Monte Carlo implementation of initial-state radiation

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Disclaimer

- Work in progress with C. Carloni Calame, S. Frixione, G. Montagna, F. Piccinini
- MC implementation of ISR within the collinear framework
  (= DGLAP equation as a guideline and inclusive PDFs as a benchmark)
- YFS discussed later this afternoon (Z. Was, A. Price)

1. Recast DGLAP in a MC-friendly form
2. NLL ingredients
3. Numerical results
4. Conclusions

1 Recast DGLAP in a MC-friendly form

2 NLL ingredients

3 Numerical results

4 Conclusions

Structure of DGLAP equation

Adopting notation of 2309.15587

The evolution of PDFs in flavour space is given by

$$\frac{\partial F(x)}{\partial \log \mu^2} = \mathbb{O} \otimes_x F, \qquad \mathbb{O} \otimes_x F = \int_0^1 \frac{dz}{z}\, \mathbb{O}(z)\, F(x/z)\, \Theta(z \geq x)$$

In $\overline{\text{MS}}$, the operator $\mathbb{O}$ coincides with the matrix of splitting kernels.

The most general form of the operator $\mathbb{O}$ at NLO is

$$\mathbb{O}(z) = \big[\mathbb{A}(z)\big]_+ + \mathbb{B}\,\delta(1-z) + \mathbb{C}(z)$$

with

$$(\mathbb{A}(z))_{ij} = \delta_{ij} A_i(z), \quad (\mathbb{B})_{ij} = \delta_{ij} B_i, \quad (\mathbb{C}(z))_{ij} = C_{ij}(z)$$

Introducing the cutoff

Adopting notation of 2309.15587

Introduction of a cutoff ϵ to separate unresolved/resolved emissions in z-space

$$\Theta_z^> \equiv \Theta(z > 1-\epsilon), \quad \Theta_z^< \equiv \Theta(z < 1-\epsilon)$$

Exploiting the trivial identity

$$[g(z)]_+ = \left[g(z)\,\Theta_z^>\right]_+ + \left(g(z)\,\Theta_z^<\right) + \left(-\int_0^1 d\omega\, g(\omega)\Theta_\omega^<\right)\delta(1-z)$$

the operator $\mathbb{O}$ becomes

$$\mathbb{O}(z) = \left[\mathbb{A}(z)\Theta_z^>\right]_+ + \mathbb{A}(z)\Theta_z^< + \overline{\mathbb{B}}\,\delta(1-z) + \mathbb{C}(z)\Theta_z^> + \mathbb{C}(z)\Theta_z^<$$

$$\overline{\mathbb{B}} = \mathbb{B} - \int_0^1 dz\, \mathbb{A}(z)\Theta_z^<$$

DGLAP with cutoff

Adopting notation of 2309.15587

Evolution equation rewritten as

$$\frac{\partial F(x)}{\partial \log \mu^2} = \mathbb{O} \otimes_x F = \mathbb{W}\left[F\right](x) + \mathbb{Z}\left[F\right](x) + \overline{\mathbb{B}} F(x)\,,$$

with new operators defined as

$$\begin{aligned} \mathbb{W}\left[F\right](x) &= \left(\left[\mathbb{A}\Theta^{>}\right]_+ + \mathbb{C}\Theta^{>}\right) \otimes_x F\,, \\ \mathbb{Z}\left[F\right](x) &= \left(\left[\mathbb{A}+\mathbb{C}\right]\Theta^{<}\right) \otimes_x F\,. \end{aligned}$$

- Physical interpretation: $\mathbb{Z}$: resolved real emissions; $\overline{B}$: virtual emissions; $\mathbb{W}$: remainder with no clear physical meaning (it arises from the distributions in the kernels)
- Cutoff dependence of $\mathbb{W}$ must cancel that of the $\overline{\mathbb{B}}$ term (in general logarithmic).

