QED structure functions at NNLO

Gearing Up for Future Colliders, 2026

Funded by
the European Union

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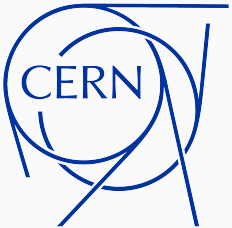
CERN, June 30, 2026



Outline

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Exploiting the Renormalization Group

From Radiators to PDFs

Motivation

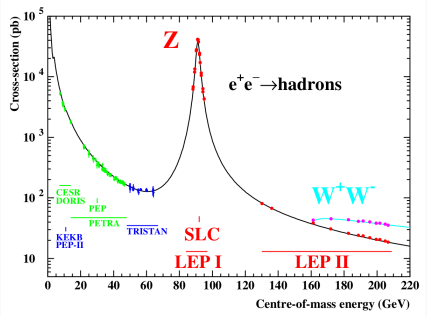
Motivation

- Corrections due to initial state radiation (ISR) can be large, especially due to large logarithmic corrections

$$L = \ln(m_Z^2/m_e^2) \approx 24.$$

- These corrections are important e.g.
  - for the prediction of the Z-boson peak
  - for $t\,\bar{t}$ production
  - associated Higgs production

  at future e^+ e^- colliders.



Factorization of Initial State Radiation

Factorization of Initial State Radiation

The initial state radiation factorizes from the born cross section:

$$\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}z} = \sum_{l,k} \underbrace{\Gamma_{li}\left(z, \frac{\mu^2}{m_e^2}\right)}_{\text{'parton distributions'}} \otimes \underbrace{\tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right)}_{\text{'hard scattering'}} \otimes \underbrace{\Gamma_{kj}\left(z, \frac{\mu^2}{m_e^2}\right)}_{\text{'parton distributions'}} + O\left(\frac{m_e^2}{s}\right)$$

with $z = s'/s$, μ the factorization scale, into:

- massless hard-scattering cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$
- process independent 'parton distributions' $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence

$e^+ e^- \to Z^*/\gamma^*$:
massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij}$ known up to three-loop order

[Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]

[Duhr, Dulat, Mistlberger (Phys.Rev.Lett. 125 (2020))]

Factorization of Initial State Radiation

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Mellin transform:

$$\Gamma_{ij}\left(N, \frac{\mu^2}{m_e^2}\right) = \int_0^1 \mathrm{d}z\, z^{N-1}\, \Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$$

Fixed Order Results

Factorization of Initial State Radiation

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}z} = \sum_{l,k} \Gamma_{li} \cdot \tilde{\sigma}_{lk} \cdot \Gamma_{kj} = \sigma^{(0)}(z) \otimes \underbrace{H_{e^+e^-}(z, L)}_{\text{'radiator'}} = \sigma^{(0)}(z) \otimes \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left(\frac{\alpha}{4\pi}\right)^i L^k c_{i,k}$$

with

$$L = \ln(m_e^2/s) \sim -24$$

$$L\frac{\alpha}{\pi} \sim -0.056$$

Original two-loop calculation for LEP:

- Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))
  - Full calculation with massive electrons in the limit $m_e^2 \ll s$.
  - Result implemented for LEP analysis in e.g.
    - `TOPAZ0` [Montagna, Nicrosini, Piccinini, Passarino (CPC 117 (1999))]
    - `ZFITTER` [Bardin et al (CPC 133 (2001))]
    - ...

- 1988: Direct calculation of the radiator up to $O(\alpha^2)$.
  [Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))]
- 2012: New calculation up to $O(\alpha^2)$ using the method of massive operator matrix elements.
  [Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))]
  Calculation of the last missing operator matrix element (OME) at $O(\alpha^2)$:
  [Ablinger, Bümlein, de Freitas, KS (Nucl.Phys.B 955 (2020))]

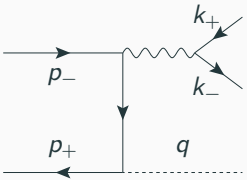
$\Rightarrow$ Calculations do not agree at $O(\alpha^2 L^0)$!

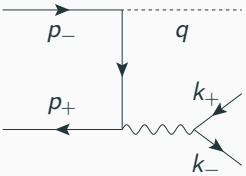
- Errors in one of the calculations?
- Breakdown of factorization?

Validation

- We revisited the original calculation, doing the expansion in m_e at the latest stage [Blümlein, De Freitas, Raab, KS (Nucl. Phys. B956 (2020))] and found agreement with [Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))] .
- In Berends et al the $\mathcal{O}(\alpha^2)$ corrections have been split up into four distinct processes:
  - Process I, photon radiation ✓
  - Process II, non-singlet fermion pair production ✗
  - Process III, pure-singlet fermion pair production ✗
  - Process IV, interference between non-singlet and pure-singlet fermion pair production ✓
- In the following we will present some details on process II.

The Non-Singlet Case





- The two diagrams given above are contributing to the non-singlet process.
- The electron pair in the final state completely factorizes from the phase space integration.
- This property allows to derive a compact one dimensional integral representation.

The Non-Singlet Case

- The non singlet contribution can be given by

$$\frac{d\sigma^{(2),\mathrm{II}}(z)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \int_{4m_e{}^2}^{s(1-\sqrt{z})^2} ds'' \frac{16}{3s\,s''{}^2}\sqrt{1-\frac{4m_e{}^2}{s''}}(2m_e{}^2+s'')\Bigg\{$$
$$-\frac{\lambda^{1/2}(s,s',s'')\left[2s\,s'\,s''+m_e{}^2\left(s^2+(s'-s'')^2\right)+4s\,m_e{}^4\right]}{s\,s'\,s''+m_e{}^2\left(s^2+(s'-s'')^2-2s\,(s'+s'')\right)}$$
$$+\frac{(s'+s'')^2+4m_e{}^2(s-s'-s'')+s^2-8m_e{}^4}{\beta(s-s'-s'')}\ln\left(\frac{s-s'-s''+\beta\lambda^{1/2}(s,s',s'')}{s-s'-s''-\beta\lambda^{1/2}(s,s',s'')}\right)\Bigg\}$$

with

$$\beta=\sqrt{1-\frac{4m_e{}^2}{s}}, \qquad \lambda(s,s',s'')=s^2+s'{}^2+s''{}^2-2ss'-2s's''-2ss''.$$

- The two square roots cannot be rationalized simultaneously.

