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Some thoughts on ISR at e^+e^- colliders

Gearing up for future colliders, CERN, 30/6/2026

For the majority of tasks, the standard strategy at e^+e^- colliders relied on semi-analytical, "parton"-level, high-accuracy, inclusive, process-specific results (*evaluations*)

To connect these with data, the usage of numerical, "particle"-level, exclusive programs (*simulations*), either process-specific or not, is unavoidable: effects not accounted for by evalutations must be deconvoluted

- In the infinite-precision limit, this strategy cannot work: non-factorizable effects are always relevant. Thus, it may be extremely problematic also in a realistic large-accuracy scenario
- Regardless of that, a result is as good as its least good component

  (excellent evaluations are worthless if employed with poor simulations)

Note that often deconvolutions are seen as an inevitable fact of life:

> factor. In particular the perturbative calculations of the trivial QED effects, which have to be removed from the experimental data, are considered to be a major challenge for almost all quantities to be measured at FCC-ee. The task of this paper

Abstract of 1903.09895, Jadach&Skrzypek

A sensible strategic plan for FCCee should assume this *not to be* the case

To my untrained eye, EWPOs appear as an absolutist's goal

A ghost from the past

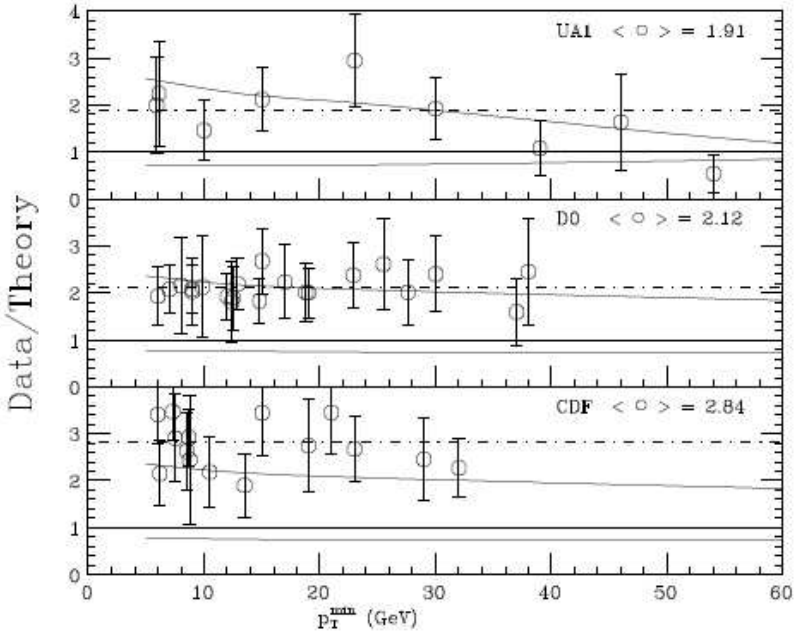
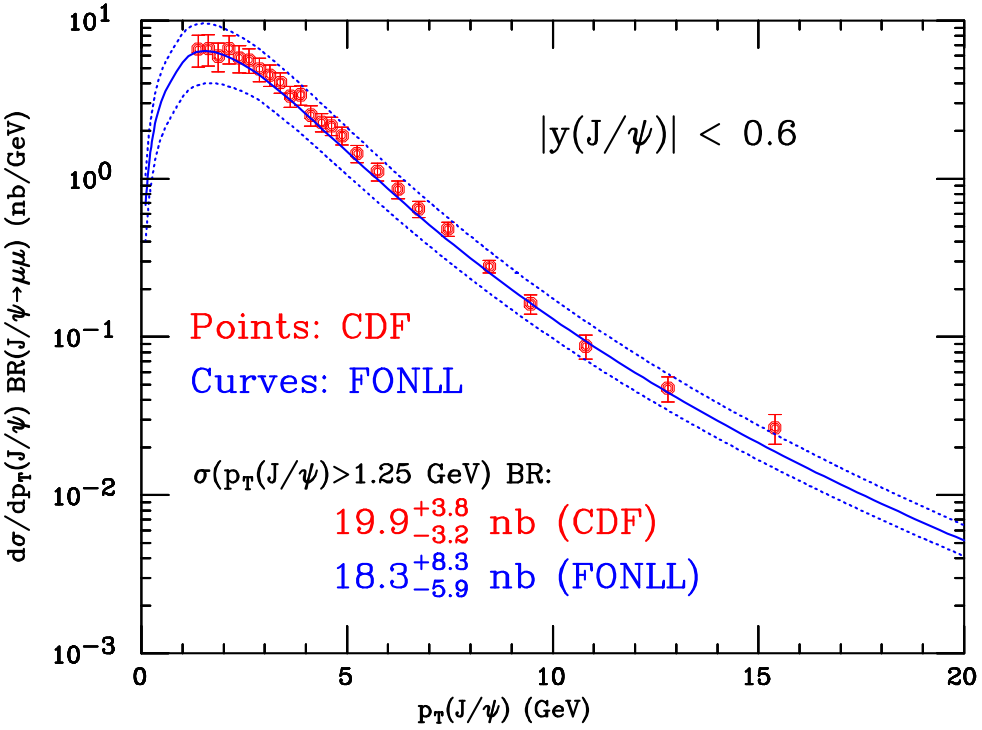


Figure 23: *Linear comparison between experimental data and theory for the integrated b-quark p_T distribution. See the text for details.*



- Left: b-quark *data*, as extracted by experiments
- Right: B-meson-driven cross section within acceptance cuts, simulated by theorists

The b-quark crisis was an artefact

My conclusion, informed in part by our collective experience at hadron colliders, is to:

Try and embed into simulations as much parton-level perturbative information as possible

This will limit the amount of deconvolutions (if any). Evaluations may or may not be relevant as such (the underpinning computations will be)

Note that deconvoluted data are unlikely to be useful after a few years,
which is not the case for data within acceptance

A very relevant (counter??)example

A cross section with beamstrahlung:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl} \int dy_+ dy_- \, \mathcal{B}_{kl}(y_+, y_-) \, d\bar{\sigma}_{kl}(y_+ P_{e^+}, y_- P_{e^-}) \,,$$

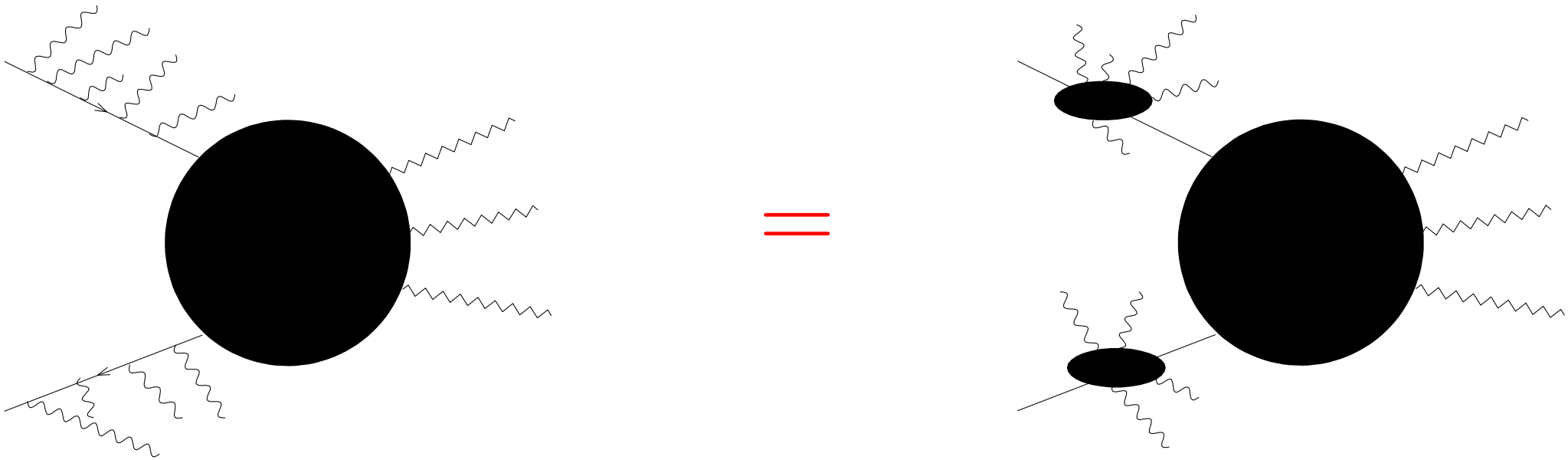
$$k \in \{e^+, \gamma\}, \qquad l \in \{e^-, \gamma\},$$

where, if you like collinear factorisation:

$$d\bar{\sigma}_{kl}(p_k, p_l) = \sum_{ij} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2, m^2) \,.$$