Sudakov factor

Adopting notation of 2309.15587

Introducing the Sudakov form factor

$$\mathbb{S}(\mu^2) = \exp\left[\int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2}\overline{\mathbb{B}}(\kappa^2)\right].$$

It is possible to rewrite the evolution as

$$\frac{\partial}{\partial\mu^2}\left(\frac{1}{\mathbb{S}(\mu^2)}F(\mu^2)\right) = \frac{1}{\mu^2\,\mathbb{S}(\mu^2)}\left(\mathbb{W}\left[F\right](\mu^2) + \mathbb{Z}\left[F\right](\mu^2)\right),$$

which in integrated form reads (by setting $\mathbb{S}(\mu_0^2) = 1$ and $\mathbb{S}(\kappa_1^2;\kappa_2^2) \equiv \mathbb{S}(\kappa_1^2)/\mathbb{S}(\kappa_2^2)$)

$$F(\mu^2) = \mathbb{S}(\mu^2)\,F(\mu_0^2) + \int_{\mu_0^2}^{\mu^2}\frac{d\kappa^2}{\kappa^2}\mathbb{S}(\mu^2;\kappa^2)\left(\mathbb{W}\left[F\right](\kappa^2) + \mathbb{Z}\left[F\right](\kappa^2)\right)$$

In a Monte Carlo approach, the $\mathbb{W}$ term is discarded

Solution in Mellin space

Adopting notation of 2309.15587

$$F_N(\mu^2) = \sum_{k=0}^{\infty} F_N^{(k)}(\mu^2)\,,$$

$$\begin{aligned} F_N^{(0)}(\mu^2) &= \mathbb{S}(\mu^2)\, F_N(\mu_0^2)\,, \\ F_N^{(k)}(\mu^2) &= \int_{\mu_0^2}^{\mu^2} \left[\prod_{p=1}^{k} \frac{d\kappa_p^2}{\kappa_p^2}\, \Theta\left(\kappa_{p-1}^2 \le \kappa_p^2 \le \kappa_{p+1}^2\right) \mathbb{S}(\kappa_{p+1}^2; \kappa_p^2) \mathbb{Z}_N(\kappa_p^2) \right] \mathbb{S}(\kappa_1^2; \mu_0^2)\, F_N(\mu_0^2)\,, \end{aligned}$$

$$\kappa_0^2 = \mu_0^2\,, \qquad \kappa_{k+1}^2 = \mu^2\,.$$

Physical interpretation: emission at κ_p^2, no emissions between κ_p^2 and κ_{p+1}^2, and so on.
In QCD usually this equation reads *backwards*; here it reads *forwards* (from μ_0^2 to μ^2)

Solution in x-space

Adopting notation of 2309.15587

$$
\begin{aligned}
F^{(0)}(x,\mu^2) &= \mathbb{S}(\mu^2)\, F(x,\mu_0^2)\,, \\
F^{(k)}(x,\mu^2) &= \int_0^1 \left[\prod_{q=1}^{k+1} dz_q\right] \delta\left(x - \prod_{q=1}^{k+1} z_q\right) \\
&\quad \times \int_{\mu_0^2}^{\mu^2} \left[\prod_{p=1}^{k} \frac{d\kappa_p^2}{\kappa_p^2}\, \Theta\left(\kappa_{p-1}^2 \le \kappa_p^2 \le \kappa_{p+1}^2\right) \mathbb{S}(\kappa_{p+1}^2;\kappa_p^2)\right. \\
&\quad \left. \times \left(\mathbb{W}(z_p,\kappa_p^2) + \mathbb{Z}(z_p,\kappa_p^2)\right)\right] \mathbb{S}(\kappa_1^2;\mu_0^2)\, F(z_{k+1},\mu_0^2)\,.
\end{aligned}
$$

Note the additional integration variable to sample the initial condition (if different from a δ-function as in $\overline{\text{MS}}$ at NLO)