The Non-Singlet Case

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} &= \frac{\sigma^{(0)}(s')}{s}\,a^2\left\{\frac{64}{3}z(1-z)(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_7}+\frac{256}{3}z\rho(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_6}\right.\\
&+\frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2}\tilde{\mathrm{H}}_{d_8,d_7}\\
&+\frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3}\tilde{\mathrm{H}}_{d_8,d_6}\\
&+\frac{16}{9(1-z)^2}\Big[(1+z)^2\big(4-9z+4z^2\big)+2\big(9-16z+13z^2-2z^3\big)\rho+32\rho^2\Big]\tilde{\mathrm{H}}_{d_2}\\
&+\frac{512z\rho}{9(1-z)^4}\Big[3(1-z)^4z-(1-z)^3\big(4+z^2\big)\rho-2\big(9-29z+38z^2-17z^3+3z^4\big)\rho^2\\
&-4(2-z)\big(3+6z-5z^2\big)\rho^3+16\big(7-8z+9z^2\big)\rho^4+128(3-z)\rho^5\Big]\tilde{\mathrm{H}}_{d_4}\\
&-\frac{16}{9(1-z)^4}\Big[3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8\big(2-16z+9z^2\\
&+4z^3-5z^4+2z^5\big)\rho+16z\big(12-13z+18z^2-z^3\big)\rho^2+32\big(1+22z-7z^2\big)\rho^3\Big]\tilde{\mathrm{H}}_{d_1}\\
&-\frac{128z}{9(1-z)^4}\Big[1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2\big(7-55z+54z^2\\
&+16z^3-17z^4+3z^5\big)\rho-4\big(39-16z+16z^2+4z^3+5z^4\big)\rho^2\\
&+16\big(8-23z+22z^2+9z^3\big)\rho^3+128\big(7+2z-z^2\big)\rho^4\Big]\tilde{\mathrm{H}}_{d_5}-\frac{64}{3}(2z+(1-z)\rho)\tilde{\mathrm{H}}_{d_3}\\
&+\left[\frac{16}{3\sqrt{1-4\rho}}(1+z-4\rho)\tilde{\mathrm{H}}_{v_4}+\frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}}\tilde{\mathrm{H}}_{d_8}\right]\\
&\times\ln\left(\frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}\right)\Bigg\}
\end{aligned}
$$

The Non-Singlet Case

- The explicit expansion of the analytical result in the limit $m_e^2 \ll s$ gives

$$\begin{aligned}\frac{d\sigma^{(2),\mathrm{II}}(z)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ & \frac{8}{3}\frac{1+z^2}{1-z}L^2 - \left[\frac{16}{9}\frac{11-12z+11z^2}{1-z} - \frac{16}{3}\frac{1+z^2}{1-z}\mathrm{H}_0\right. \\ & \left. - \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1\right] L + \frac{32}{9(1-z)^3}\left(7-13z+8z^2-13z^3+7z^4\right) \\ & - \frac{16z}{9(1-z)^4}\left(3-36z+94z^2-72z^3+19z^4\right)\mathrm{H}_0 - \frac{8z^2}{3(1-z)}\mathrm{H}_0^2 \\ & + \left(\frac{32}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{2+z^2}{1-z}\mathrm{H}_0\right)\mathrm{H}_1 + \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1^2 + \frac{16z^2}{3(1-z)}\mathrm{H}_{0,1} \\ & - \frac{16\left(2+3z^2\right)}{3(1-z)}\zeta_2\Bigg\} + \mathcal{O}\left(\frac{m_e^2}{s}\right),\end{aligned}$$

  with H the harmonic polylogarithms evaluated at argument z.

- The result contains higher denominator powers which have not been obtained by Berends et al.
- Where does this disagreement come from?

The Non-Singlet Case

Where does this disagreement come from?

- The disagreement can be traced back to neglected of initial state masses.
- The non-singlet phase space for particles with different masses in the initial and final state is given by

$$\frac{d\sigma^{(2),\mathrm{II}}}{ds'} = \frac{\sigma^0(s')}{s}\, a^2 \int_{4m_f{}^2}^{s(1-\sqrt{z})^2} ds'' \frac{16}{3s\, s''{}^2}\sqrt{1-\frac{4m_f{}^2}{s''}}(2m_f{}^2+s'')\Bigg\{$$
$$-\frac{\lambda^{1/2}(s,s',s'')\left[2s\,s'\,s''+m_i{}^2\left(s^2+(s'-s'')^2\right)+4s\,m_i{}^4\right]}{s\,s'\,s''+m_i^2\left(s^2+(s'-s'')^2-2s\,(s'+s'')\right)}$$
$$+\frac{(s'+s'')^2+4m_i{}^2(s-s'-s'')+s^2-8m_i{}^4}{\beta(s-s'-s'')}\ln\left(\frac{s-s'-s''+\beta\lambda^{1/2}(s,s',s'')}{s-s'-s''-\beta\lambda^{1/2}(s,s',s'')}\right)\Bigg\},$$

with $\beta=\sqrt{1-4m_i{}^2/s}$.

The Non-Singlet Case

Where does this disagreement come from?

