Universität Zürich[UZH]

Scattering Amplitudes for 3-loop Higgs+Jet Production

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LHC Higgs WG
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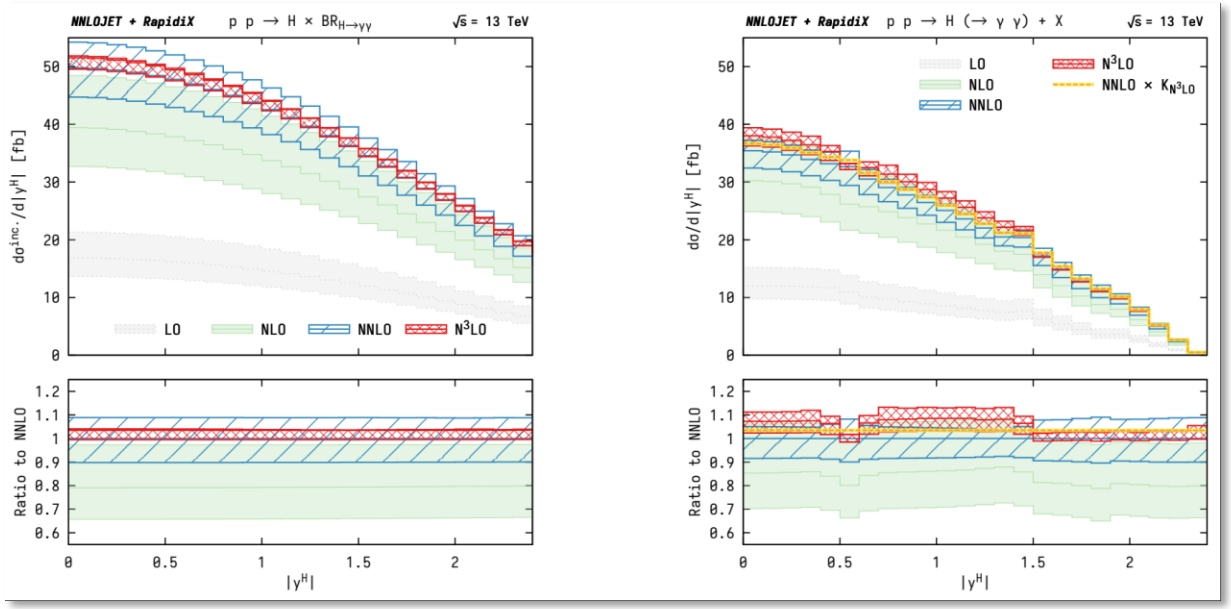
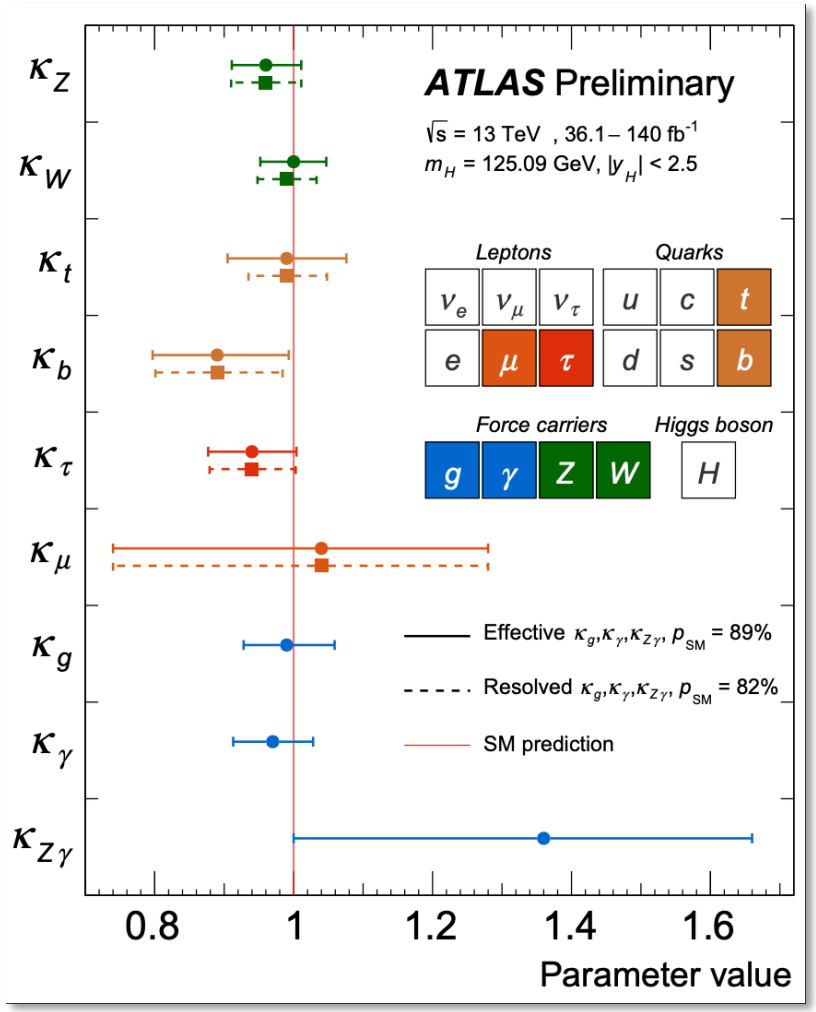
In collaboration with *Xin Guan* and *Bernhard Mistlberger*
Based on: [arXiv: **2504.06490**]