Ambiguities

Adopting notation of 2309.15587

The virtual split $\overline{\mathbb{B}} = \overline{\mathbb{B}}^{>} + \overline{\mathbb{B}}^{<}$ is arbitrary; only $\overline{\mathbb{B}}^{<}$ is exponentiated into $\mathbb{S}$, $\overline{\mathbb{B}}^{>}$ goes into $\mathbb{W}$. With the physical requirement that $\overline{\mathbb{B}}^{>} \to 0$ as $\epsilon \to 0$, different choices differ only by cutoff-power-suppressed terms. In terms of $\mathbb{A}$ and $\mathbb{B}$ this implies

$$\overline{\mathbb{B}}^{>} = \mathbb{B}^{>}\,, \qquad \overline{\mathbb{B}}^{<} = \mathbb{B}^{<} - \int_0^1 dz\, \mathbb{A}(z)\Theta_z^{<}\,,$$

with

$$\mathbb{B} = \mathbb{B}^{>} + \mathbb{B}^{<}\,, \qquad \lim_{\epsilon\to 0} \mathbb{B}^{>} = 0\,.$$

Example: in LO QCD, the quark Sudakov form factor usually features $B^{>} = 0$ and

$$B^{<} = -\frac{\alpha_S}{2\pi} C_F \int_0^{1-\epsilon} \frac{1+z^2}{1-z}\, dz$$

(but this is arbitrary to some degree)

Example: Non-singlet at LL

In the non-singlet case, i.e. a linear equation, at LL with fixed α it reduces to the BabaYaga algorithm (in its first implementation)

$$\begin{aligned} F(x,\mu^2) &= S(\mu^2,m^2)\Bigg\{\delta(1-x)+\int_{m^2}^{\mu^2}\frac{d\mu_1^2}{\mu_1^2}\frac{\alpha}{2\pi}\int_0^{1-\epsilon}dyP(y)\delta(x-y) \\ &+ \int_{m^2}^{\mu^2}\frac{d\mu_1^2}{\mu_1^2}\int_{m^2}^{\mu_1^2}\frac{d\mu_2^2}{\mu_2^2}\left(\frac{\alpha}{2\pi}\right)^2\int_0^{1-\epsilon}dx_1\int_0^{1-\epsilon}dx_2P(x_1)P(x_2)\delta(x-x_1x_2)+\cdots\Bigg\} \end{aligned}$$

with Sudakov (which factors out) given by

$$S(\mu_1^2,\mu_2^2)=\exp\left[-\frac{\alpha}{2\pi}\int_{\mu_1^2}^{\mu_2^2}\frac{d\kappa^2}{\kappa^2}\int_0^{1-\epsilon}dzP(z)\right],\quad P(x)=\frac{1+x^2}{1-x}$$

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Initial conditions

Now known up to NNLO (see Kay's talk). Up to NLO

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z)\,, \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z)\,, \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z)\,, \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0\,,
\end{aligned}$$

The $\overline{\text{MS}}$ scheme is such that

$$K_{ee}(z) = K_{\gamma e}(z) = 0$$

Another factorization scheme: Δ

Maximally simplify initial conditions by choosing

$$
\begin{aligned}
K_{ee}^{(\Delta)}(z) &= \left[\frac{1+z^2}{1-z}\Big(2\log(1-z)+1\Big)\right]_+ , \\
K_{\gamma e}^{(\Delta)}(z) &= \frac{1+(1-z)^2}{z}\Big(2\log z+1\Big) .
\end{aligned}
$$

That is, when $\mu_0^2 = m^2$, the initial condition at NLO is identical to the LO one.
In flavour space, we can collect the scheme-changing terms in a matrix

$$
\mathbb{K}^{(\Delta)} = \begin{pmatrix} K_{ee}^{(\Delta)}(z) & 0 & 0 \\ K_{\gamma e}^{(\Delta)}(z) & 0 & K_{\gamma e}^{(\Delta)}(z) \\ 0 & 0 & K_{ee}^{(\Delta)}(z) \end{pmatrix} .
$$