$\mathcal{B}_{kl}$ is the *beamstrahlung* function, which parametrises how particles k and l emerge from the beams to undergo hard collisions

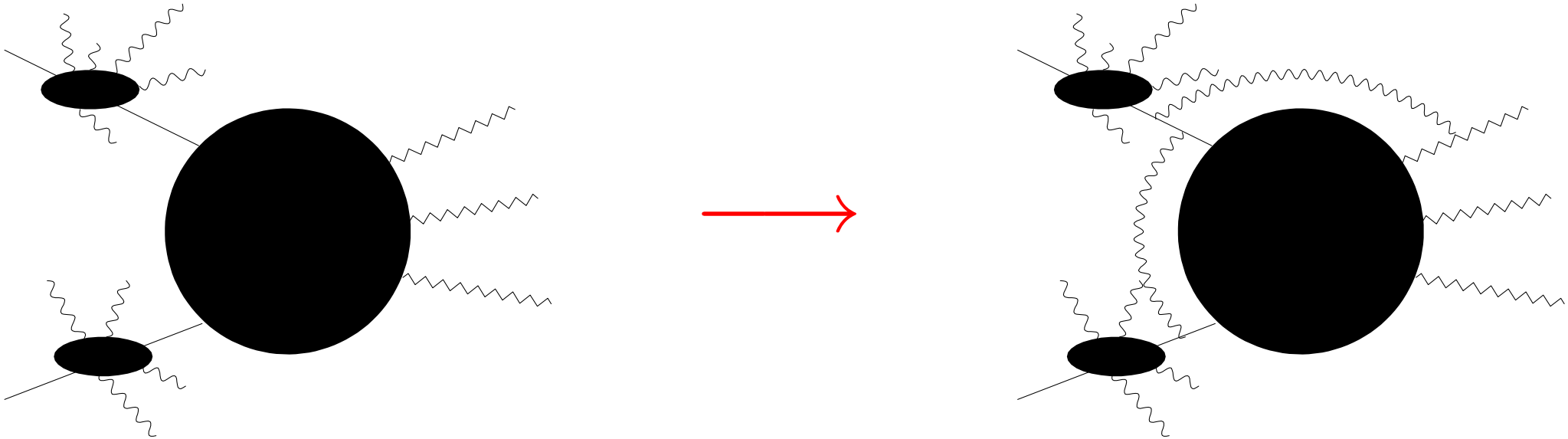
Collinear factorisation in pictures



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

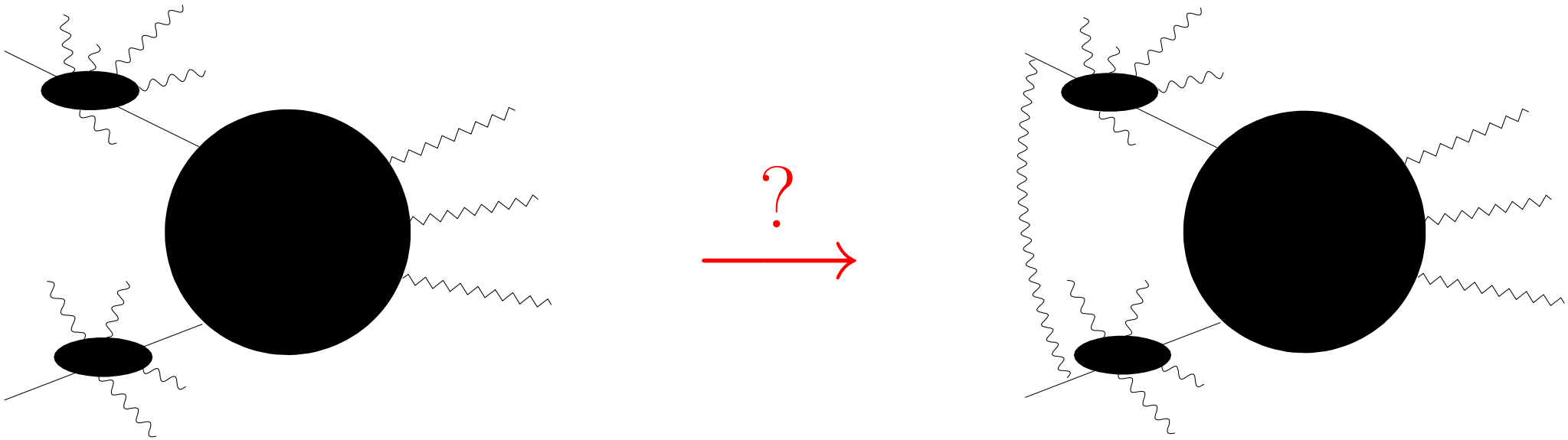
PDFs collect (universal) small-angle dynamics

Then, compute matrix elements



$$d\sigma = \text{PDF} \star \text{PDF} \star \; d\hat{\sigma}(\textstyle\sum_k \mathcal{O}(\alpha^k))$$

But: are we able to handle this?



$d\sigma$ = beamstr.$\star$ PDF $\star$ PDF $\star$ $d\hat{\sigma}$

"Handling" there means: unambiguously separate beamstrahlung from ISR effects, in keeping with the unambiguous separation between ISR and matrix-element results

Unambiguous does not mean strictly physical: as in ISR vs ME separation, conventions are not only perfectly acceptable, but
should be an essential part of the setup

My take is that, if the answer is yes, we can deconvolve beamstrahlung effects. Otherwise, we are better off with including them in our simulations, without attempting their *independent* fully accurate understanding

One does not have to figure out e.g. cancellations

The beamstrahlung equation encompasses all cases. Indeed, the absence of beamstrahlung can be written as:

$$\mathcal{B}_{kl}(y_+, y_-) = \delta_{ke^+}\delta_{le^-}\delta(1-y_+)\,\delta(1-y_-)$$

The general case formalises beam-beam interactions by inducing a beamstrahlung function where the variables are not separable. Fortunately, one can show that the following is a good approximation

$$\mathcal{B}_{kl}(y_+, y_-) \approx \sum_{n=1}^{M} b_{n,kl}^{(e^+)}(y_+)\, b_{n,kl}^{(e^-)}(y_-)$$

for suitable basis functions $b_{n,kl}^{(e^\pm)}(y)$, and for relatively small values of M

The whole thing normally depend on collider operating conditions

In light of what has been said before, is:

$$\mathcal{B}_{kl}(y_+, y_-) \approx \sum_{n=1}^{M} b_{n,kl}^{(e^+)}(y_+)\, b_{n,kl}^{(e^-)}(y_-)$$

- good enough for some 10^{-5} relative precision goals?
- with $M = ?$
- for acceptance simulations?
- for deconvolutions?
- reliable when $(k, l) \neq (e^+, e^-)$?