- The disagreement can be traced back to neglected of initial state masses.
- Neglecting initial state masses this expression reduces to

$$\frac{d\sigma^{(2),\mathrm{II}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\, a^2 \int_{4m_f{}^2}^{s(1-\sqrt{z})^2} ds'' \frac{16}{3s\, s''{}^2} \sqrt{1-\frac{4m_f{}^2}{s''}} (2m_f{}^2 + s'') \Bigg\{$$

$$- 2\lambda^{1/2}(s,s',s'') + \frac{s^2+(s'+s'')^2}{s-s'-s''} \ln\left(\frac{s-s'-s''+\lambda^{1/2}(s,s',s'')}{s-s'-s''-\lambda^{1/2}(s,s',s'')}\right)\Bigg\}.$$

- This formula is the starting point of Berends et al.
  (It was derived in Kniehl et al, Phys. Lett. B209 (1988).)
- It is valid for the production of heavy particles, like muons, but not for electron pair production.

The Non-Singlet Case

- For the non-singlet term one finds the difference term:

$$\delta_{II} = \frac{8}{3}\int_0^1 \frac{\mathrm{d}y}{y}\sqrt{1-y}(2+y)\left[\frac{(1-z)(1-(4-z)z)y}{4z+(1-z^2)y} - \frac{1+z^2}{1-z}\ln\left(1+\frac{(1-z)^2 y}{4z}\right)\right]$$

$$= -\frac{128}{9}\left[3+\frac{1}{(1-z)^3}-\frac{2}{(1-z)^2}-2z\right] - 16\left[1+\frac{5z}{3}+\frac{8}{9}\frac{1}{(1-z)^4}-\frac{20}{9}\frac{1}{(1-z)^3}\right.$$

$$\left.+\frac{4}{9}\frac{1}{(1-z)^2}\right]\ln(z) + \frac{8}{3}\frac{1+z^2}{1-z}\left[\frac{10}{9}-\frac{14}{3}\ln(z)-\ln^2(z)\right],$$

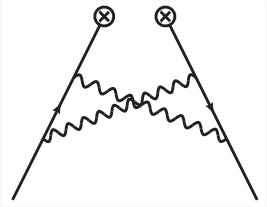
- In this case the difference can be completely attributed to the neglection of initial state electron masses.

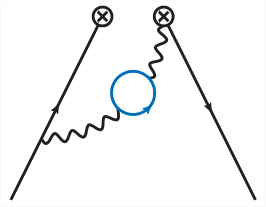
$\Rightarrow$ Our results agree with the ones obtained through factorization.

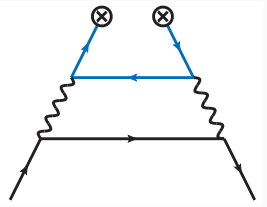
New caculations

Recently our calculations have been validated in a framework based in soft collinear effective theory (SCET):

- [Stahlhofen; JHEP 11 (2025) 171]
  - Recalculation of Γ_{ee} and $\Gamma_{\gamma e}$.
  - Added the contributions of a second heavy lepton flavor with different mass.
- [Schnubel, Szafron; JHEP 12 (2025) 167]
  - Recalculation of all OMEs (also the ones with massless external partons).
  - Independent check of OMEs used for the VFNS in PDF fits [Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07,'09] .







Exploiting the Renormalization Group

Renormalization Group Equations

Factorization allows for separate renormalization group equations:

- massless cross sections $\tilde{\sigma}_{ij}$

$$\left[\left(\frac{\partial}{\partial\lambda} - \beta(a)\frac{\partial}{\partial a}\right)\delta_{kl}\delta_{jm} + \frac{1}{2}\gamma_{kl}(N)\delta_{jm} + \frac{1}{2}\gamma_{jm}(N)\delta_{kl}\right]\tilde{\sigma}_{lj}(N) = 0$$

- massive operator matrix elements Γ_{ij}

$$\left[\left(\frac{\partial}{\partial\Lambda} + \beta(a)\frac{\partial}{\partial a}\right)\delta_{jl} + \frac{1}{2}\gamma_{kl}(N)\right]\Gamma_{li}(N) = 0$$

with $\lambda = \ln(s'/\mu^2)$, $\Lambda = \ln(\mu^2/m_e^2)$, the QED β-function $\beta(a)$ and $a = \alpha/4\pi$

- Here the usual anomalous dimensions, i.e. Mellin transforms of the splitting functions, contribute:

$$\gamma_{ij}(N) = -\int_0^1 \mathrm{d}z\; z^{N-1}P_{ij}(z)$$

Higher order results for the radiators

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}z} = \sum_{l,k} \Gamma_{li} \otimes \tilde{\sigma}_{lk} \otimes \Gamma_{kj} \quad = \sigma^{(0)}(z) \otimes \underbrace{H_{e^+e^-}(z, L)}_{\text{'radiator'}} = \sigma^{(0)}(z) \otimes \sum_{i=0}^{\infty} \sum_{k=0}^{i} a^i \, L^k \, \boxed{c_{i,k}}$$

The radiators:

$$c_{1,1} = -\gamma_{ee}^{(0)},$$

$$c_{1,0} = \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)},$$

$$c_{2,2} = \frac{1}{2}\gamma_{ee}^{(0)2} + \frac{\beta_0}{2}\gamma_{ee}^{(0)} + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)},$$

$$\dots$$

$$\begin{aligned} c_{3,1} = {} & -\gamma_{ee}^{(2)} - 2\Gamma_{ee}^{(0)}\gamma_{ee}^{(1)} - \Gamma_{ee}^{(0)}\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(1)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(1)} - \beta_1\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(1)}\tilde{\sigma}_{ee}^{(0)} \\ & - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{ee}^{(0)} - 2\Gamma_{ee}^{(0)}\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(1)}\tilde{\sigma}_{e\gamma}^{(0)} - \Gamma_{\gamma e}^{(0)}\gamma_{\gamma\gamma}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{\gamma e}^{(1)} + \beta_0\Big[-2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} \\ & - 2\tilde{\sigma}_{ee}^{(1)} - 2\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\Big] - \gamma_{ee}^{(0)}\left[\Gamma_{ee}^{(0)2} + 2\Gamma_{ee}^{(1)} + 2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} + \tilde{\sigma}_{ee}^{(1)} + \Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\right], \end{aligned}$$

$$\dots$$

The Radiators

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} \underbrace{H_{e^+e^-}(z, L)}_{\text{'radiators'}} = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^{i} a^i L^k c_{i,k}$$