Higgs Boson

- Standard Model → New Physics
- High precision comparison between experimental results and theoretical predictions;
  - A few percent level;
- Differential observables contain richer information than only inclusive rates.



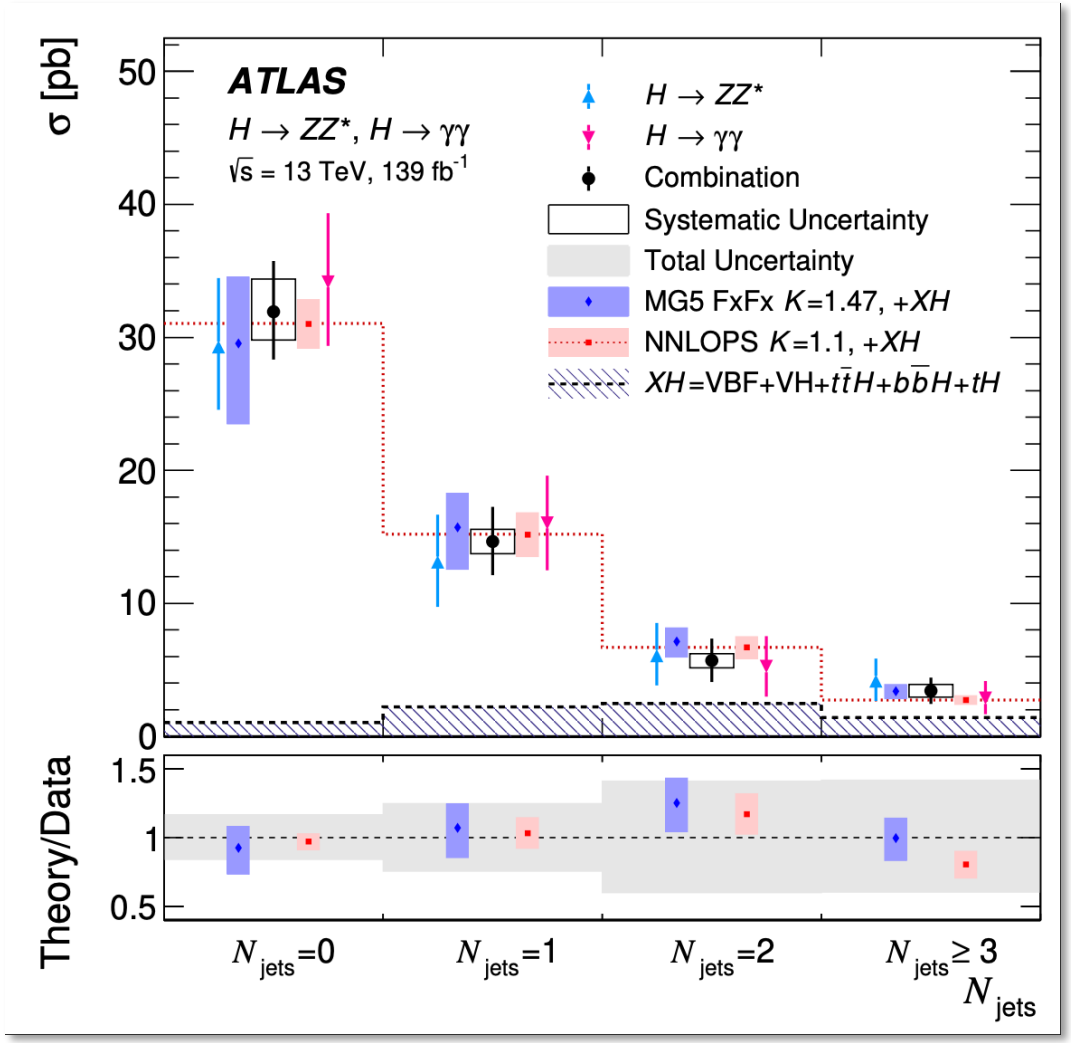
Comparison between inclusive (left) and fiducial (right) predictions for the rapidity distribution of the Higgs boson up to N3LO. X. Chen, T. Gehrmann, et al. *(2021)*



