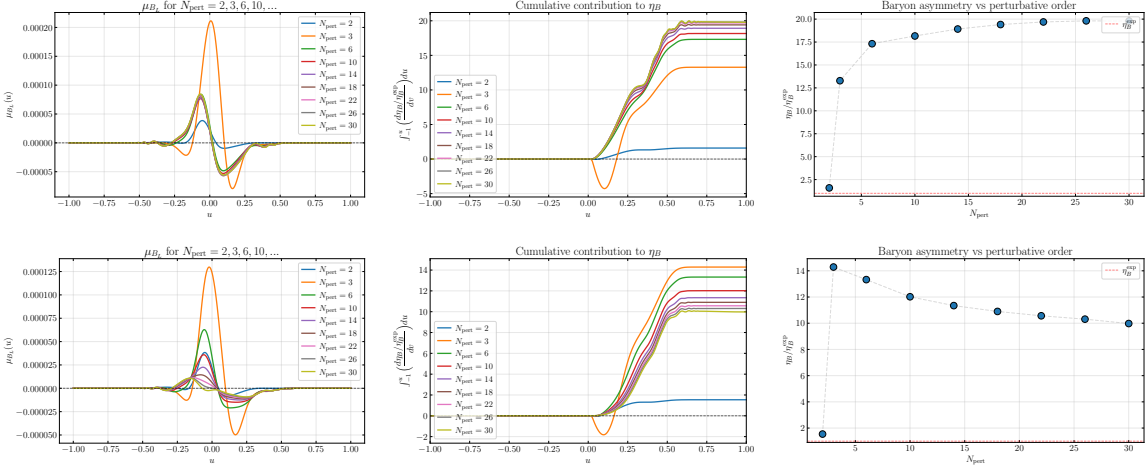


Figure: *Top row* (constant–R closure): (left) $\mu_{B_L}(u)$ for $N_{\rm PERT} = 2, 6, \ldots, 30$; (middle) cumulative BAU $\int_{-1}^{u} (d\eta_B/\eta_B^{\rm exp})\, dv$; (right) $\eta_B/\eta_B^{\rm exp}$ versus $N_{\rm PERT}$. *Bottom row* (variance truncation).

Appendix: Universal kernel

Key idea: All transport coefficients appearing in the WKB fluid equations (diffusion constants, relaxation kernels, CP–odd sources) can be written as **universal, Lorentz-invariant thermal integrals** depending only on the local ratio $x = m/T$ and the wall velocity v_w.

Define $w = E/T$, $\tilde{p}_w = \sqrt{w^2 - x^2}$, $y = \cos\theta$. Boost to the wall frame: $\tilde{E} = \gamma_w(w - v_w y \tilde{p}_w)$, $\tilde{p}_z = \gamma_w(y\tilde{p}_w - v_w w)$. Every coefficient takes the generic form

$$\langle p_z^n / E^m \, V \, \mathcal{F}_{0w}^{(k)} \rangle = T^{n-m-k-1} K(\mathcal{F}_0^{(k)}; V; n, m)$$

where K is a **universal 2D integral** over (w, y). The weight V selects the structure: $V = 1$ (scalar), $V_s = |\tilde{p}_z|/E_{0z}$ (spin), V_h (helicity).

$$D_\ell(x, v_w) = K\big(f'_{0w};\, 1;\, \ell, \ell\big)\,, \qquad T^2 Q_\ell(x, v_w) = \frac{1}{2} K\big(f''_{0w};\, 1;\, \ell - 1, \ell\big)\,,$$

$$T^2 Q_\ell^e(x, v_w) = \frac{1}{2} K\big(f'_{0w};\, 1;\, \ell - 1, \ell\big)\,, \qquad T^2 Q_\ell^{8o}(x, v_w) = \frac{1}{2}\, K\big(f'_{0w};\, V;\, \ell - 2, \ell\big)\,,$$

$$T^4 Q_\ell^{9o}(x, v_w) = \frac{1}{4}\left[K\big(f'_{0w};\, V;\, \ell - 2, \ell + 2\big) - \gamma_w\, K\big(f''_{0w};\, V;\, \ell - 2, \ell + 1\big)\right].$$

These kernels are **model independent**: they depend only on (x, v_w). They are **precomputed once**, tabulated, and interpolated in `BARYONET` for all solvers and all models.