Your generic cross section, sufficiently inclusive:

$$\sigma = \alpha^b \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

This is symbolic, and only useful to expose the presence of:

$$\ell = \log \frac{Q^2}{\langle E_\gamma \rangle^2} \quad \text{(soft)} \qquad L = \log \frac{Q^2}{m^2} \quad \text{(collinear)}$$

Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = m_Z$

$$\begin{aligned} L = 24.18 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.06 \\ 0 \le m_{ll} \le m_Z\,, \quad \ell = 8.29 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.02 \\ m_Z - 1\ \text{GeV} \le m_{ll} \le m_Z\,, \quad \ell = 13.66 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.034 \end{aligned}$$

Your generic cross section, sufficiently inclusive:

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Numerology: consider the production of $Z \to ll$ at:

- $\sqrt{Q^2} = 500$ GeV

$$\begin{aligned} L = 24.59 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} L = 0.068 \\ 0 \le m_{ll} \le m_Z \,, \quad \ell = 1.46 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.0036 \\ m_Z - 1 \text{ GeV} \le m_{ll} \le m_Z \,, \quad \ell = 4.51 \quad &\Longrightarrow \quad \frac{\alpha}{\pi} \ell = 0.01 \end{aligned}$$

It takes a lot of brute force (i.e. fixed-order results to some $\mathcal{O}(\alpha^n)$) to overcome the enhancements due to L and ℓ.

It is always convenient to first improve by means of factorisation formulae:

$$d\sigma(L,\ell) = \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu \quad (1)$$

$$= \mathcal{K}_{coll}(L;\ell)\otimes d\hat{\sigma}(\ell) \quad (2)$$

Use of:

(1) YFS (resummation of ℓ)

(2) collinear factorisation (resummation of L)

Common features: $\mathcal{K}$ is an *all-order* universal factor; β and $d\hat{\sigma}$ are process-specific and computed order by order

(still brute force, but to a lesser extent)

Ideally:

$$\sigma = \alpha^b \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

In practice (fixed order):

$$\sigma_{FO} = \alpha^b \sum_{n=0}^{n_{\max}} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

In practice (collinear resummed):

$$\sigma_{CR} = \alpha^b \sum_{i=0}^{\infty} L^i \sum_{n=i_{\min}(i)}^{i} \alpha^n \sum_{j=0}^{n} \widetilde{\varsigma}_{n,i,j} \ell^j$$

In practice (soft resummed):

$$\sigma_{SR} = \alpha^b \sum_{j=0}^{\infty} \ell^j \sum_{n=j_{\min}(j)}^{j} \alpha^n \sum_{i=0}^{n} \widetilde{\varsigma}_{n,i,j} L^i$$

Ideally:

$$\sigma = \alpha^b \sum_{n=0}^{\infty} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

In practice (fixed order):

$$\sigma_{FO} = \alpha^b \sum_{n=0}^{n_{\max}} \alpha^n \sum_{i=0}^{n} \sum_{j=0}^{n} \varsigma_{n,i,j} L^i \ell^j$$

In practice (collinear resummed, soft improved):

$$\sigma_{CRi} = \alpha^b \sum_{i=0}^{\infty} L^i \sum_{n=i_{\min}(i)}^{i} \alpha^n \sum_{j=0}^{n} \widetilde{\varsigma}_{n,i,j} \ell^j + \left(\sigma_{FO} - \sigma_{CR}\right)\Big|_{\alpha^k, k \leq n_{\max}}$$

In practice (soft resummed, collinear improved):

$$\sigma_{SRi} = \alpha^b \sum_{j=0}^{\infty} \ell^j \sum_{n=j_{\min}(j)}^{j} \alpha^n \sum_{i=0}^{n} \widetilde{\varsigma}_{n,i,j} L^i + \left(\sigma_{FO} - \sigma_{SR}\right)\Big|_{\alpha^k, k \leq n_{\max}}$$

YFS

Aim: soft resummation for:

$$\left\{ e^+(p_1) + e^-(p_2) \longrightarrow X(p_X) + \sum_{i=0}^{n} \gamma(k_n) \right\}_{n=0}^{\infty}$$

Achieved with:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{soft}(\ell;L)\beta(L)d\mu \\ &= e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n\left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma} \end{aligned}$$

This is symbolic, and stands for both the EEX and CEEX approaches [hep-ph/0006359 Jadach, Ward, Was] that build upon the original YFS work [Ann.Phys.13(61)379]

EEX: exclusive (in the photons) exponentiation, matrix element level

CEEX: coherent exclusive (in the photons) exponentiation, amplitude level, including interference

YFS

Aim: soft resummation for:

$$\left\{ e^+(p_1) + e^-(p_2) \longrightarrow X(p_X) + \sum_{i=0}^{n} \gamma(k_n) \right\}_{n=0}^{\infty}$$

Achieved with:

$$d\sigma(L,\ell) \quad = \quad e^{Y(p_1,p_2,p_X)} \sum_{n=0}^{\infty} \beta_n \left(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}p_X; \{k_i\}_{i=0}^{n}\right) d\mu_{X+n\gamma}$$

- Y essentially universal (process dependence only through kinematics); resums ℓ
- The soft-finite β_n are process-specific, and are constructed by means of local subtractions involving matrix elements and eikonals (i.e. *not* BN)

$$\beta_n = \alpha^b \sum_{i=0}^{n} \alpha^i \sum_{j=0}^{i} c_{n,i,j} L^j$$

- For a given n, matrix elements have different multiplicities, hence the need for the kinematic mapping $\mathcal{R}$

Collinear factorisation

Aim: collinear resummation for:

$$\left\{ k(p_k) + l(p_l) \longrightarrow X(p_X) + \sum_{i=0}^{n} a_i(k_n) \right\}_{n=0}^{\infty} \qquad a_i = e^{\pm}, \gamma \dots$$

with initial-state particles stemming from beams:

$$(k,l) = (e^+, e^-)\,, \quad (k,l) = (e^+, \gamma)\,, \quad (k,l) = (\gamma, e^-)\,, \quad (k,l) = (\gamma, \gamma)\,, \dots$$

Master formula:

$$\begin{aligned} d\sigma(L,\ell) &= \mathcal{K}_{coll}(L;\ell) \otimes d\hat{\sigma}(\ell) \\ \longrightarrow\ d\sigma_{kl} &= \sum_{ij} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2)\, \Gamma_{j/l}(z_-, \mu^2, m^2) \\ &\qquad \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2; p_X, \{k_i\}_{i=0}^n) \end{aligned}$$

- $\Gamma_{\alpha/\beta}$ universal (the PDF); resums L
- The collinear-finite $d\hat{\sigma}_{ij}$ are process-specific, and are the standard short-distance matrix elements, constructed order by order (*with* BN). May or may not include resummation of other large logs (including ℓ)

Note that, in the factorisation theorem, I have:

- Set the masses to zero on the r.h.s.
- Turned on PDF evolution ($\mu_0 \sim m \to \mu$)
- Discarded power-suppressed terms (in the mass) on the l.h.s.

$$d\bar{\sigma}_{e^+e^-\to X} = d\sigma_{e^+e^-\to X} + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right), \qquad p \geq 1\,.$$

YFS vs collinear factorisation

Both are systematically improvable in perturbation theory:
in YFS the β_n's (fixed-order), in collinear factorisation both the PDFs (logarithmic accuracy) and the $d\hat{\sigma}$'s (fixed-order, resummation)

YFS: very little room for systematics. Exceptions are the kinematic mapping $\mathcal{R}$, and the quark masses (when the quarks are radiators). Renormalisation schemes??