From https://cds.cern.ch/record/2937634/files/ATLAS-CONF-2025-006.pdf

Why $H + J$?

- **Additional final state jet**: requirement for any observables dependent on the Higgs boson having **non-vanishing transverse momentum**, broaden observables: $p_T^H, p_T^{j_1}, y^{j_1}$ …
- **High p_T^H**: Probe possible **new particles** that may enter into the top-quark loop or any other **new physics**;
- Other aspects: number of jets, discriminate between different Higgs production modes…



Differential $pp \to H + X$ cross-sections as a function of number of jets.
ATLAS. *(2022)*

$H + J$ status

➢ **Experiment**

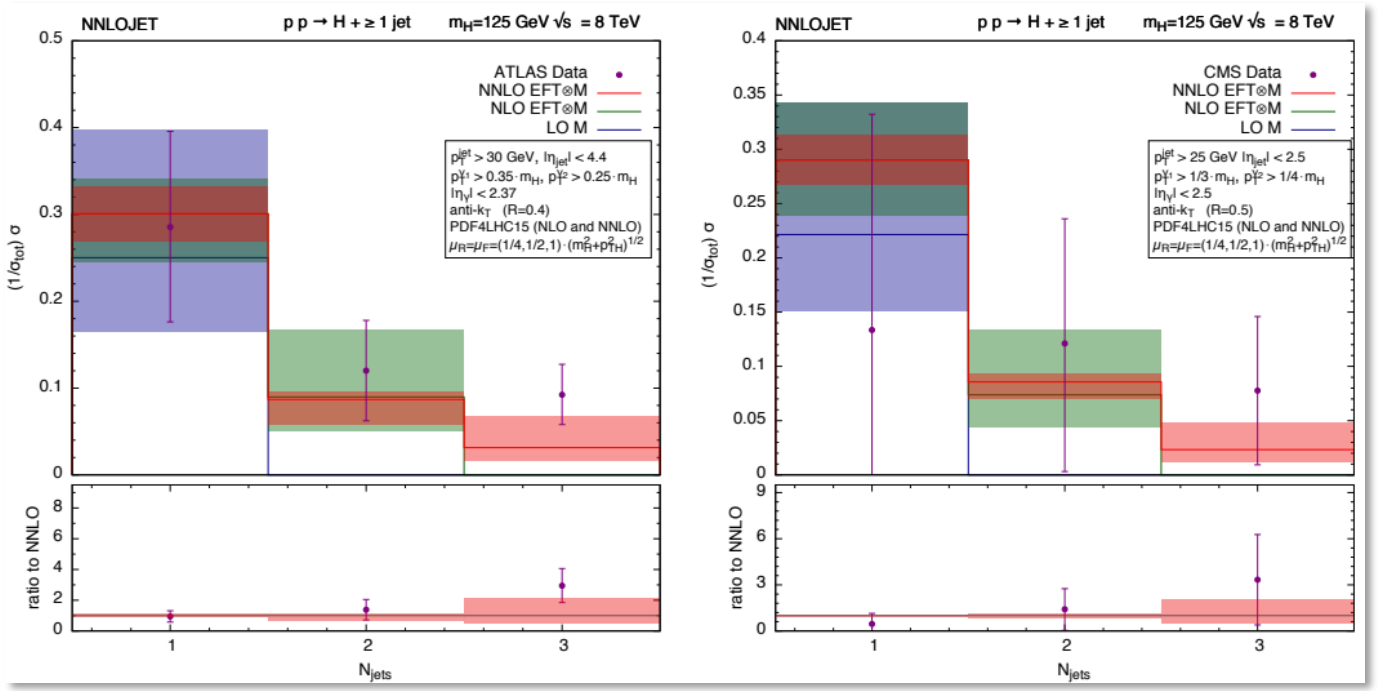
- Current precision: $10 - 15\%$ uncertainty, dominated by statistical errors; ATLAS. *(2020)* ATLAS. *(2022)*
- Expected precision at HL-LHC: 9.5%, with a sample of $3000 fb^{-1}$ of data, if the remaining systematic errors remain the same;