Numerical solution of DGLAP at NLL

In our previous works, we solved DGLAP in Mellin space by introducing an evolution operator

$$\Gamma_N(\mu) = \mathbb{E}_N(\mu, \mu_0)\, \Gamma_{0,N}(\mu_0)\,, \qquad \mathbb{E}_N(\mu_0, \mu_0) = I\,.$$

and writing an equation for this operator. In $\overline{\text{MS}}$ we have

$$\begin{aligned} \frac{\partial \mathbb{E}_N(\mu, \mu_0)}{\partial \log \mu^2} &= \sum_{k=0}^{\infty} \left(\frac{\alpha(\mu)}{2\pi}\right)^{k+1} \mathbb{P}_N^{[k]}\, \mathbb{E}_N(\mu, \mu_0) \\ &= \frac{\alpha(\mu)}{2\pi} \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi}\, \mathbb{P}_N^{[1]}\right] \mathbb{E}_N(\mu, \mu_0) + \mathcal{O}(\alpha^2)\,. \end{aligned}$$

which can be solved in closed form for the non-singlet, or via path-ordering methods in the general case (also in the presence of running α). At the end, an inverse Mellin transform is used to obtain the x-space result.

Evolution in a generic scheme

Equation for evolutor becomes

$$\frac{\partial \mathbb{E}_{K,N}(\mu^2,\mu_0^2)}{\partial \log \mu^2} = \frac{\beta(\alpha(\mu))}{2\pi}\mathbb{K}_N \left(I + \frac{\alpha(\mu)}{2\pi}\mathbb{K}_N\right)^{-1} \mathbb{E}_{K,N}(\mu^2,\mu_0^2) + \sum_{k=0}^{\infty}\left(\frac{\alpha(\mu)}{2\pi}\right)^{k+1}\left(I + \frac{\alpha(\mu)}{2\pi}\mathbb{K}_N\right)\mathbb{P}_N^{[k]}\left(I + \frac{\alpha(\mu)}{2\pi}\mathbb{K}_N\right)^{-1}\mathbb{E}_{K,N}(\mu^2,\mu_0^2)$$

In the Δ scheme, it is safe to expand perturbatively the second term, but *not* the first one (it leads to a non-integrable singularity at $z = 1$). In x-space, we have

$$\frac{\partial F_K(\mu^2)}{\partial \log \mu^2} = \mathbb{O}(\mu^2,\mu_0^2) \otimes F_K(\mu^2)$$

$$\mathbb{O}(\mu^2,\mu_0^2) = M^{-1}\left[\frac{\partial \mathbb{E}_{K,N}(\mu^2,\mu_0^2)}{\partial \log \mu^2}\left(\mathbb{E}_{K,N}(\mu^2,\mu_0^2)\right)^{-1}\right]$$

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We applied all the preceding formalism and NLL ingredients to recast the *non-singlet* DGLAP evolution as an exclusive generation of photons. Crucial ingredients are

- the expression of $\mathbb{O}(\mu^2, \mu_0^2)$ in the $\overline{\text{MS}}$ and Δ schemes:

$$\mathbb{O}(\mu^2, \mu_0^2) = \frac{\beta(\alpha(\mu^2))}{2\pi} M^{-1} \left[\mathbb{K}_N \left(I + \frac{\alpha(\mu)}{2\pi} \mathbb{K}_N \right)^{-1} \right] + \frac{\alpha(\mu^2)}{2\pi} M^{-1} \left[\mathbb{P}_N^{[0]} \right] + \left(\frac{\alpha(\mu^2)}{2\pi} \right)^2 M^{-1} \left[\mathbb{P}_N^{[1]} + \left[\mathbb{K}, \mathbb{P}_N^{[0]} \right] \right]$$

  (in the non-singlet, everything is scalar and the last commutator is zero)

- the expression for the NLL Sudakov in both schemes i.e.