Collinear factorisation: systematic variations much larger. At the LL (used in phenomenology so far) a rigorous definition of uncertainties is impossible (parameters are arbitrary), and comparisons with YFS are largely fine tuned

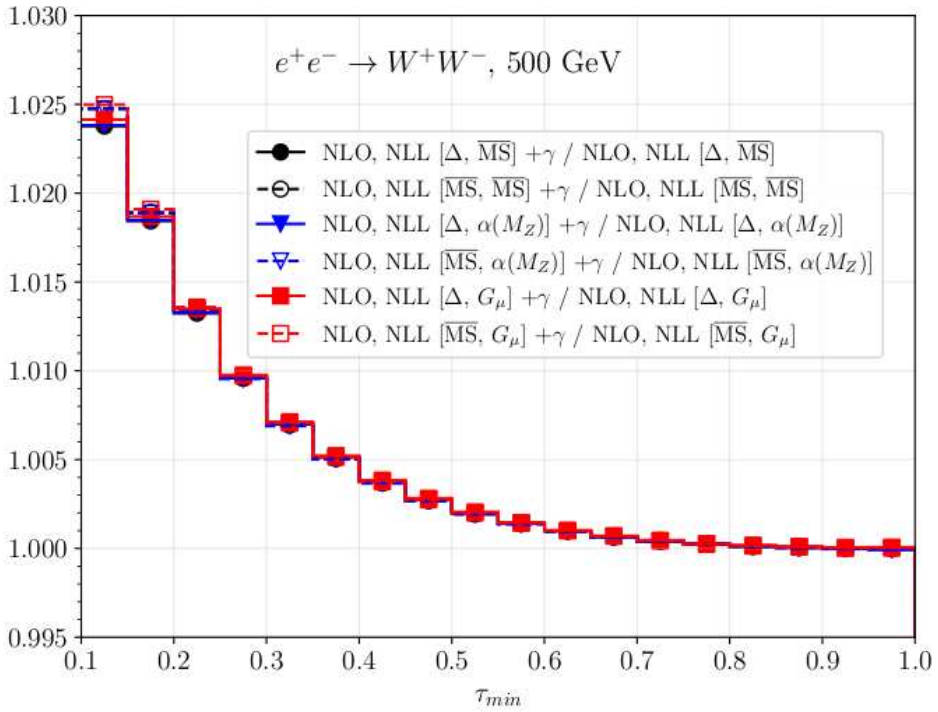
YFS: the computations of β_n require an ad-hoc (i.e. non-BN/KLN) approach

Collinear factorisation: the computations of $d\hat{\sigma}_{ij}$ are as dictated by BN/KLN

YFS: Subprocesses not initiated by e^+e^- are not resummed

Collinear factorisation: Subprocesses not initiated by e^+e^- are also resummed

Example: $\gamma\gamma \to W^+W^-$, $\sqrt{s} = 500$ GeV



$$\sigma(\tau_{min}) = \int d\sigma\, \Theta\left(\tau_{min} \leq \frac{M^2_{W^+W^-}}{s}\right)$$

⟵ Ratio with/without $\gamma\gamma$

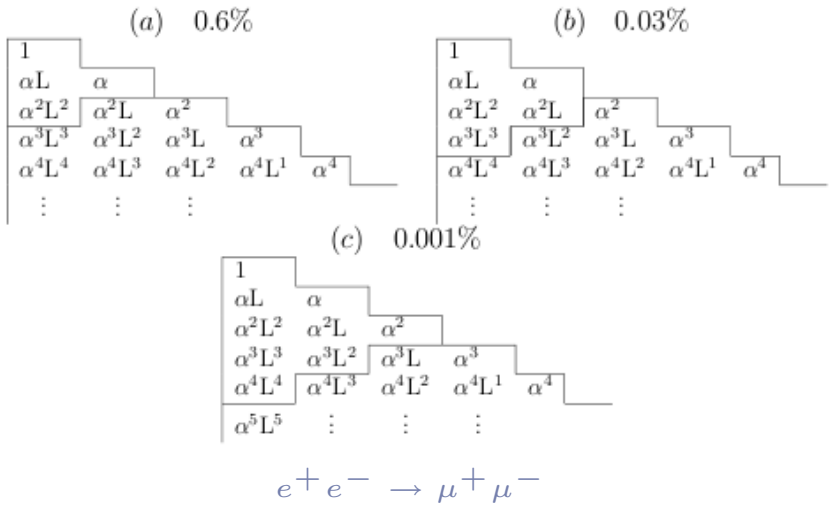
2207.03265

Impact of specific channels very much dependent on process, energy, cuts, etc.

Impossible to tell beforehand, whence the importance of accurate simulations

an aside

YFS and collinear factorisation differ on naming conventions !



1903.09895, Jadach&Skrzypek

- $\mathcal{O}(\alpha L)$ and $\mathcal{O}(\alpha^2 L)$ in YFS *typically* mean the corresponding coefficients in the β_n terms
- $\mathcal{O}(\alpha L)$ and $\mathcal{O}(\alpha^2 L)$ in collinear factorisation mean the whole LL and NLL towers, respectively

Folk wisdom about collinear factorisation: while YFS is exclusive in the photons (*true* in EEX and CEEX), collinear factorisation is inclusive (*not true* in general)

Folk wisdom about collinear factorisation: while YFS is exclusive in the photons (*true* in EEX and CEEX), collinear factorisation is inclusive (*not true* in general)

Firstly, one is always exclusive in the photons (possibly) emerging from the hard process ($d\hat{\sigma}_{ij}$)

Secondly, whether one is exclusive also in the photons associated with ISR depends on the implementation of the factorisation formula

- MC integration as is: inclusive (or modelled as e.g. in Whizard)
- Integration through recursion (e.g. parton shower): exclusive.
  Examples: Pythia8, Herwig, Sherpa, Babayaga

Folk wisdom about collinear factorisation: while YFS is exclusive in the photons (*true* in EEX and CEEX), collinear factorisation is inclusive (*not true* in general)

Open problems in *precise*, exclusive e^+e^- simulations: extend QCD matching techniques (MC@NLO, Powheg) to ISR QED*; extend logarithmic accuracy of shower to NLL; extend matching beyond NLO

Before these can be considered $\longrightarrow$

*LL solutions in pp collisions in HORACE [see e.g. hep-ph/0609170] and Powheg [see e.g. 1302.4606]

All physics simulations based on collinear factorisation done so far are based on a LL-accurate picture

This is not tenable at high energies/high statistics:

- accuracy is insufficient (see e.g. W^+W^- production)
- systematics not well defined

Fortunately, the upgrade of PDFs from LL to NLL accuracy is now established

z-space LO+LL PDFs $(\alpha \log(Q^2/m^2))^k$:

$\sim$ 1992

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes
- for e^-

z-space NLO+NLL PDFs $(\alpha \log(Q^2/m^2))^k + \alpha\,(\alpha \log(Q^2/m^2))^{k-1}$:

$\longrightarrow$ 1909.03886, 1911.12040, 2105.06688, 2207.03265 (Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao)

- $0 \leq k \leq \infty$ for $z \simeq 1$
- $0 \leq k \leq 3$ for $z < 1 \iff \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, γ, and light quarks
- both numerical and analytical
- factorisation schemes: $\overline{\rm MS}$ and Δ (that has DIS-like features)

New: z-space NNLO+NNLL PDFs (Frixione, Schonwald, in preparation)

Is this needed?

$$\Gamma_{NS}^{(LO)}(z,\mu^2) \stackrel{z\to 1}{\longrightarrow} \frac{e^{-\gamma_{\rm E}\xi_0}e^{\hat{\xi}_0}}{\Gamma(1+\xi_0)}\,\xi_0(1-z)^{-1+\xi_0}$$