➢ **Theory**

- QCD: $NNLO_{HTL}$, $\sim 5\%$ uncertainty;

X. Chen, J. Cruz-Martinez, et al. *(2016)*

N^3LO_{HTL} calculation is still crucial!

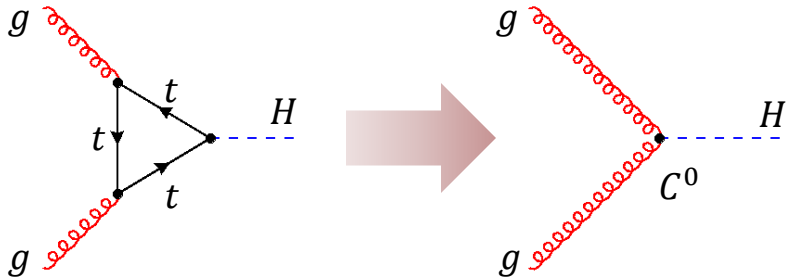


Jet multiplicity in $H + J$ production, normalized to the total fiducial cross section.

X. Chen, J. Cruz-Martinez, et al. *(2016)*

Theoretical efforts towards N^3LO_{HTL} calculation

➢ **Heavy-Top-Limit (HTL):** $m_t \gg \sqrt{s_{12}}/2, \quad \mathcal{L}_{eff} = \mathcal{L}_{SM,5} - \frac{C^0}{4} H G_a^{\mu\nu} G_{a,\mu\nu}.$

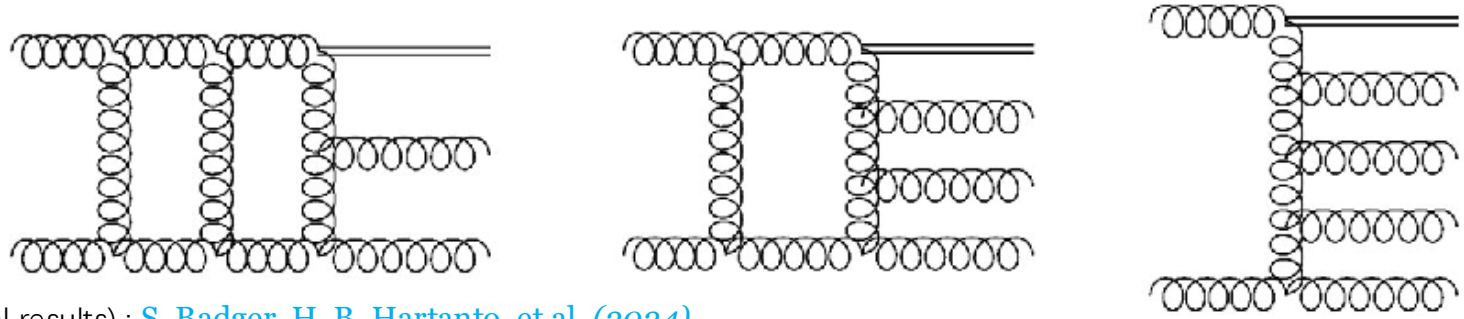


➢ **Virtual correction:**

- Three-loop (**this talk**);
- Lower-loop at high-order in the ϵ expansion;

S. D. Badger, E. W. N. Glover. *(2006)*
S. Badger, E. W. N. Glover, et al. *(2009)*
L. J. Dixon, Y. Sofianatos. *(2009)*
T. Gehrmann, M. Jaquier, et al. *(2011)*
T. Gehrmann, P. Jakubčík, et al. *(2023)*