Appendix: Linearised collision operators (I)

- Consider a generic process $i_1\, i_2 \to f_1\, f_2$ and track the evolution of species i_1.
- Collision operator:

$$C[f_j] = -\frac{1}{2E_{i_1}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(\sum p)\, \mathcal{P}[f_{i_{1,2}}, f_{f_{1,2}}]\,,$$

  with the statistical factor

$$\mathcal{P} = f_{i_1} f_{i_2} (1 \mp f_{f_1})(1 \mp f_{f_2}) - f_{f_1} f_{f_2} (1 \mp f_{i_1})(1 \mp f_{i_2})\,.$$

- Split distributions in the wall frame:

$$f_i = f_i(\mu_i) + \delta f_i\,,$$

  where $f_i(\mu_i)$ is (boosted) equilibrium with small chemical potential and δf_i is the perturbation.

Linearisation and MB limit: i) Expand $\mathcal{P}$ to first order in μ_i and δf_i. ii) Use the identity $1 \mp f = f\, \exp[\gamma_w (E + v_w p_z)/T]$ and energy–momentum conservation. iii) In the **Maxwell–Boltzmann limit** (and dropping Boltzmann-suppressed pieces), the linearised collision term simplifies and is controlled by the combinations

$$\frac{\delta f_{i_1}}{f^0_{i_1}} + \frac{\delta f_{i_2}}{f^0_{i_2}} - \frac{\delta f_{f_1}}{f^0_{f_1}} - \frac{\delta f_{f_2}}{f^0_{f_2}}\,,$$

implementing a consistent truncation of the collision operator.

Appendix: Linearised collision operators (II)

Out-of-equilibrium part δf: The linearised collision operator can be organised as

$$C[f] \;\simeq\; -\Gamma(p_{i_1})\,\delta f_{i_1} \;+\; \Delta\mathcal{F}\,,$$

where:

- $\Gamma(p_{i_1})$ is the **thermal scattering rate** for $i_1 i_2 \to f_1 f_2$,
- $\Delta\mathcal{F}$ encodes the perturbations of the other species.

Chemical potential part μ_i: For $f_i(\mu_i) \simeq f_i^0[1 + (1 \mp f_i^0)\xi_i]$, with $\xi_i = \mu_i/T$, the statistical factor gives

$$\mathcal{P}[\mu] \;\simeq\; f_{i_1}^0 f_{i_2}^0\,(\xi_{i_1} + \xi_{i_2} - \xi_{f_1} - \xi_{f_2})\,,$$

the familiar detailed-balance combination. This leads to number-changing terms $\propto \sum_{i,j} \widehat{\Gamma}_{i\to j}\, s_{ij}\, \xi_j$ in the zeroth and first moments.

Thermal averaging and moments: Replace $\Gamma(p)$ by per-particle rates $\widehat{\Gamma}$ and take velocity moments:

$$\langle \delta C \rangle \simeq -K^{(0)}\kappa^X \sum_{i,j} \widehat{\Gamma}_{i\to j} s_{ij}\,\xi_j, \qquad \left\langle \frac{p_z}{E_0}\delta C \right\rangle \simeq -K^{(1)}\kappa^X \sum_{i,j} \widehat{\Gamma}_{i\to j} s_{ij}\,\xi_j - \kappa^X \widehat{\Gamma}_{\rm tot}\, u_1\,.$$

Physically: **(i)** inelastic processes relax specific combinations of chemical potentials, **(ii)** elastic + inelastic scatterings *damp the fluid velocity* perturbation u_1.

Appendix: Diffusion constants (I)

Pedagogical rederivation of **diffusion constants** for relativistic particles in a hot plasma, starting from the Boltzmann equation and clarifying the role of the **transport rate** Γ_t.