- The radiators do not depend on the factorization scale
  $\rightarrow$ there are no collinear singularities for massive electrons
- The analytic structures directly translate to the different ingredients
- Radiators are distributions in z-space:

$$c_{i,j}(z) = c_{i,j}^{\delta}\delta(1-z) + c_{i,j}^{+} + c_{i,j}^{\mathrm{reg}}$$

$$c_{3,3}^{\delta} = \frac{572}{9} - \frac{704}{3}\zeta_2 + \frac{512}{3}\zeta_3,$$

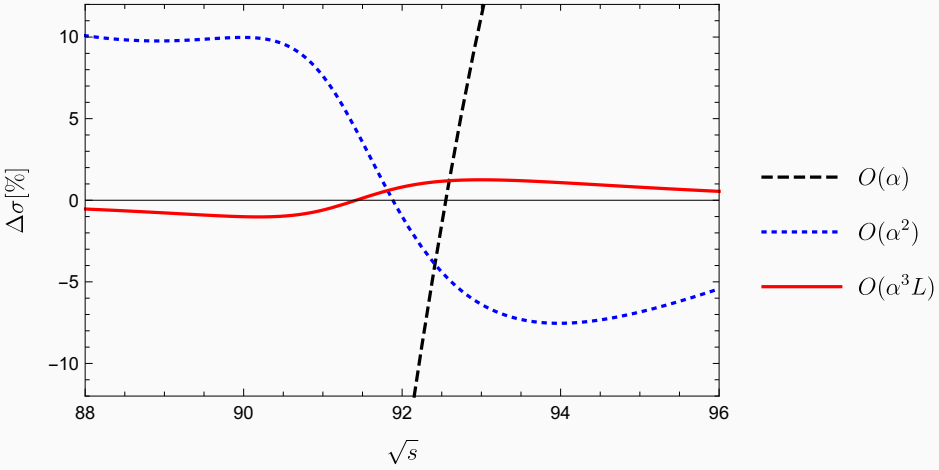
$$c_{3,3}^{+} = \left(\frac{5744}{27} - 256\zeta_2\right)\mathcal{D}_0 + \frac{1408}{3}\mathcal{D}_1 + 256\mathcal{D}_2,$$

$$\mathcal{D}_k = \left(\frac{\ln^k(1-z)}{1-z}\right)_+,$$

$$c_{3,3}^{\mathrm{reg}} = \left\{ \frac{16\mathrm{H}_0 P_{104}}{9(z-1)} - \frac{4P_{131}}{27z} + \frac{8(3-19z^2)\mathrm{H}_0^2}{3(z-1)} + \left[\frac{16P_{105}}{9z} - \frac{128(1+z^2)\mathrm{H}_0}{z-1}\right]\mathrm{H}_1 - 128(1+z)\mathrm{H}_1^2 - \frac{352}{3}(1+z)\mathrm{H}_{0,1} + \frac{736}{3}(1+z)\zeta_2 \right\}$$

Numerical Results

- $\Delta\sigma$ is the change in the total cross section from one order to the other for $z_0 = 4m_\tau^2$



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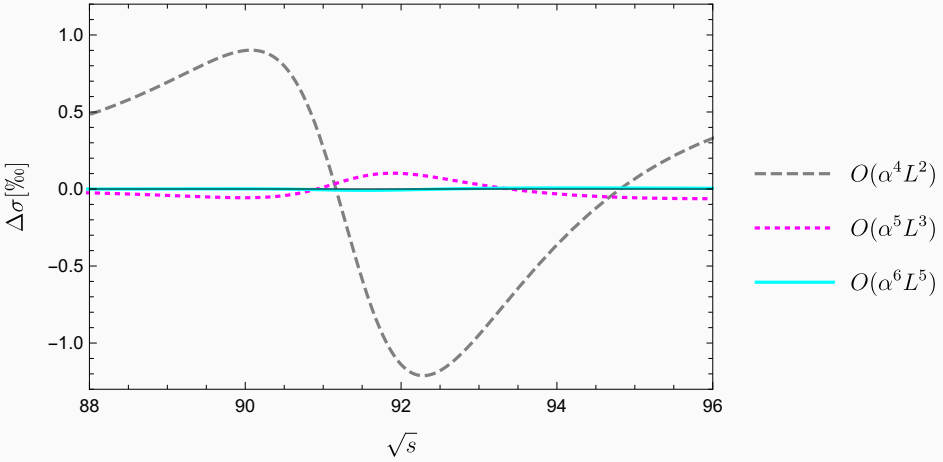


From Radiators to PDFs

From Radiators to PDFs

$$\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}z} = \sum_{l,k} \underbrace{\Gamma_{li}\left(N, \frac{\mu^2}{m_e^2}\right)}_{\text{'parton distributions'}} \cdot \underbrace{\tilde{\sigma}_{lk}\left(N, \frac{s'}{\mu^2}\right)}_{\text{'hard scattering'}} \cdot \underbrace{\Gamma_{kj}\left(N, \frac{\mu^2}{m_e^2}\right)}_{\text{'parton distributions'}} + O\left(\frac{m_e^2}{s}\right)$$

- Radiators are useful, when convolutions can be performed analytically.
- For complicated processes hard scattering will only be known numerically.
- Exploit the collinear factorization and treat Γ_{ij} as parton distribution functions:
  1. $L = \ln(m_e^2/\mu^2)$ can be resummed by DGLAP equations.
     NLL: [Bertone, Cacciari, Frixione, Stagnitto '19]
  2. We have freedom in choosing an arbitrary factorization scheme.
     NLL: [Frixione '21]
  3. We need to handle the limit $z \to 1$ precisely.