➢ **Real emissions:**



$Hb\bar{b}gg$ (partial results) : S. Badger, H. B. Hartanto, et al. *(2024)*

➢ **Subtraction scheme, event generator…**

Work-flow

➢ Work-flow for high-order calculation:

Generating Integrands
($\mathcal{O}$(few minutes to hours))

$$\mathcal{A} = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i$$

Feynman diagrams

Integrals Reduction
($\mathcal{O}$(hours to days))

$$I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} M_j$$

Integration-By-Parts(IBP) method

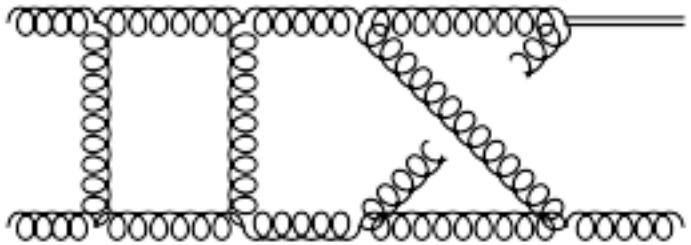
Evaluating master integrals
($\mathcal{O}$(days) typically in frontier)

$$I_i(p_i \cdot p_j, m_i^2) = \int \frac{d^D l_1 \cdots d^D l_h}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}}$$

Differential equations

Generating amplitudes

- Generating amplitudes: QGRAF: P. Nogueira. *(1993)*
  - All possible processes: $gg \to Hg, qg \to Hq, q\bar{q} \to Hg$.
  - Kinematic variables: $t = (k_1 - k_3)^2, u = (k_2 - k_3)^2, m_H^2 (= 1)$.
  - **Generalized leading color limit**: treat N_c and n_f as large, neglect $1/N_c^2$ suppressed terms:
    - ✓ 3 gluons: $\{N_c^6, N_c^5 n_f, N_c^4 n_f^2, N_c^3 n_f^3\}$;
    - ✓ 1 gluon and 2 (anti-)quarks: $\{N_c^5, N_c^4 n_f, N_c^3 n_f^2, N_c^2 n_f^3\}$;
- Number of the squared amplitudes: $\sim 10^5$;
- Number of families (after using external symmetries): 46;
- Number of integrals: $\sim 8 \times 10^5$;



Example diagram for the leading color, three-loop scattering amplitude of three gluons and a Higgs boson.

Integral Reduction

➢ **Integration-By-Parts (IBP):** K. G. Chetyrkin, F. V. Tkachov. *(1981)*

$$\int \prod_{i=1}^{L} \frac{d^D l_i}{(2\pi)^D} \frac{\partial}{\partial l_j^\mu} \left(r^\mu \prod_{\alpha=1}^{N} \frac{1}{\mathcal{D}_\alpha^{\nu_\alpha}} \right) = 0 \quad \xrightarrow{\text{change the indices}} \quad \text{linear equations among Feynman integrals}$$

- **Intermediate Expression Swell**: the rational functions with very high powers of kinematic variables and the dimension D appear during calculation, even though the final expressions are simpler!

➢ **Finite fields:** A. V. Manteuffel, R. M. Schabinger. *(2014)*
T. Peraro. *(2019)*

- For fixed values of variables, rational functions can be mapped to modular numbers by a large prime ($\sim 2^{64}$) in a finite field;
- **Solve the linear system multiple times** for different values of variables and different primes, then **reconstruct** the analytic coefficients;
- **Advantages:**
  - ✓ Efficiency of numerical computation;
  - ✓ Preservation of analytic structure;

Challenges for solving IBP

➢ **Bottleneck 1: memory usage**

- **Trimming**: select linearly independent relations from original IBP system, which is the most **memory-consuming** process;
- For frontier problem: can be $\geq \boldsymbol{O(TB)}$ and even more, which is unfeasible for most clusters;