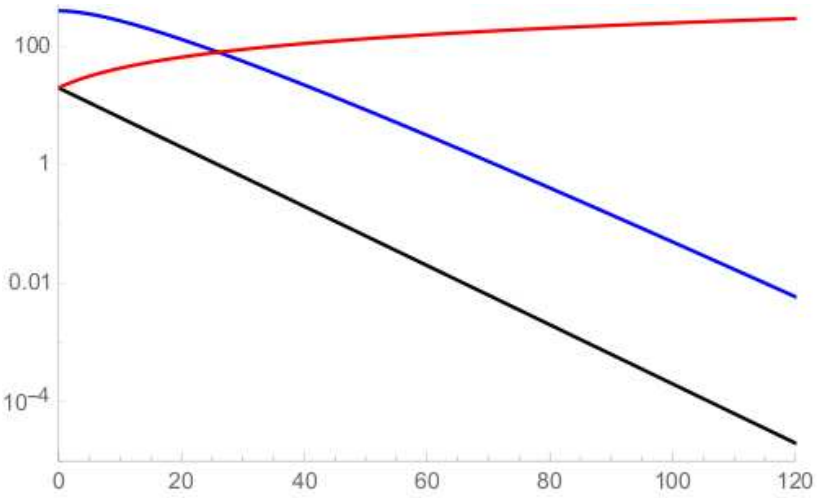
This is (essentially) the famous Gribov-Lipatov solution, while at the NLO:

$$\Gamma_{NS}^{(NLO,\overline{\rm MS})}(z,\mu^2) \stackrel{z\to 1}{\longrightarrow} \frac{e^{-\gamma_{\rm E}\xi_1}e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\,\xi_1(1-z)^{-1+\xi_1}$$
$$\times\left\{1+\frac{\alpha}{2\pi}\left(2-\frac{\pi^2}{3}+\frac{3}{2}L_0+2(1-L_0)d_1(\xi_1)-2d_2(\xi_1)\right.\right.$$
$$\left.\left.-\Big[2(1-L_0)-4d_1(\xi_1)\Big]\log(1-z)-2\log^2(1-z)\right)\right\}$$

with

$$d_k(\xi)=\frac{1}{\mathcal{G}_d(\xi,0)}\,\frac{\partial^k\mathcal{G}_d(\xi,\delta)}{\partial\delta^k}\bigg|_{\delta=0}, \qquad \mathcal{G}_d(\xi,\delta)=\frac{e^{-\gamma_{\rm E}(\xi-\delta)}}{\Gamma(\xi-\delta)}$$

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0}\log\frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha}{2\pi}\log\frac{\mu^2}{\mu_0^2} + \ldots \\
\xi_0 &= 2t \\
\hat{\xi}_1 &= \frac{3}{2}\,t \\
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_{_F} + \frac{4\pi b_1}{b_0}\right) \\
&= 2\left[1 - \frac{\alpha(\mu)}{\pi}\left(\frac{5}{9}n_{_F} + \frac{\pi b_1}{b_0}\right)\right]t \\
&\qquad + \frac{\alpha(\mu)}{\pi}\left(\frac{10}{9}\pi b_0 n_{_F} + 2b_1\pi^2\right)t^2 + \mathcal{O}(t^3) \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\left[1 + \frac{\alpha(\mu)}{\pi}\left(\frac{\lambda_1}{3} - \frac{\pi b_1}{b_0}\right)\right]t \\
&\qquad - \frac{\alpha(\mu)}{\pi}\left(\frac{\pi b_0}{2}\,\lambda_1 - \frac{3}{2}\,\pi^2 b_1\right)t^2 + \mathcal{O}(t^3)
\end{aligned}
$$



$$\xi = 0.05$$

x-axis: y as in $z = 1 - 10^{-y}$

$$\frac{1}{\xi} - \int_0^z dx (1-x)^{-1+\xi}$$

$$\frac{1}{\xi^2} + \int_0^z dx \log(1-x)(1-x)^{-1+\xi}$$

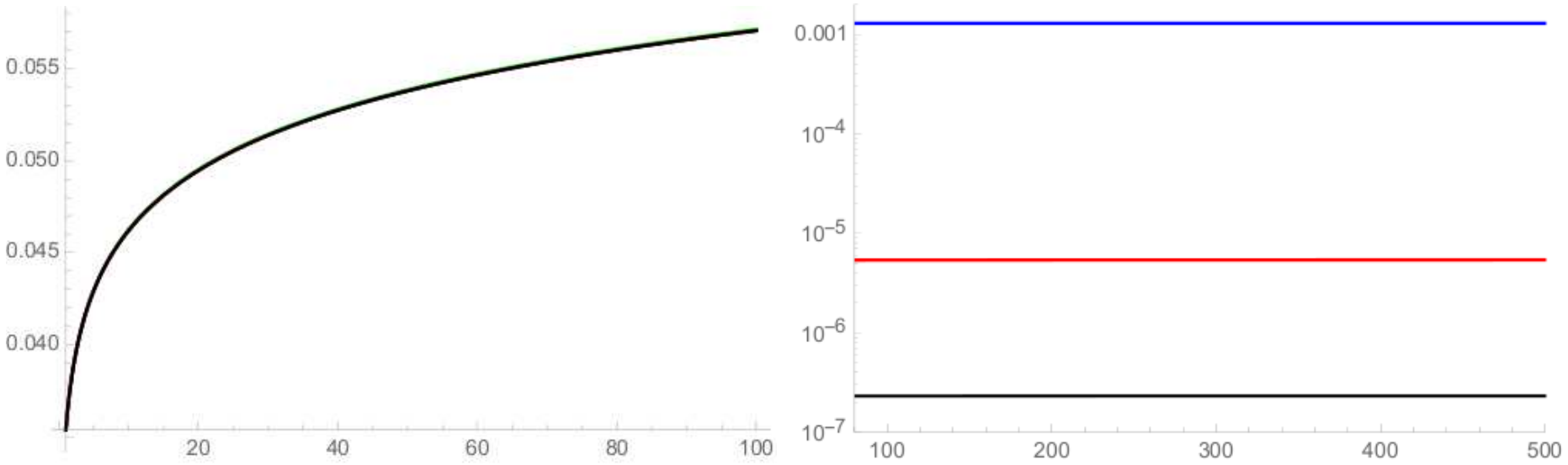
red=blue/black

Which is to say, one *must* integrate up to exactly $z = 1$ to get it right

(or at least to $1 - 10^{-120}$ to get to the target relative precision of 10^{-5})

And this while fighting (see the red curve) a factor ~ 100 stemming from the unphysical $\overline{\mathrm{MS}}$ logs (at the NLO; and it gets worse)

ξ parameters



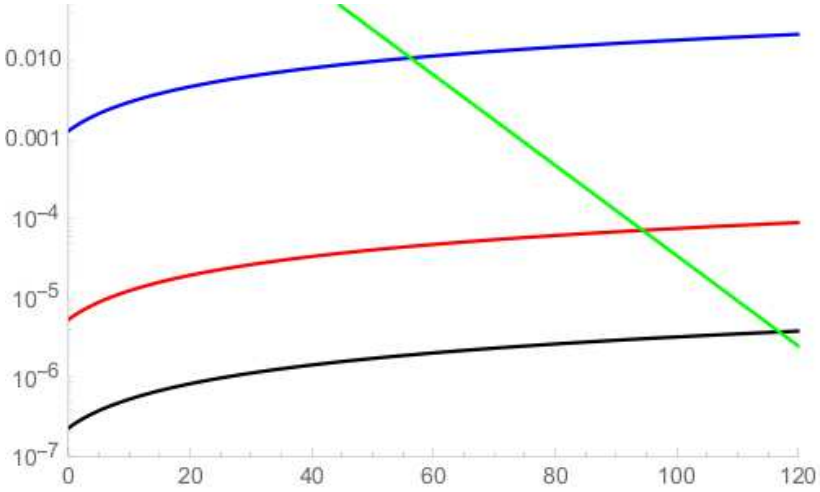
- Left: $\xi_2^{(res)}$ vs μ
- Right: $\left|1 - \xi_i/\xi_2^{(res)}\right|$ vs μ

  Blue: $i = 0$ (LO)

  Red: $i = 1$ (NLO)

  Black: $i = 2$ (NNLO)

Corrections are as expected. *But:*



x-axis: y as in $z = 1 - 10^{-y}$

$$\mathcal{I}(\xi) = \frac{1}{\xi} - \int_0^z dx (1-x)^{-1+\xi}$$

- $\mathcal{I}(\xi_2^{(res)})$
- $\left|1 - \mathcal{I}(\xi_0)/\mathcal{I}(\xi_2^{(res)})\right|$
- $\left|1 - \mathcal{I}(\xi_1)/\mathcal{I}(\xi_2^{(res)})\right|$
- $\left|1 - \mathcal{I}(\xi_2)/\mathcal{I}(\xi_2^{(res)})\right|$

If the blue/red/black line is above the green line, the relative accuracy in the computation of the total rate is smaller w.r.t. what is required at that point, i.e. the corresponding N^{k}LO PDF is insufficient to achieve the desired level of precision (relative to a resummed NNLO result, the best so far)