Factorization schemes

$$\frac{\mathrm{d}\sigma_{e^+e^-}}{\mathrm{d}z} = \Gamma_e^{\mathrm{T}}\left(N, \frac{\mu^2}{m_e^2}\right) \cdot \tilde{\sigma}\left(N, \frac{s'}{\mu^2}\right) \cdot \Gamma_e\left(N, \frac{\mu^2}{m_e^2}\right)$$

- Rotation matrix R is (in principle) arbitrary.
- $\overline{\mathrm{MS}}$: usual factorization scheme, only subtract $1/\epsilon$ poles
- Δ': demand that $\Gamma_e^{\Delta} \overset{N\to\infty}{=} \mathcal{O}(N^{-1}) \overset{z\to 1}{=} \mathcal{O}((1-z)^0)$

$$\Gamma_e = \begin{pmatrix} \Gamma_{ee}^{\mathrm{NS}} \\ \Gamma_{ee}^{\mathrm{S}} \\ \Gamma_{\gamma e} \end{pmatrix}$$

Note that:

$$\ln(\bar{N}) \overset{z\to 1}{=} -\left[\frac{1}{1-z}\right]_+ ,$$

$$\ln^2(\bar{N}) \overset{z\to 1}{=} \left[\frac{2\ln(1-z)}{1-z}\right]_+ - \zeta_2\delta(1-z) \, , \ldots$$

$$\tilde{\sigma} = \begin{pmatrix} \sigma_{ee}^{\mathrm{NS}} & 0 & 0 \\ 0 & \sigma_{ee}^{\mathrm{S}} & \sigma_{\gamma e} \\ 0 & \sigma_{\gamma e} & \sigma_{\gamma\gamma} \end{pmatrix}$$

Factorization schemes

$$\mathrm{d}\sigma_{e^+e^-} = \underbrace{\Gamma_e^{\mathrm{T}}\left(N, \frac{\mu^2}{m_e^2}\right)\cdot R^{\mathrm{T}}(N)}_{\hat{\Gamma}_e^{\mathrm{T}}} \cdot \underbrace{R^{-1\,\mathrm{T}}(N)\cdot\tilde{\sigma}\left(N, \frac{s'}{\mu^2}\right)\cdot R^{-1}(N)}_{\hat{\tilde{\sigma}}} \cdot \underbrace{R(N)\cdot\Gamma_e\left(N, \frac{\mu^2}{m_e^2}\right)}_{\hat{\Gamma}_e}$$

- Rotation matrix R is (in principle) arbitrary.
- $\overline{\mathrm{MS}}$: usual factorization scheme, only subtract $1/\epsilon$ poles
- Δ': demand that $\Gamma_e^{\Delta} \overset{N\to\infty}{=} \mathcal{O}(N^{-1}) \overset{z\to 1}{=} \mathcal{O}((1-z)^0)$

$$\Gamma'_e = R\cdot\begin{pmatrix}\Gamma_{ee}^{\mathrm{NS}}\\ \Gamma_{ee}^{\mathrm{S}}\\ \Gamma_{\gamma e}\end{pmatrix}$$

Note that:

$$\ln(\bar{N}) \overset{z\to 1}{=} -\left[\frac{1}{1-z}\right]_+ ,$$

$$\ln^2(\bar{N}) \overset{z\to 1}{=} \left[\frac{2\ln(1-z)}{1-z}\right]_+ - \zeta_2\delta(1-z) \ , \ldots$$

$$\tilde{\sigma}' = R^{-1\,\mathrm{T}}\cdot\begin{pmatrix}\sigma_{ee}^{\mathrm{NS}} & 0 & 0\\ 0 & \sigma_{ee}^{\mathrm{S}} & \sigma_{\gamma e}\\ 0 & \sigma_{\gamma e} & \sigma_{\gamma\gamma}\end{pmatrix}\cdot R^{-1}$$

Asymptotic MS solution – NLL

Non-singlet and singlet coincide in $z \to 1$ limit; the photon PDF is more complicated

- Asymptotic solution ($z \to 1$) of the non-singlet PDF:

$$\Gamma_{\rm NLL}^{\rm NS} \xrightarrow{z\to 1} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\frac{7}{4} - \frac{\pi^2}{6} - d_2(\xi_1) + \left(L_0 - 1 + 2 d_1(\xi_1) \right) \boxed{\ln(1-z)} - \boxed{\ln^2(1-z)} \right] \right\}$$

$L_0 = \ln(\mu_0^2/m_e^2)$, vanishes for the starting scale $\mu_0 = m_e$

$$d_k(\xi) = \frac{1}{G_d(\xi,0)} \left. \frac{\partial^k G_d(\xi,\delta)}{\partial \delta^k} \right|_{\delta=0}, \quad G_d(\xi,\delta) = \frac{e^{\delta-\gamma_{\rm E}}}{\Gamma(\xi-\delta)}$$

$$\xi_1 = 2t - \frac{\alpha(\mu)}{4\pi^2 b_0} \left(1 - e^{-2\pi b_0 t}\right) \left(\frac{20}{9} + \frac{4\pi b_1}{b_0} \right)$$

$$\hat{\xi}_1 = \frac{3}{2} t + \frac{\alpha(\mu)}{4\pi^2 b_0} \left(1 - e^{-2\pi b_0 t}\right) \left(\frac{5}{24} - \frac{13\pi^2}{18} + 6\zeta_3 - \frac{3\pi b_1}{b_0} \right)$$

$$t = \frac{1}{2\pi b_0} \ln \left(\frac{\alpha(\mu)}{\alpha(\mu_0)} \right)$$

Asymptotic Δ solution – NLL

Non-singlet and singlet coincide in $z \to 1$ limit; the photon PDF is more complicated