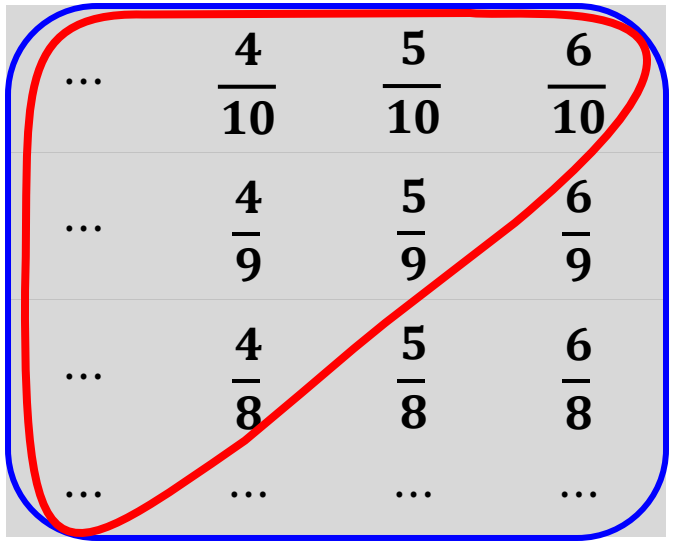
➢ **Bottleneck 2: time-consumption**

- $H + J$: the most time-consuming part is from **sampling**;
- Example: **one of the most complicated families**:
  - ✓ Need $\sim 6 \times 10^6$ sampling points (using Laporta basis as master integrals);
  - ✓ $t_{IBP} = 3600s$ (**have used improved seeding**), solving IBP system numerically for each sampling point in the finite field;
  - ✓ Estimated total time-consumption is $\boldsymbol{\sim 6 \times 10^6 cpu \cdot h}$ $(\approx 100\ kernels \times 2500\ days)$.

Unaffordable!

Technique I for reduction: Improved seeding

- Seeding: change indices to generate IBP relations;
- **Traditional seeding:**
  - Seeding integrals with uniform rank across all sectors: the blue rectangle;
- **Improved seeding:** Blade: X. Guan, X. Liu, et al. *(2024)*; M. Driesse, G. U. Jakobsen, et al. *(2024)*
  - A smaller set of seed integrals is sufficient: the red triangle;
    - ✓ More seeds for high-level sectors;
    - ✓ Fewer seeds for low-level sectors;
- Performance (tested on **rank-3** target integrals):
  - **Memory usage reduced by $6\times\sim 10\times$,** feasible for many clusters;
  - Fewer equations: reduced by $\sim 20\times$;
  - **Time-consumption for solving IBP system**: $\sim 5\times$;

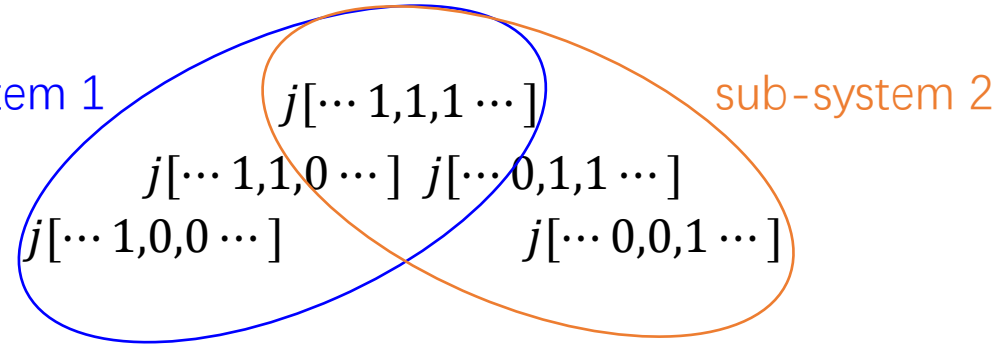


$\frac{r}{n}$: n is the number of propagators, r is the rank (total power in the numerator).

Technique II for reduction: Spanning sector

➢ **Further improvement: spanning sector**

- Divide the problem into smaller subsystems;



- Solve spanning sectors separately and merge results;

sub-system 1: $I = c_1 j_1 + c_2 j_2$
sub-system 2: $I = c_2 j_2 + c_3 j_3$
} merge: $I = c_1 j_1 + c_2 j_2 + c_3 j_3$