Definition of D: Number current decomposed as $\mathbf{J} = n\,\mathbf{u} + \mathbf{J}_d, \qquad \mathbf{J}_d = -D\,\boldsymbol{\nabla} n.$

- Start from Boltzmann equation

$$\frac{p^\mu}{p^0}\partial_\mu f(x,p) = -C[f](x,p),$$

with a slowly varying chemical potential $\xi(x) = \mu(x)/T$.

- Linearise around equilibrium

$$f = f_0 + \delta f, \qquad f_0 = \frac{1}{e^{\beta p^0 + \xi(x)} + 1},$$

and use the anisotropic ansatz

$$\delta f_{\mathbf{p}} = g(p)\,\frac{p_x}{p^0}.$$

Transport rate: Inserting the ansatz into the linearised Boltzmann equation gives

$$f_0'(p^0)\,\partial_x \xi \;\simeq\; -g(p)\,\Gamma_t(p),$$

which **defines** the momentum–dependent transport rate $\Gamma_t(p)$, i.e. the relaxation rate of the anisotropic part of the distribution.

Appendix: Diffusion constants (II)

Master formula for D: For relativistic fermions, $n \simeq n_0 - \dfrac{T^3}{12}\,\xi, \qquad J_z = \displaystyle\int \frac{d^3p}{(2\pi)^3}\,\delta f\,\frac{p_z}{p^0}$. Combining these yields

$$D = \frac{12}{T^3}\int \frac{d^3p}{(2\pi)^3}\,\frac{-f_0'(p^0)}{\Gamma_t(p)}\left(\frac{p_z}{p^0}\right)^2,$$

i.e. **diffusion = thermal average of** $1/\Gamma_t$.

Transport rate from $2\to 2$ scatterings:For a generic $2\to 2$ process, linearising and projecting onto the transport harmonic gives

$$\Gamma_t(p) = \frac{1}{2p_0^3}\int_{\{p',k,k'\}} (2\pi)^4\delta^{(4)}\Big(\sum p\Big)\,|\mathcal{M}|^2\,(-f_0')\,(p\cdot p').$$

Dominant contribution: t-**channel exchange of gauge bosons** with $|\mathcal{M}|^2 \propto (t-M^2)^{-2}$; IR divergences at $t\to 0$ are regulated by **Debye masses** $M_{B,W,g}^2$. In the CM frame and for small-angle scattering this yields a **leading-log** structure

$$\Gamma_t \;\propto\; g^4\,\frac{T^3}{p^2}\,\ln\!\Big(\frac{\text{hard scale}}{M^2}\Big) \quad\Rightarrow\quad D^{-1} = \frac{5}{448\pi^3}\,A\,g^4\,T\,\ln\!\Big(\frac{2x_2x_4T^2}{M^2}\Big),$$

where A encodes colour/hypercharge factors and M^2 is the Debye mass of the exchanged gauge boson.

Appendix: Diffusion constants (III)

Gauge channels and species-dependent D: For SM fermions, three gauge channels contribute $D_B^{-1}, \quad D_W^{-1}, \quad D_G^{-1}$ from $U(1)_Y$, $SU(2)_L$ and $SU(3)_c$. Each has the LL form $D_X^{-1} \sim \alpha_X^2\, T\, \ln\!\left(\frac{T^2}{M_X^2}\right)$, with different coefficients for B, W, g. The total inverse diffusion constant for a given species is a **sum over channels**, weighted by its quantum numbers (hypercharge, weak isospin, colour). Examples:

$$D_{\ell_R}^{-1} = D_B^{-1}, \quad D_{\ell_L}^{-1} = D_W^{-1} + Y_{\ell_L}^2 D_B^{-1},$$

$$D_{q_{L,R}}^{-1} = Y_{q_{L,R}}^2 D_B^{-1} + \epsilon_{L,R} D_W^{-1} + D_G^{-1}.$$

Higgs diffusion constant D_h: Same strategy as for fermions, but with **Bose–Einstein** statistics and **scalar** matrix elements:

$$-f_B'(E) \quad \text{and} \quad |\mathcal{M}|^2 \propto (s-u)^2/(t-M^2)^2.$$

For SM Higgs self-scattering, the LL estimate is

$$D_h \;\sim\; \mathcal{O}\!\left(\frac{10}{T}\right),$$

with additional scatterings on light fermions and gauge bosons further reducing D_h.