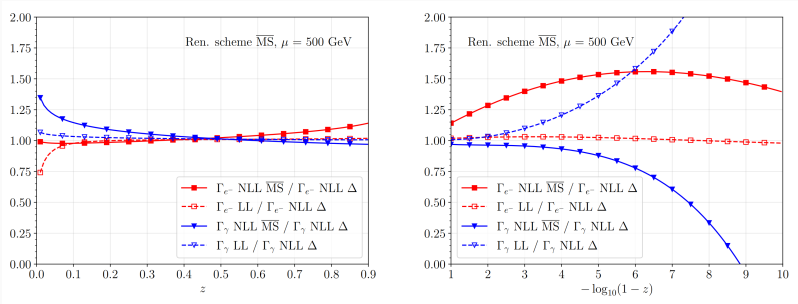
- Asymptotic solution ($z \to 1$) of the non-singlet PDF:

$$\Gamma_{\rm NLL}^{\rm NS,\Delta} \overset{z\to 1}{\to} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \left[\frac{\alpha(\mu)}{\alpha(\mu_0)} + \frac{\alpha(\mu_0)}{\pi} L_0 \left(\frac{3}{4} - d_1(\xi_1) \boxed{\ln(1-z)} \right) \right]$$

- In the Δ scheme the PDFs are finite for $z \to 1$.
- The need for soft resummation is completely shifted to the cross-section.

NLL automatized

- Scheme choice can have large impact on the PDF.



- NLL evolution is automatized in `eMELA` [Bertone, Cacciari, Frixione, Stagnitto, Zhao, Zaro '22] and e.g. linked to `MadGraph5`. [Frixione, Mattelaer, Zaro, Zhao '21]

Asymptotic MS solution – NNLL

Non-singlet and singlet coincide in $z \to 1$ limit; the photon PDF is more complicated

- All ingredients for NNLL evolution are available.
- Asymptotic solution ($z \to 1$) of the non-singlet PDF:

$$\Gamma^{\rm NS}_{\rm NNLL} \overset{z\to 1}{\to} \frac{e^{-\gamma_{\rm E}\xi_2} e^{\hat{\xi}_2}}{\Gamma(1+\xi_2)} \xi_2 (1-z)^{-1+\xi_2} \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\ldots + (L_0 - 1 + 2d_1(\xi_2)) \boxed{\ln(1-z)} - \boxed{\ln^2(1-z)} \right] \right.$$
$$\left. + \left(\frac{\alpha(\mu_0)}{\pi} \right)^2 \left[\ldots \boxed{\ln^2(1-z)} - (L_0 - 1 + 2d_1(\xi_2)) \boxed{\ln^3(1-z)} + \frac{1}{2} \boxed{\ln^4(1-z)} \right] \right\}$$

$$\xi_2 = \xi_1 + \mathcal{O}(\alpha^2)$$
$$\hat{\xi}_2 = \hat{\xi}_1 + \mathcal{O}(\alpha^2)$$

Asymptotic Δ solution – NNLL

Non-singlet and singlet coincide in $z \to 1$ limit; the photon PDF is more complicated

- All ingredients for NNLL evolution are available (ongoing work with Stefano Frixione).
- Asymptotic solution ($z \to 1$) of the non-singlet PDF:

$$\Gamma^{\mathrm{NS},\Delta}_{\mathrm{NNLL}} \overset{z\to 1}{\to} \frac{e^{-\gamma_E \xi_2} e^{\hat{\xi}_2}}{\Gamma(1+\xi_2)} \xi_2 (1-z)^{-1+\xi_2} \left[\left(\frac{\alpha(\mu)}{\alpha(\mu_0)} \right)^2 + L_0 \left(- \boxed{\ln^3(1-z)} + \ldots \right) \right]$$

Impact on the cross-section

Result for the scheme independent radiator: $\mu = \mu_0 = m_e$, $\bar{N} = Ne^{\gamma_E}$

$$H_{e^+e^-} \overset{N\to\infty}{=} 1 + \frac{\alpha(\mu)}{\pi}\left[2\ln(\bar{N}) - 2 + \frac{\pi^2}{3}\right] + \left(\frac{\alpha(\mu)}{\pi}\right)^2\left[-\frac{4}{9}\ln^3(\bar{N}) + \frac{8}{9}\ln^2(\bar{N}) + \left(\frac{2\pi^2}{3} - \frac{164}{27}\right)\ln(\bar{N}) - \frac{73\zeta_3}{18} - \frac{\pi^4}{30} - \frac{31\pi^2}{144} + \frac{3181}{324} - \pi^2\ln(2)\right]$$

- $\overline{\text{MS}}$

$$\Gamma_{ee}^{\text{NS}} \overset{N\to\infty}{=} 1 + \frac{\alpha(\mu)}{\pi}\left[-\ln^2(\bar{N}) + \ln(\bar{N}) + 1 - \frac{\pi^2}{6}\right] + \left(\frac{\alpha(\mu)}{\pi}\right)^2\left[\frac{1}{2}\ln^4(\bar{N}) - \ln^3(\bar{N}) + \left(\frac{1}{6}\pi^2 - \frac{1}{2}\right)\ln^2(\bar{N}) - \left(\frac{1}{6}\pi^2 - \frac{13}{27}\right)\ln(\bar{N}) - \frac{5\zeta_3}{24} - \frac{11\pi^4}{720} - \frac{\pi^2}{48} + \frac{32077}{10368} - \frac{1}{2}\pi^2\ln(2)\right]$$

$$\sigma_{ee}^{\text{NS}} \overset{N\to\infty}{=} 1 + \frac{\alpha(\mu)}{\pi}\left[2\ln^2(\bar{N}) + \frac{2\pi^2}{3} - 4\right] + \left(\frac{\alpha(\mu)}{\pi}\right)^2\left[2\ln^4(\bar{N}) - \frac{4}{9}\ln^3(\bar{N}) + \left(\frac{4}{3}\pi^2 - \frac{82}{9}\right)\ln^2(\bar{N}) - \frac{28}{27}\ln(\bar{N}) - \frac{131\zeta_3}{36} + \frac{23\pi^4}{120} - \frac{361\pi^2}{144} + \frac{2041}{192}\right]$$