Lessons learned

- If the target relative precision is 10^{-5}, one must work at the NNLO
- Regardless, values of z insanely close to one must be accurately handled
- This is *extremely* difficult to do in $\overline{\mathrm{MS}}$, the more so the higher the perturbative accuracy

A different factorisation scheme must be employed

The Δ scheme (NLO: 2105.06688, Frixione; NNLO and beyond: in preparation, Frixione&Schonwald) is a viable alternative which:

- is defined multiplicatively
- eliminates all $\log(1-z)$ contributions

With:

$$f(x) = \frac{1 + \frac{\alpha(\mu)}{\pi}(x^2 - x + C)}{1 + \frac{\alpha(\mu_0)}{\pi}(x^2 - x + C)}\,, \qquad C = \frac{\pi^2}{6} - 1$$

one finds:

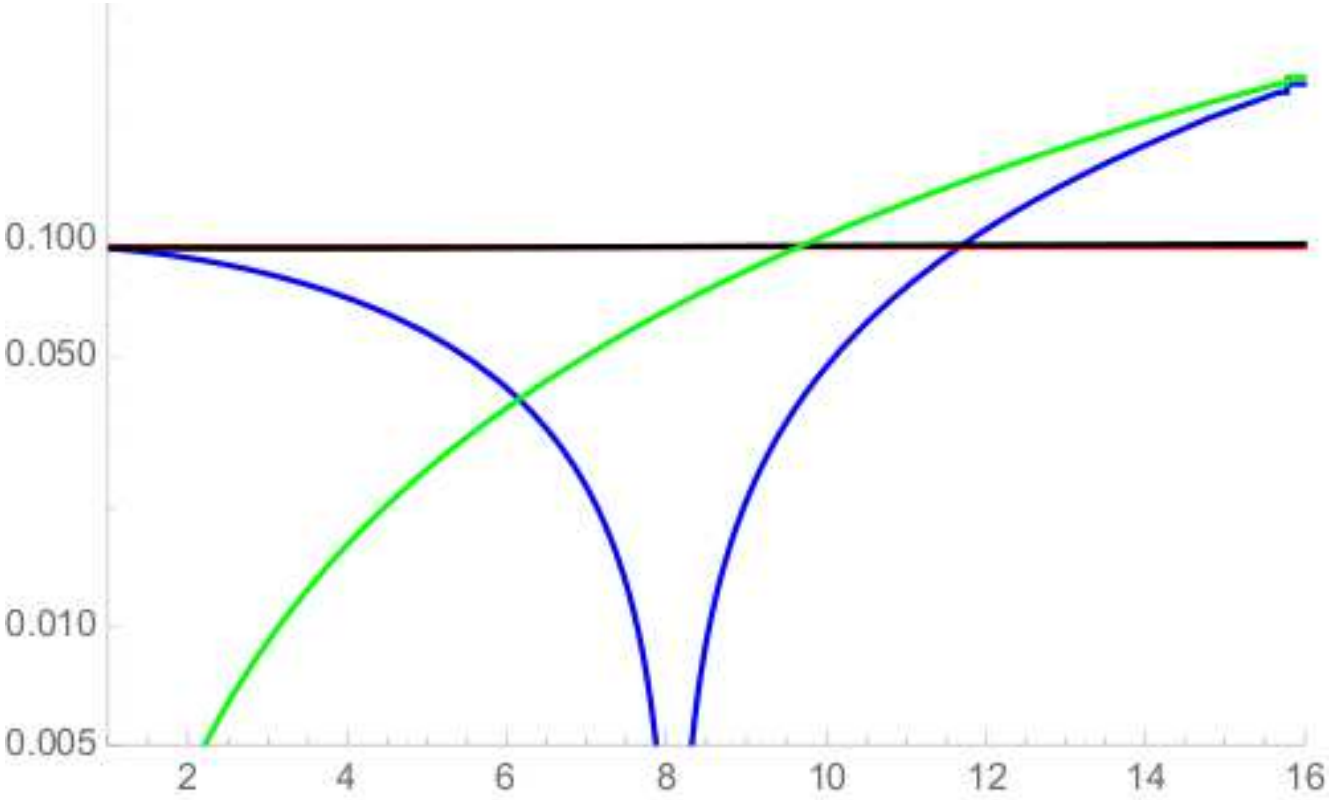
$$\Gamma_{NS}^{(NLO,\Delta)}(z) \overset{z\to 1}{\longrightarrow} e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1} \frac{\xi_1 (1-z)^{-1+\xi_1}}{\Gamma(1+\xi_1)} \frac{1}{\mathcal{G}_d(\xi,0)} \mathcal{G}_d\left(\xi, \frac{d}{dx}\right) f(x)\Big|_{x=-\log(1-z)}$$

$$\overset{z\to 1}{\longrightarrow} e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1} \frac{\xi_1 (1-z)^{-1+\xi_1}}{\Gamma(1+\xi_1)} \frac{\alpha(\mu)}{\alpha(\mu_0)}$$

Whatever form one chooses, all $\log(1-z)$ act as suppression factors in the Δ scheme

The difference w.r.t. $\overline{\rm MS}$ is striking

Let's take a look



black: LO

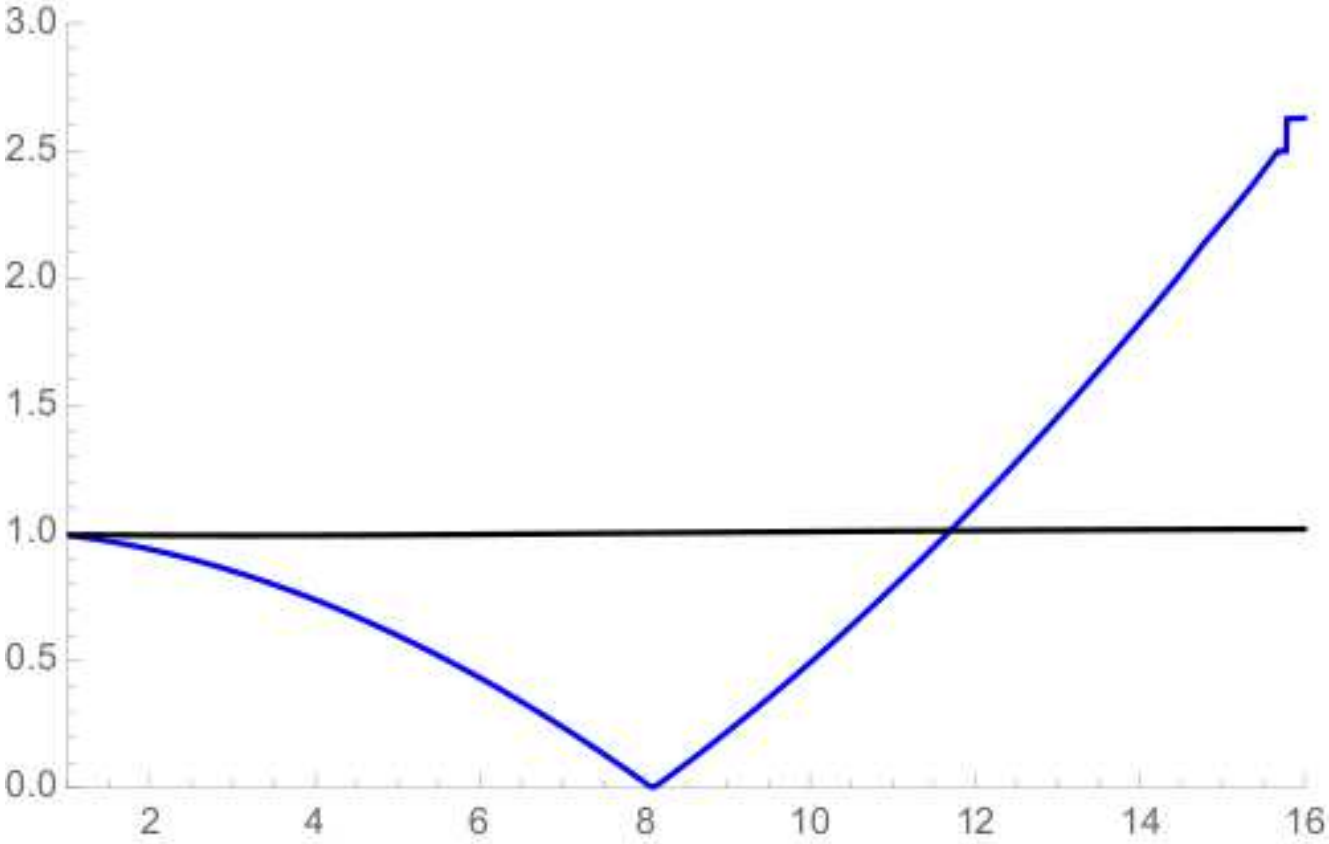
blue: $\left|\mathrm{NLO}\ \overline{\mathrm{MS}}\right|$