Appendix: Yukawa–mediated relaxation rates (I)

Goal Derive from first principles the **per-particle** relaxation rates $\widehat{\Gamma}_y$ induced by Yukawa interactions in a hot plasma, keeping all phase space, group factors and IR regulators explicit.

Definition: Per-particle rate (our conventions):

$$\widehat{\Gamma} = \frac{12}{T^3} \int_{\{p,p',k,k'\}} f_p \tilde{f}_k (1 + \tilde{f}_{p'})(1 - f_{k'})\, \delta^{(4)}(p + k - p' - k')\, |\mathcal{M}|^2,$$

where $f, \tilde{f}$ are fermion/boson distributions.

Relevant $2 \to 2$ processes: Chirality/flavour relaxation via Yukawa y_ψ is dominated by near-forward

$$\psi + X \;\longleftrightarrow\; \psi' + \phi,$$

with $X \in \{B, W, g\}$ a bath gauge boson, ϕ the Higgs, and ψ, ψ' chiral fermions. After spin/colour sums, for small-angle t-channel exchange:

$$|\mathcal{M}_X|^2 = \frac{A_X\, y_\psi^2\, s\,(-t)}{(t - M^2)^2}, \qquad A_X = c_X (4\pi\alpha_X),$$

with channel-dependent coefficients $c_B = 4Y_\psi^2$, $\quad c_W = 6$, $\quad c_g = \dfrac{32}{3}$. M^2 is an **asymptotic (thermal) gauge-boson mass** acting as IR regulator.

Appendix: Yukawa–mediated relaxation rates (II)

Leading-log structure: Angular integration over the near-forward region gives the same kernel $\mathcal{I}(a)$ as in the diffusion-constant derivation:

$$\mathcal{I}(a) = \int_0^2 dy \frac{y}{(y+a)^2} \simeq \ln\left(\frac{2}{a}\right) + \mathcal{O}(1), \quad a = \frac{M^2}{4pk}.$$

The rate simplifies to

$$\widehat{\Gamma} = \frac{12}{T^3} \int_{\{p,k\}} f_p \tilde{f}_k \, \mathcal{J}, \quad \mathcal{J} = \frac{A_X y_\psi^2}{8\pi} \mathcal{I}\left(\frac{M^2}{4pk}\right).$$

Thermal averaging: Work in the LL regime $\lambda \equiv 4T^2/M^2 \gg 1$, expand $\ln(1+\lambda xy) \simeq \ln(\lambda xy)$. Use fermionic/bosonic moments and "slow-log" approximation:

$$\int dx\, x^m f(x) \ln\frac{1}{x} \approx \ln\frac{1}{x_{m+1}} \int dx\, x^m f(x),$$

with suitable effective points x_{m+1}. After performing the integrals explicitly one finds, schematically,

$$\widehat{\Gamma} \;=\; \frac{\zeta(3)\, A_X y_\psi^2}{64\pi^3}\, T \left[\ln\left(\frac{4x_2 T^2}{M^2}\right) + \text{subleading constants} \right],$$

where x_2 encodes the thermal averaging and the subleading constants are numerically small compared to the LL piece.

Appendix: Yukawa–mediated relaxation rates (III)

HTL thermal masses in the log: The IR scale entering the log is the **asymptotic fermion thermal mass**

$$m_\psi^2 = \frac{T^2}{8}\Big[\sum_a g_a^2 C_2^{(a)}(\psi) + g'^2 Y_\psi^2\Big],$$

with $a \in \{SU(3)_c, SU(2)_L\}$ and hypercharge Y_ψ. For SM chiral multiplets this gives $M_{q_L}, M_{u_R}, M_{d_R}, M_{\ell_L}, M_{\ell_R} \sim \mathcal{O}(0.1\text{–}0.6)\,T$, leading to

$$\mathsf{LL}_i = \ln\Big(\frac{4x_2 T^2}{M_i^2}\Big) \sim 3\text{–}6.$$

Final leading-log rates & colour normalisation:Plugging everything in, one obtains