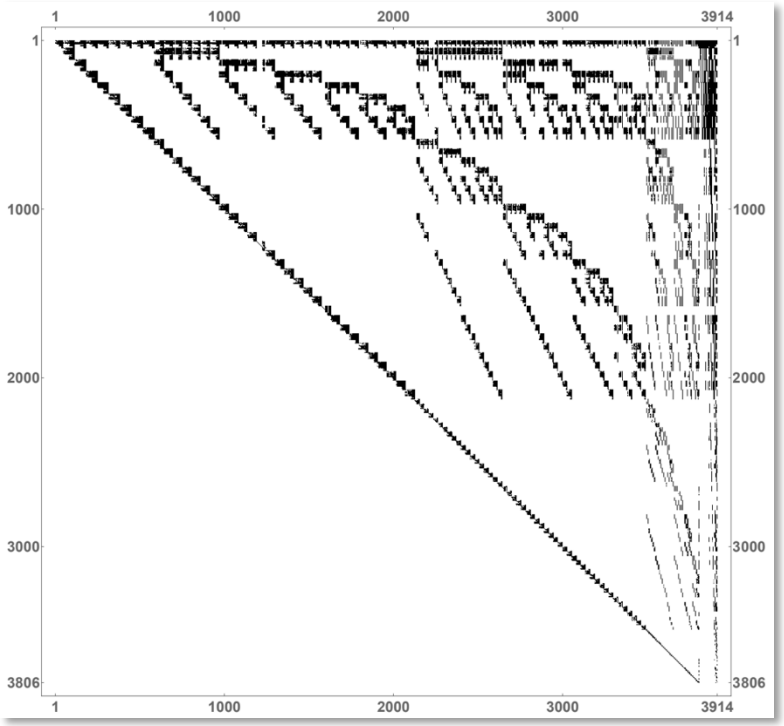
- Application: **rank-6 integrals in the most complicated family**: 29 spannings, 64 (non-zero) sectors in the largest one, out of 396 (non-zero) total sectors;

	Without spanning sector	With spanning sector
Memory usage	$270GB$	$76GB$
t_{IBP}	$3680s$	$1845s$(sum of 29 spannings)

Memory-feasible!

Technique III for reduction: Block-Triangular form

- How to reduce the solving time?
  - reduce the size of linear system (traditionally, IBP);
- **Block-triangular form:** X. Guan, X. Liu, et al. *(2019)*
  - Idea:
    - ✓ Make an ansatz for the (generalized) target integrals;
    - ✓ Use IBP as input to solve for the coefficients for the ansatz;
  - **Advantages:**
    - ✓ Includes only target integrals;
    - ✓ Orders of magnitude fewer equations;
    - ✓ Block-by-block structure, which makes solving faster;



Example figure from 1912.09294, each row represents a relation and each column represents an integral.

- **Performance in the most complicated spanning:**

#Eqs in improved IBP	#Eqs in block-triangular form	t_{IBP}	t_{BL}	$N_{fit\ btform}$	$N_{reconstruct}$
900292	12376	$166s$	$0.9s$	624	241781

184 × faster!

Summary of reduction

➢ Application of several techniques reduced memory usage and time-consumption significantly

- In the finite field: **for the most complicated family**

Original IBP: $\sim 3TB$? Ram,$\sim 18000?\ s/pt$.

Improved seeding
+spanning sector

$100GB$ Ram, $1800s/pt$.

Block-triangular form

$\leq O(GB)$ Ram, $18.6s/pt$.

- **Total resources consumption for all families: $5 \times 10^5 cpu \cdot h$;**
- **If using traditional method: $\sim 10^9 cpu \cdot h$** with **$\sim 3TB$ Ram**, which is impossible.

Master integrals

➢ **Canonical differential equations** J. M. Henn. *(2013)*

$$d\vec{I} = \epsilon\left(d\tilde{A}\right)\vec{I}, \qquad \tilde{A} = \sum_k A_k \ln \alpha_k$$

- **Second reduction**: reduce Laporta basis to canonical basis S. D. Vita, P. Mastrolia, et al. *(2014)* D. D. Canko, N. Syrrakos. *(2021)* J. M. Henn, J. Lim, et al. *(2023)* D. D. Canko, N. Syrrakos. *(2023)* T. Gehrmann, J. Henn, et al. *(2024)*
  - ✓ Confirm the differential equations against literatures;
- Boundary conditions: agree with existing results; J. Henn, B. Mistlberger, et al. *(2020)*
- 20 symbol letters, two of which involve square roots:

Validation of results

➢ **Cancellation of all the poles (UV&IR)**

S. Catani. *(1998)*
O. Almelid, C. Duhr, et al. *(2015)*
T. Becher, M. Neubert. *(2019)*

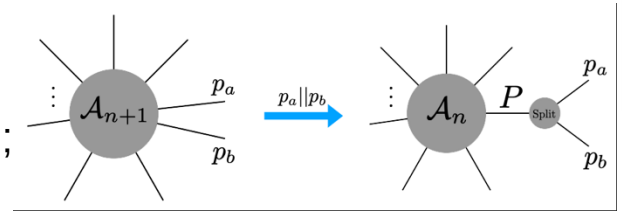
- Finite remainder

$$\mathcal{A}_X^F = Z(\alpha_s(\mu^2), \{p_i\})\mathcal{A}_X(\alpha_s, C^0)$$
$$= Z(\alpha_s(\mu^2), \{p_i\})\mathcal{A}_X\left(Z_{\alpha_s} N_\epsilon \alpha_s^R, Z_{C^0} C^{0,R}\right)$$

$$\mathcal{R}^{(n)} = \sum_{i=0}^{n} \left(\frac{\alpha_s}{\pi}\right)^i Re\left(\frac{\mathcal{M}_{H\to ggg}^{F,(i)}}{\mathcal{M}_{H\to ggg}^{F,(0)}}\right), \quad \mathcal{M}_X^{F,(i)} = \sum_{spin,color} \mathcal{A}_X^{F,(i)} \mathcal{A}_X^{\dagger F,(0)}$$

➢ **Two-parton collinear limit:**

- Universal splitting amplitudes;



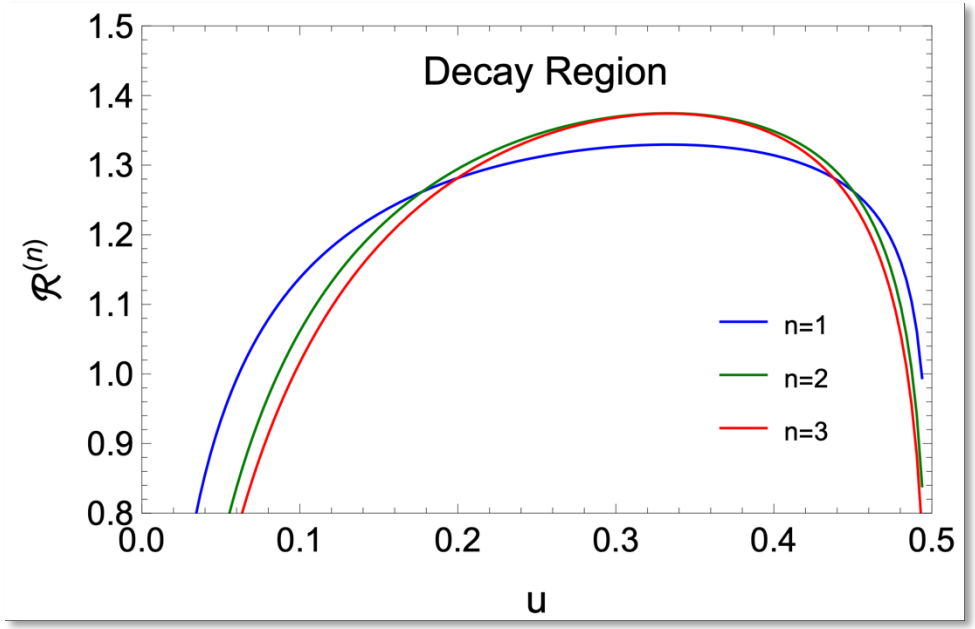
X. Guan, F. Herzog, et al. *(2024)*

➢ **One-parton soft limit:**

- Universal soft current;

F. Herzog, Y. Ma, et al. *(2023)*
W. Chen, M.-X. Luo, et al. *(2023)*

➢ **Numerical evaluation via AMFlow;** AMFlow: X. Liu, Y.-Q. Ma. *(2022)*



Example: different orders in the decay region for the line $q^2 = 1$ and $t = u$ as a function of u.

Summary and outlook

➢ **Summary**

- We compute the **three-loop QCD corrections** to the scattering amplitude of **a Higgs boson and three partons** interfered with its tree level counterpart in the **generalized leading color limit**;
- Combining several cutting-edge **integral reduction techniques** enables us to achieve this results.

➢ **Outlook**

- Sub-leading color contribution;
- Other ingredients to $H+J$ production at N^3LO_{HTL}.

Thanks for your attention!