red: NLO Δ

green: $\propto \log^2(1-z)$

Plotted as a function of x, $z = 1 - 10^{-x}$, no prefactor

Remember: this is QED, where $\alpha \simeq 7{\cdot}10^{-3}$

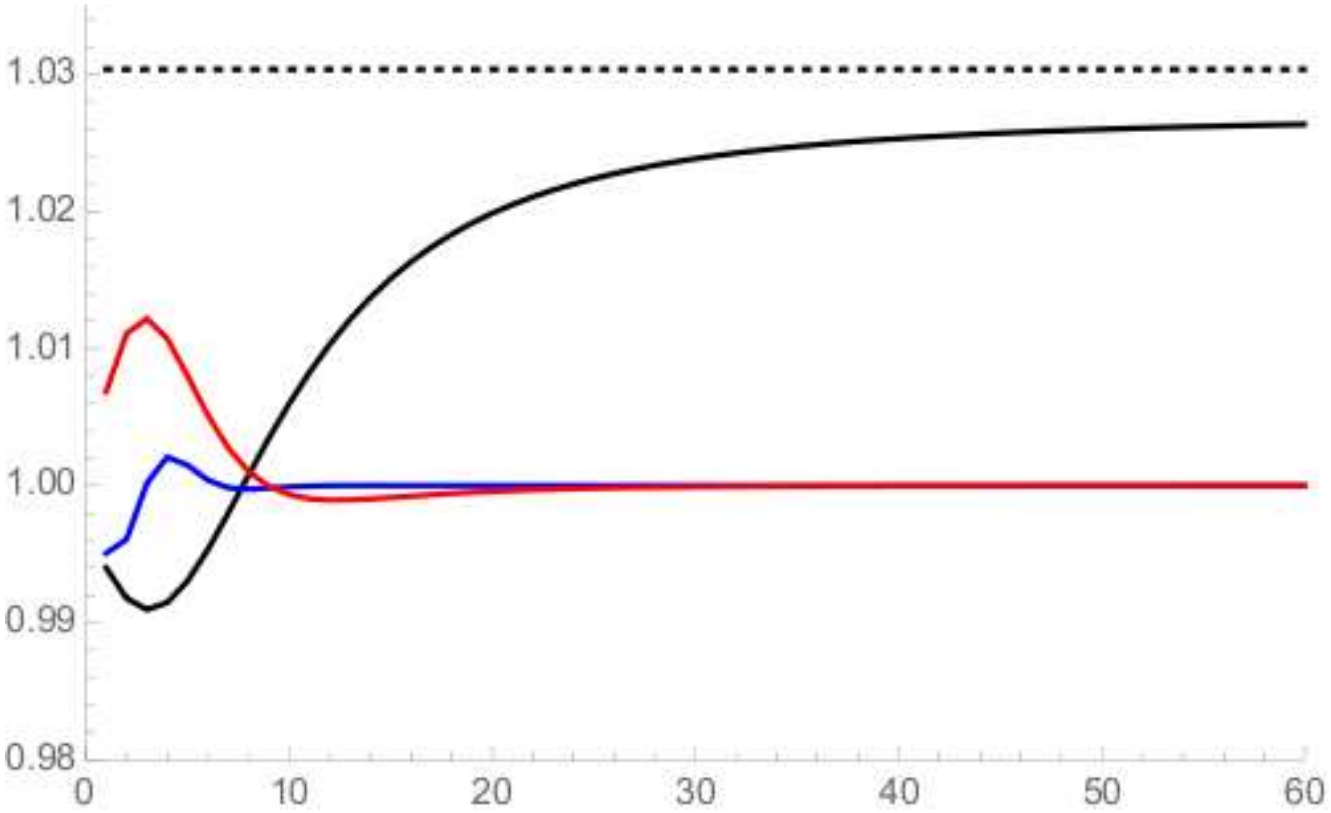


black: (NLO Δ)/LO

blue: $\left|\text{NLO } \overline{\text{MS}}\right|$/LO

Plotted as a function of x, $z = 1 - 10^{-x}$, no prefactor

Remember: this is QED, where $\alpha \simeq 7{\cdot}10^{-3}$



black dotted: $\alpha(\mu)/\alpha(\mu_0)$

black: NLO Δ exact

blue: (NLO Δ)/(NLO Δ)

8 terms/exact

$10^7(\#-1)+1$

red: (NLO Δ)/(NLO Δ)

2 terms/exact

$10^2(\#-1)+1$

Plotted as a function of x, $z = 1 - 10^{-x}$, no prefactor

Let's look (2207.03265) at a few sample NLO results for:

$$\begin{aligned} e^+e^- &\longrightarrow q\bar{q} \\ e^+e^- &\longrightarrow t\bar{t} \\ e^+e^- &\longrightarrow W^+W^- \end{aligned}$$

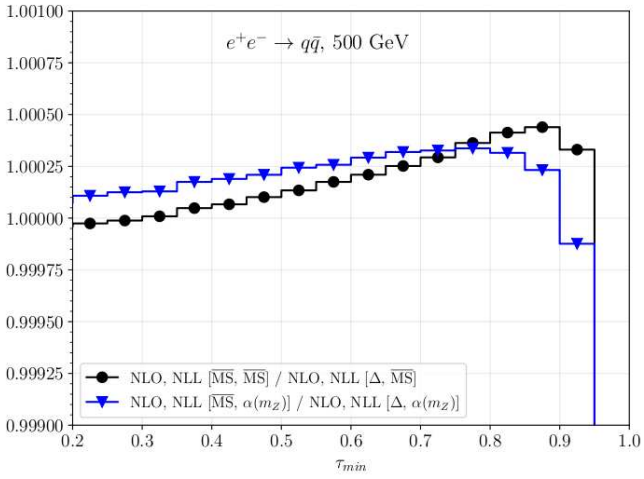
with $q\bar{q}$ production (massless quarks) restricted to ISR QED radiation.
The other two are in the SM

What is plotted:

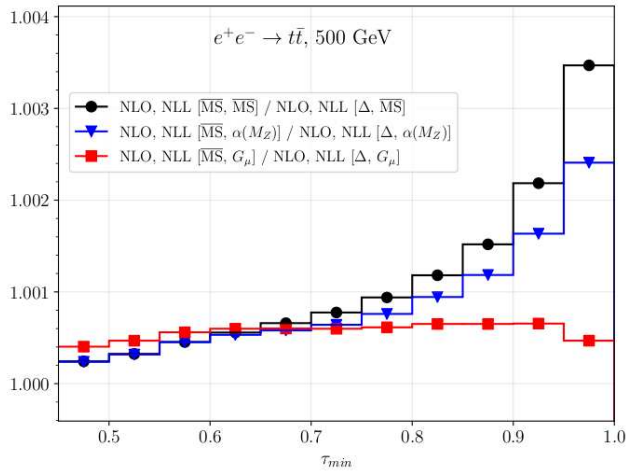
$$\sigma(\tau_{min}) = \int d\sigma\, \Theta\left(\tau_{min} \leq \frac{M^2_{p\bar{p}}}{s}\right), \qquad p = q\,, t\,, W^+$$