$$\widehat{\Gamma}_{\ell_R} = 2\,\widehat{\Gamma}_{\ell_L} \;\simeq\; 0.32\,\alpha_w\, y_\ell^2\, T, \qquad \widehat{\Gamma}_{q_L} \simeq 0.27\,\alpha_s\, y_{q_L}^2\, T, \qquad \widehat{\Gamma}_{q_R} \simeq 0.40\,\alpha_s\, y_{q_R}^2\, T,$$

up to the (explicit) LL factors and mild scheme dependence. These are **colour-summed** rates. The colour-averaged rate entering the diffusion network is

$$\widehat{\Gamma}_y = \frac{1}{N_c}\,\widehat{\Gamma}, \qquad N_c = 3.$$

Appendix: Yukawa helicity–flipping rate $\widehat{\Gamma}_M$ (fermions)

Goal Compute the **mixing (helicity-flip) rate** $\widehat{\Gamma}_M$ that damps L–R coherence in EW transport, using the CTP (Schwinger–Keldysh) formalism with HTL quasiparticles.

Setup: Chiral doublet in a slowly varying Higgs background:

$$\mathcal{L} \supset y_f\, \phi_b(z)\, \bar{\psi}_L \psi_R + \text{h.c.}$$

Off–diagonal correlator S_{LR} obeys a Kadanoff–Baym equation with collision kernel built from $\mathrm{Im}\,\Sigma^R$. Wigner transform and gradient expansion around the wall (VEV–insertion): quasiparticle poles

$$\varepsilon_i(k) = \omega_i(k) - i\Gamma_i, \quad \omega_i = \sqrt{k^2 + m_{i,\text{th}}^2}, \quad i = L, R.$$

Thermal weight for fluctuations: $h(x) = \dfrac{e^x}{(1+e^x)^2} = -\dfrac{\partial n_F}{\partial x}$.

Master structure: KB commutator and contour integration produce denominators $\varepsilon_R^* - \varepsilon_L$ (retarded–advanced) and $\varepsilon_L + \varepsilon_R$ (particle–hole). After angular integration and using susceptibilities ($n_f = \chi_f \mu_f$ with $\chi_f = T^2/6$), the mixing rate can be written as

$$\widehat{\Gamma}_M = C(\mu, \gamma)\, \frac{m_f^2}{T},$$

with $m_f^2 = y_f^2 \phi_0^2$, $\mu = m_{\text{th}}/T$, $\gamma = \Gamma/T$, and $C(\mu,\gamma)$ a dimensionless momentum integral depending only on HTL masses and widths.

Appendix: Fermionic $\widehat{\Gamma}_M$ – widths and numerical values

Input: fermion damping widths: Single–particle widths from $\mathrm{Im}\,\Sigma^R$ of HTL quasiparticles:

$$\Gamma_f \;\simeq \sum_{i\in\{SU(3),SU(2),U(1)_Y\}} a_i\,\frac{g_i^2\,C_i(R_f)}{4\pi}\,T \;+\; \mathcal{O}(y_f^2 T),$$

with $C_i(R_f)$ the Casimir (Y_f^2 for $U(1)_Y$) and $a_i = \mathcal{O}(1)$.

Typical SM values (EW scale)

$$\Gamma_{\text{quark}} \simeq 0.16\,T, \qquad \Gamma_{\ell_L} \simeq 0.028\,T, \qquad \Gamma_{\ell_R} \simeq 0.01\,T,$$

entering the complex poles $\varepsilon = \omega - i\Gamma$ and thus $C(\mu,\gamma)$.