Impact on the cross-section

Result for the scheme independent radiator: $\mu = \mu_0 = m_e,\ \bar{N} = Ne^{\gamma_{\mathbb{E}}}$

$$H_{e^+e^-} \overset{N\to\infty}{=} 1 + \frac{\alpha(\mu)}{\pi}\left[2\ln(\bar{N}) - 2 + \frac{\pi^2}{3}\right] + \left(\frac{\alpha(\mu)}{\pi}\right)^2\left[-\frac{4}{9}\ln^3(\bar{N}) + \frac{8}{9}\ln^2(\bar{N}) + \left(\frac{2\pi^2}{3} - \frac{164}{27}\right)\ln(\bar{N}) - \frac{73\zeta_3}{18} - \frac{\pi^4}{30} - \frac{31\pi^2}{144} + \frac{3181}{324} - \pi^2\ln(2)\right]$$

- Δ-scheme

$$\Gamma_{ee}^{\mathrm{NS},\Delta} \overset{N\to\infty}{=} 1$$

$$\sigma_{ee}^{\mathrm{NS},\Delta} \overset{N\to\infty}{=} 1 + \frac{\alpha(\mu)}{\pi}\left[2\ln(\bar{N}) + \frac{\pi^2}{3} - 2\right] + \left(\frac{\alpha(\mu)}{\pi}\right)^2\left[-\frac{4}{9}\ln^3(\bar{N}) + \frac{8}{9}\ln^2(\bar{N}) + \left(\frac{2\pi^2}{3} - \frac{164}{27}\right)\ln(\bar{N}) - \frac{73\zeta_3}{18} - \frac{\pi^4}{30} - \frac{31\pi^2}{144} + \frac{3181}{324} - \pi^2\ln(2)\right]$$

Take-Away

- In the $\overline{\mathrm{MS}}$ scheme we observe a cancellation of leading logs in the large-z behavior between PDFs and cross-sections.
- The cancellations can be avoided by changing the factorization scheme (Δ scheme).
- In the Δ scheme no large-z resummation is necessary for the PDFs.
- Convolutions in the Δ scheme are easier to compute numerically:
  1. PDFs are smooth functions for $z \to 1$.
  2. No large cancellation between PDFs and cross-sections have to be compensated.

Path towards higher precision

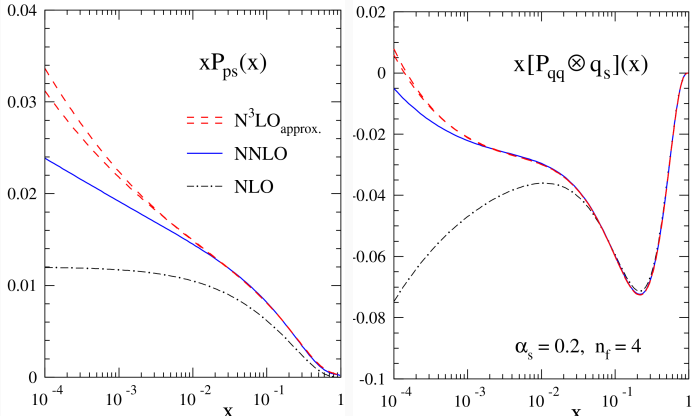
FCC$_{ee}$ wants to measure some observables to sub-permille precision.
How can we improve the structure functions?

- Finish NNLL evolution.
- Include more massive fermion flavors:
  - Relevant starting from NNLO, and already achieved [Stahlhofen '25] .
  - Are they relevant for $e^+ e^-$ colliders?
- Do we need N^{3}LO QED corrections?
  1. The radiators suggest a size which might become of relevance.
  2. The fixed order Drell-Yan cross-sections for Z-pole physics are available.
  3. The calculation is technically feasible.
- We should also consider mixed QED-QCD effects (at NNLO?).
  1. How do we handle the QCD evolution to $\mu \sim m_e$?

Do we need N^{3}LO QED corrections?

- For N^{3}LO ISR we need:
  1. The OMEs (initial conditions) to N^{3}LO (massive three loop calculation).
  2. The splitting functions to N^{3}LO (massless four loop calculation).
- For the full splitting functions approximations based on moments and physical constraints are available [Moch,Ruijl,Ueda,Vermaseren,Vogt '17, '23; Falcioni, Herzog, Moch, Pelloni, Vogt '25] .
- Uncertainties for small x not so important for QCD, but might be significant for ISR.
- Full results for NS became available recently [Gehrmann, Manteuffel, Sotnikov, Yang '26] .