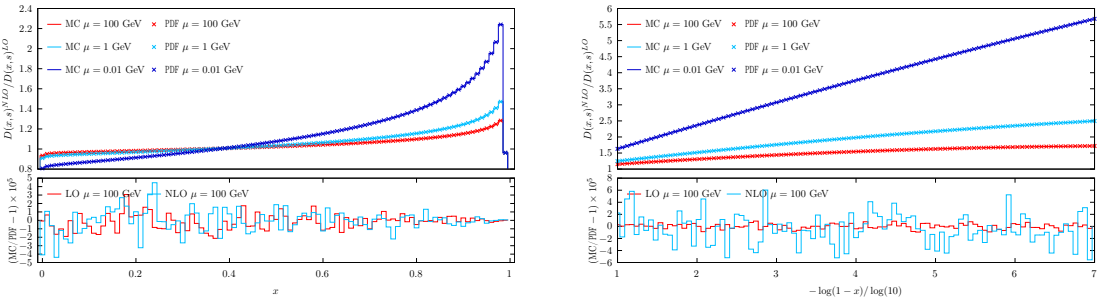
$$S(\mu^2) = \exp \int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2} \overline{B}(\kappa^2)$$

- initial conditions, running of α, ...

Recovering PDFs

PRELIMINARY

Simplest case: $\overline{\text{MS}}$ **(initial condition and evolution), fixed** $\alpha \sim 1/137$

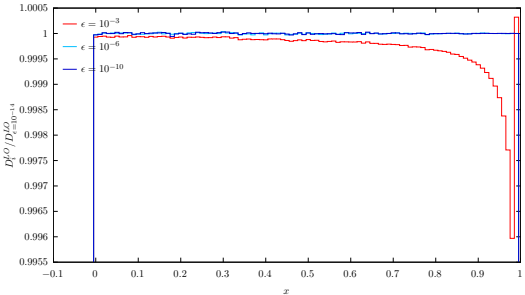


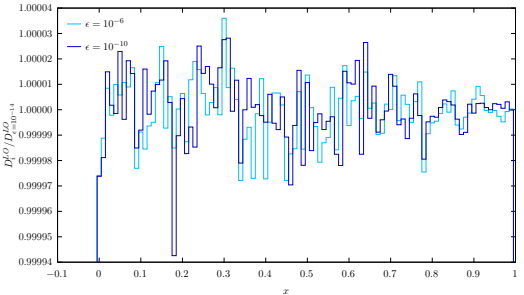
Comparing the original PDF with the MC implementation in the linear and logarithmic (close to 1) ranges.

Recovering PDFs

PRELIMINARY

Simplest case: $\overline{\text{MS}}$ **(initial condition and evolution), fixed** $\alpha \sim 1/137$



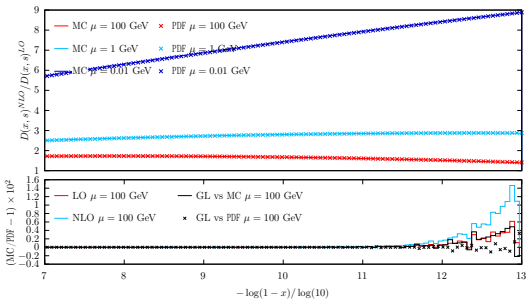


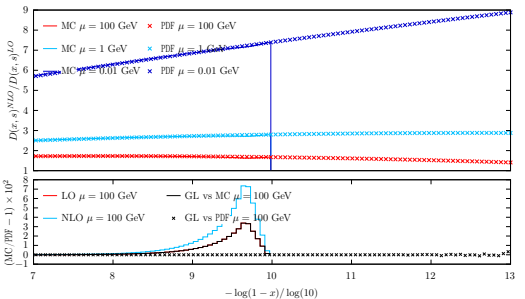
Study of ϵ independence in the linear range