Numerical coefficients $C(\mu,\gamma)$: Evaluating the master integral with HTL masses and widths:

$$C_t \simeq 0.78 \Rightarrow \quad \widehat{\Gamma}_M^t \simeq 0.78\,\frac{m_t^2}{T}, \qquad C_{\tau_L} \simeq 0.21 \Rightarrow \quad \widehat{\Gamma}_M^{\tau_L} \simeq 0.21\,\frac{m_\tau^2}{T}, \qquad C_{\tau_R} \simeq 0.22 \Rightarrow \quad \widehat{\Gamma}_M^{\tau_R} \simeq 0.22\,\frac{m_\tau^2}{T}.$$

Transport equations use **colour–averaged** rates:

$$\widehat{\Gamma}_m^t = \frac{1}{N_c}\widehat{\Gamma}_M^t \;\simeq\; 0.26\,\frac{m_t^2}{T}, \qquad N_c = 3.$$

Appendix: Bosonic Higgs rate $\widehat{\Gamma}_M^{(h)}$ and comparison

Higgs (bosonic) helicity–flip rate: Higgs coherence is damped by thermal W bosons in the plasma. CTP analysis of the Higgs self–energy with a W loop leads to

$$\widehat{\Gamma}_M^{(h)} = C_h(\mu_T, \gamma_W)\, \frac{m_W^2}{T},$$

with $\mu_T = m_{\infty,W}/T$, $\gamma_W = \Gamma_W/T$, and $m_W^2(z) = g^2\phi(z)^2/4$.
Bosonic thermal weight:

$$h_B(x) = \frac{e^x}{(e^x-1)^2} = x\, n_B(1+n_B)/x.$$

Narrow–width approximation: For hard transverse W bosons: $\gamma_W \simeq g^2 C_A/(4\pi) \approx 0.07 \ll 1$. In NWA, the integral reduces to an on–shell bosonic kernel and yields

$$C_h^{\mathrm{NWA}} \simeq 1.5 \quad \Rightarrow \quad \boxed{\widehat{\Gamma}_M^{(h)} \simeq 1.5\, \frac{m_W^2}{T}}.$$

Appendix: Strong sphalerons Γ_{ss} – conventions

Two conventions

- **Diffusion (per unit volume):** $\Gamma_{ss}^{\rm diff} \;=\; \dfrac{d}{dt}\dfrac{\langle\big(N_{\rm CS}(t)-N_{\rm CS}(0)\big)^2\rangle}{V} \;=\; \kappa_{ss}\,\alpha_s^5\,T^4$, with $\kappa_{ss}=\mathcal{O}(10^2)$ from hard/soft and nonperturbative physics.
- **Transport (per particle):**

$$\Gamma_{ss} \;=\; c_{ss}\,\alpha_s^4\,T,$$

obtained from $\Gamma_{ss}^{\rm diff}$ after dividing by susceptibilities and colour/flavour factors.

Key message

- $\Gamma_{ss}^{\rm diff}\sim\alpha_s^5T^4$ (lattice / real-time QCD).
- $\Gamma_{ss}\sim\alpha_s^4T$ is the **effective relaxation rate** used in transport equations; mapping between the two depends on N_c, N_f and on the definition of axial charge.

Strong sphalerons Γ_{ss} – numerical benchmark

Benchmark at the electroweak scale

- Using the transport form

$$\Gamma_{ss} \simeq 14\,\alpha_s^4\,T$$

with $\alpha_s(m_Z) \simeq 0.118$ gives

$$\boxed{\Gamma_{ss} \;\simeq\; 2.7\times 10^{-3}\,T.}$$

Nonperturbative input & extrapolation

- Lattice QCD (with $N_f = 2{+}1$) measures $\Gamma_{ss}^{\rm diff}/T^4$ at $T \sim$ few T_c, giving values $\sim 10^{-1}$–10^{-2}.
- Extrapolating with $\Gamma_{ss}^{\rm diff} \propto \alpha_s^5 T^4$ up to $T \sim 100$ GeV is consistent with the above $\mathcal{O}(10^{-3})T$ transport rate.

Takeaway

- Different numbers in the literature mainly reflect:
  - different **conventions** (diffusion vs. transport);
  - choice of α_s and renormalisation scale;
  - how axial charge and flavours are mapped into the effective Γ_{ss}.

Appendix: Weak sphalerons Γ_{ws} – symmetric phase

Diffusion vs. transport

- Symmetric-phase **diffusion rate** (per unit volume):

$$\Gamma_{ws}^{\rm diff} = \kappa_{ws}\,\alpha_w^5\,T^4.$$

- In transport equations one uses the corresponding **relaxation rate** $\Gamma_{ws} \propto \alpha_w^5 T$ after dividing by susceptibilities.