$\tau_{min} \sim 1$ is sensitive to soft emissions (not resummed)

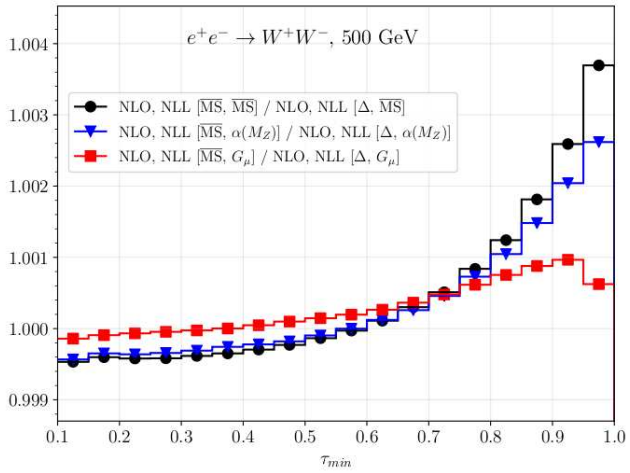
Dependence of observables on factorisation scheme



$q\bar{q}$



$t\bar{t}$



W^+W^-

$\mathcal{O}(1)$ differences for PDFs down to $\mathcal{O}(10^{-4} - 10^{-3})$ for observables

In the $\overline{\rm MS}$ scheme, huge cancellations between PDFs and short-distance cross sections

Behaviour qualitatively similar for different renormalisation schemes

Why does this cancellation work at this level? Bear in mind:

$$\begin{aligned}\Gamma_e^{(K)}(z) &= \delta(1-z) + \frac{\alpha}{2\pi}\Gamma_e^{(K)[1]} + \mathcal{O}(\alpha^2) \\ d\hat{\sigma} &= \ldots + \mathcal{K} \star d\sigma^{(0)}\end{aligned}$$

with

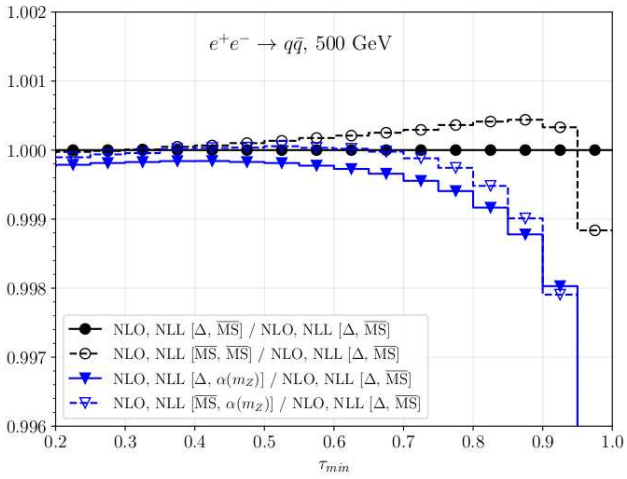
$$\begin{aligned}\mathcal{K}_{ee} &= \left[\frac{1+z^2}{1-z}\left(\log\frac{s}{\mu^2} + 2\log(1-z)\right)\right]_+ + (1-z) \\ &\quad + \left(\frac{7}{2} - \frac{3}{2}\log\frac{s}{\mu^2}\right)\delta(1-z) - K_{ee}(z)\,.\end{aligned}$$

So numerically one ends up implicitly computing

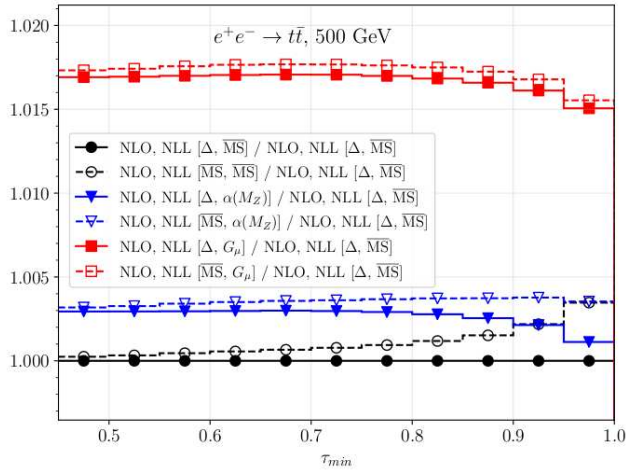
$$\left(\Gamma_e^{(K)[1]} + \mathcal{K}_{ee}\right) = \left(\frac{1+z^2}{1-z}\right)_+ \left(\log\frac{s}{m^2} - 1\right) + (1-z) + \left(\frac{7}{2} - \frac{3}{2}\log\frac{s}{\mu^2}\right)\delta(1-z)$$

- $\overline{\text{MS}}$: Double logs cancel in the sum
- Δ: Double logs cancel in individual contributions, owing to K functions

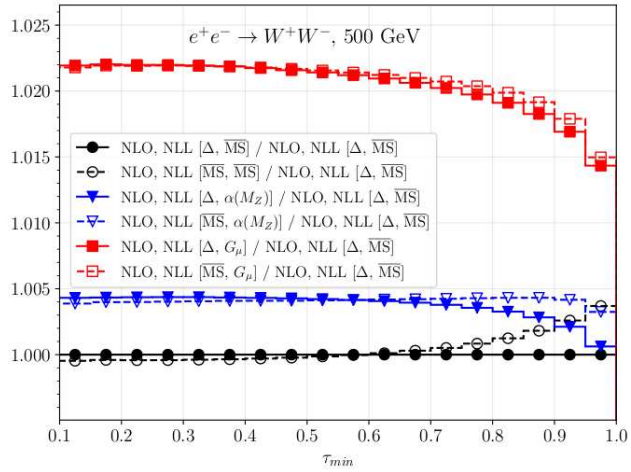
Factorisation vs renormalisation scheme dependence



$q\bar{q}$



$t\bar{t}$



W^+W^-

Renormalisation-scheme dependence (different colours) much larger than factorisation-scheme dependence (different patterns), with process-dependent features

Depending on the precision, renormalisation scheme is an informed choice; factorisation scheme always induces a systematic

Conclusions that have emerged since ~ 2022:

- The inclusion of NLL contributions into the electron PDF has an impact of $\mathcal{O}(1\%)$ (precise figures are observable and renormalisation-scheme dependent)
- This estimate does not include the effects of the photon PDF
- The comparison between $\overline{\mathrm{MS}}$- and Δ-based results shows differences compatible with non-zero $\mathcal{O}(\alpha^2)$ effects, as expected
  (but: these are potentially *large in the soft region*)
- Renormalisation-scheme dependence is of $\mathcal{O}(0.5\%)$

It is crucial that cross section computations can be performed in an arbitrary factorisation scheme

Things to do

Details are very much dependent on target precision. Rough guidelines

- In YFS, one wants $\mathcal{O}(\alpha^3 L^3)$ at least
- In collinear factorisation, one possibly wants NNNLO
- Resum the "other" log
- Understand the role of EW corrections in all of this
- One must learn how to apply (where possible) the QCD matching/merging techniques to QED
  $\Longrightarrow$ fully-exclusive simulations in collinear factorisation

A very clear lesson from the LHC: do not consider case by case, but develop general, process independent, approaches

Take-home messages

- Investigate the separation of beamstrahlung and ISR in a large-precision landscape; introduce a scheme while doing that
- Factorisation in $\overline{\rm MS}$ cannot work well at high accuracy; use a factorisation scheme such as Δ
- Cross-section computations must be Δ-compatible
- Work is required on high-accuracy QED exclusive simulations
- Deconvolutions are increasingly difficult when large precision is required: use a full simulation chain to compute observables within acceptance

It is difficult for me *not* to reach the conclusion that, at future colliders, simulations (as opposed to semi-analytical codes) will play an even more prominent role than at LEP. For this to happen, significant conceptual and technical work remains to be done