Recovering PDFs

PRELIMINARY

Simplest case: $\overline{\text{MS}}$ **(initial condition and evolution), fixed** $\alpha \sim 1/137$



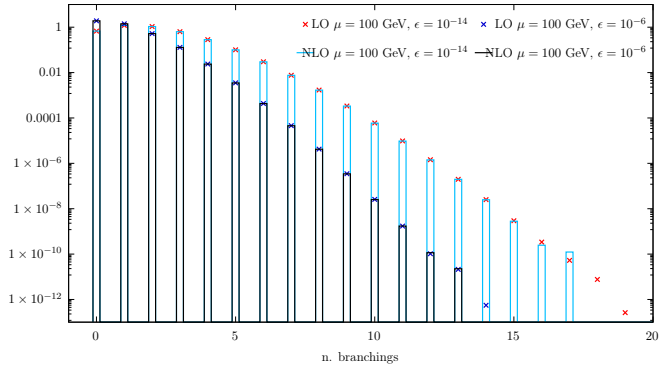


Study of ϵ independence in the logarithmic (close to 1) range
Left: $\epsilon = 10^{-14}$. Right: $\epsilon = 10^{-10}$

Recovering PDFs

PRELIMINARY

Simplest case: $\overline{\text{MS}}$ **(initial condition and evolution), fixed** $\alpha \sim 1/137$



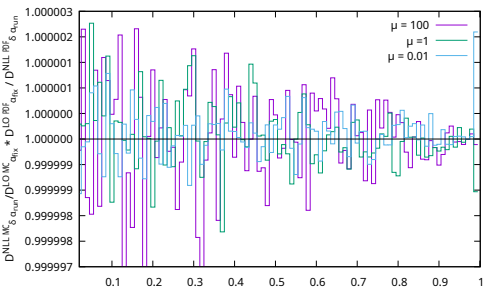
Access to more exclusive quantities e.g. the number of branchings

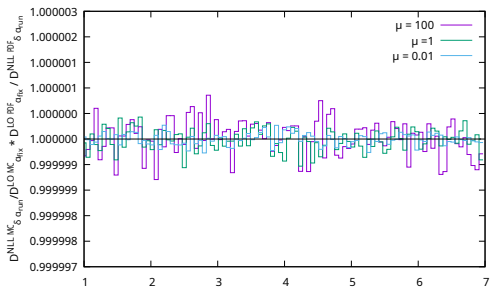
Recovering PDFs

PRELIMINARY

Same quality of plots also for:

- Δ **scheme, fixed** $\alpha \sim 1/137$ (in non-singlet with fixed α, the evolution operator is the same as $\overline{\text{MS}}$; so the only difference wrt. $\overline{\text{MS}}$ with fixed α is the initial condition)
- physical $\overline{\text{MS}}$ **and** Δ **scheme, with running** α





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Summary

Final goal: being able to profit from the higher accuracy achieved in the description of ISR, but at the same time being able to apply fiducial experimental cuts

- We recast the DGLAP equation in a form suitable for a Monte Carlo implementation
- We calculated all the theoretical ingredients to promote ISR to NLL
- We are able to reproduce the original NLL PDFs in different factorization schemes (so far limited to the non-singlet solution, with strictly collinear radiation)

Work in progress and open questions

- Better study cutoff dependence
  (One can possibly exploit the ambiguity in the definition of the Sudakov)
- Move to the full evolution (possibly including secondary $\gamma \to f + \bar{f}$ splittings)
- Add transverse degrees of freedom while preserving NLL collinear accuracy
  - In principle, the physical meaning of μ^2 is not limited to virtuality.
    Relative transverse momentum? With some rapidity or angular dependence?
- What do we require to claim NLL accuracy?
  (Apart from the basic requirement of recovering NLL PDFs)
- Matching to fixed-order? In this context, (NLO+)NLL accuracy of ISR would also require reproducing known analytical results for cross sections
- Explore the approach used by the latest versions of BabaYaga?
  (generate all photons and then assign them to legs)