Historical and modern estimates

- Early parametric forms: $\Gamma_{ws} \sim 6\,\kappa\,\alpha_w^4 T^4$ (Huet–Nelson), later refined to $\propto \alpha_w^5 T^4$.
- Many EWBG works use fixed numbers (e.g. $\Gamma_{ws} \sim 10^{-6}T$, $6.3\times10^{-6}T$) with different conventions.
- State-of-the-art nonperturbative result (D'Onofrio, Rummukainen, Tranberg) in the symmetric phase:

$$\boxed{\Gamma_{ws} = (18 \pm 3)\,\alpha_w^5\,T \;\simeq\; 8\times10^{-7}\,T}$$

for $\alpha_w \simeq 0.0338$.

Weak sphalerons Γ_{ws} – broken phase & 3D EFT

3D EFT description (broken phase)

- Dimensionally reduced $SU(2)$ gauge–Higgs theory with parameters

$$x = \frac{\lambda_3}{g_3^2}, \qquad y = \frac{m_3^2}{g_3^4}.$$

- Broken-phase rate:

$$\Gamma_{ws}^{\rm br}(T) \propto T^4 \exp\Big[-\hat{S}_{3D}(x,y)\Big], \qquad \hat{S}_{3D} = C_{\rm sph}\, v_3(x,y),$$

with $C_{\rm sph} \simeq 29$ and v_3 the 3D Higgs condensate.

Regimes (Li et al. in `arXiv:2506.01585`)

- **A: unbroken / weakly broken** v_3 small, $\Gamma_{ws}^{\rm br} \sim \Gamma_{ws}^{\rm sym}$: full washout.
- **B: partially broken** Intermediate suppression, baryon violation continues but reduced.
- **C: strongly broken** Large v_3, sphalerons *freeze out* behind the wall, asymmetry preserved.

SM-like strong transition (Scenario C)

- For $v_3 \simeq 1.2-1.3$,

$$\hat{S}_{3D} = C_{\rm sph} v_3 \simeq 29 \times (1.2-1.3) \approx 35-38,$$

in line with the classic 4D estimate $E_{\rm sph}/T \simeq 37\, h/T$.

- $\Rightarrow$ **exponentially strong** suppression of $\Gamma_{ws}^{\rm br}$ and efficient preservation of the baryon asymmetry.

Appendix: Numerical implementation of the fluid BVP

Boundary value problem: Fluid network for $\mu_i(z)$, $u_i(z)$ is a BVP with

$$\mu_i(\pm\infty) = 0, \qquad u_i(\pm\infty) = 0.$$

Numerically: integrate on a finite interval

$$z \in [-\tilde{z}_{\max} L_w,\ +\tilde{z}_{\max} L_w], \qquad \tilde{z}_{\max} \sim 10^4,$$

so that asymptotic regions are effectively reached.

Lagrange-multiplier implementation: Introduce one **constant Lagrange multiplier per constraint** (asymptotic BCs). Augment the first–order system with source terms and algebraic equations:

$$\partial_{\tilde{z}} \tilde{\mu}_i = \cdots + \lambda_{\mu_i}(\tilde{z} + \tilde{z}_{\max}), \qquad \partial_{\tilde{z}} u_i = \cdots + \lambda_{u_i}(\tilde{z} - \tilde{z}_{\max}),$$

$$\partial_{\tilde{z}} \lambda_{\mu_i} = 0, \qquad \partial_{\tilde{z}} \lambda_{u_i} = 0,$$

so that λ's are constant along z and enforce the chosen BCs. The opposite signs at $\pm\tilde{z}_{\max}$ are chosen for **numerical stability** (push the solution toward zero at both ends).

Result: By doubling the system with constant Lagrange multipliers, the BVP is converted into a form where `solve_bvp` can impose *both* boundary conditions at the two ends of the interval exactly, yielding stable and simultaneous solutions for all $\mu_i(z)$ and $u_i(z)$ across the wall.