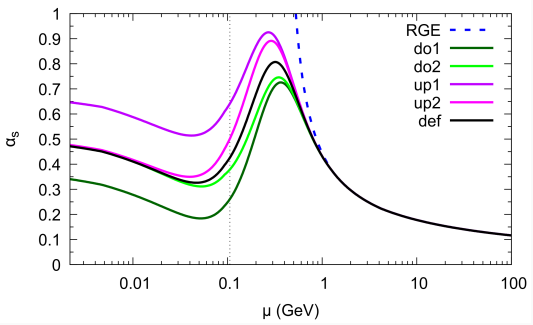
$$f(x) \otimes g(x) = \int_x^1 \frac{\mathrm{d}z}{z} g(z) f\left(\frac{x}{z}\right)$$



Mixed QCD-QED effects

- Introducing mixed QED-QCD effects requires the evolution of α_s down to $\mu \sim m_e$.
- Two approaches in the literature:
  1. $\alpha_s(\mu) = 0$ for $\mu < Q_0$ with $Q_0 = \mathcal{O}(0.5\,\text{GeV})$
     [Han, Ma, Xie '21; Garosi, Marzocca, Trifinopoulos '23]
  2. Use a parametrized running of α_s in the infrared [Fixione, Stagnitto '23]
     (inspired by dispersion relations [Webber; Dokshitzer, Marchesini, Webber '95])
- Rather large differences between the two schemes.

Taken from [Fixione, Stagnitto '23] for a TeV muon collider:



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- Rather large differences between the two schemes.

Taken from [Fixione, Stagnitto '23] for a TeV muon collider:

$\sigma(p_T^{cut} = 10\ \text{GeV})$ [pb]	an.	tr.
$\mathcal{O}(\alpha_s^2)$	$18.33^{+1.30\%}_{-1.25\%}$	$15.00^{+10.23\%}_{-10.99\%}$
γ-ind.	$8.24^{+0.68\%}_{-0.91\%}$	$7.56^{+3.71\%}_{-3.75\%}$
total	$26.58^{+1.11\%}_{-1.15\%}$	$22.57^{+8.04\%}_{-8.56\%}$

Table 4: Total dijet rates for $p_T^{cut} = 10$ GeV, in pb.

$\sigma(p_T^{cut} = 100\ \text{GeV})$ [fb]	an.	tr.
$\mathcal{O}(\alpha_s^2)$	$41.38^{+0.03\%}_{-0.03\%}$	$41.17^{+0.46\%}_{-0.85\%}$
γ-ind.	$90.03^{+0.01\%}_{-0.02\%}$	$89.67^{+0.24\%}_{-0.32\%}$
total	$136.91^{+0.00\%}_{-0.00\%}$	$136.35^{+0.28\%}_{-0.48\%}$

Table 5: Total dijet rates for $p_T^{cut} = 100$ GeV, in fb.

Conclusions and Outlook

Conclusions

- Collinear factorization allows to define perturbative parton distributions for e^+e^- collisions.
- Renormalization group equations allow to resum large logarithms $L = \ln(\mu^2/m_e^2)$.
- NLL evolution already publically available.
- Inputs at NNLO available and soon NNLL evolution.

Outlook

- How to improve precision further?
  1. Going to higher orders in QED.
  2. Including mixed QED-QCD effects.
  3. Can we add soft resummation on top of collinear resummation?

Backup

Technical Details

Mellin Space Calculations

- Usually it is more convenient to work in Mellin space:

$$M[f(z)](N) = \int_0^1 \mathrm{d}z z^{N-1} f(z)$$

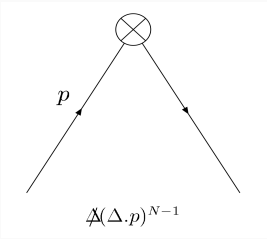
- Here the convolution $\otimes$

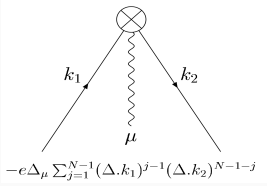
$$f(z) \otimes g(z) = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 f(z_1) g(z_2) \delta(z - z_1 z_2)$$

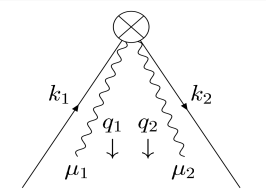
factorizes:

$$M[f(z) \otimes g(z)](N) = M[f(z)](N) \cdot M[g(z)](N)$$

Factorization in the Asymptotic Region





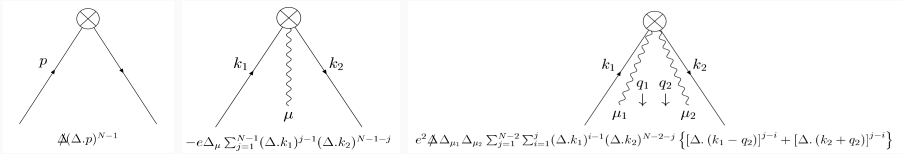


$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e| O_F^{\mathsf{NS,S}} |e\rangle ,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma| O_F^{\mathsf{S}} |\gamma\rangle ,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e| O_V^{\mathsf{S}} |e\rangle ,$$

$$O_{F;\mu_1,\ldots,\mu_N}^{\mathsf{NS,S}} = i^{N-1}\mathsf{S}\left[\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N}\psi\right] - \text{traces},$$
$$O_{V;\mu_1,\ldots,\mu_N}^{\mathsf{S}} = 2i^{N-2}\mathsf{S}\left[F_{\mu_1\alpha} D_{\mu_2} \ldots D_{\mu_{N-1}} F_{\mu_N}^{\alpha}\right] - \text{traces}$$

- The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$.
- In the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process
    Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))

The Missing Operator Matrix Element $\Gamma_{\gamma e}$



$\not{\Delta}(\Delta.p)^{N-1}$ $\quad -e\not{\Delta}\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$ $\quad e^2\not{\Delta}\Delta_{\mu_1}\Delta_{\mu_2}\sum_{j=1}^{N-2}\sum_{i=1}^{j}(\Delta.k_1)^{i-1}(\Delta.k_2)^{N-2-j}\left\{[\Delta.(k_1-q_2)]^{j-i}+[\Delta.(k_2+q_2)]^{j-i}\right\}$ $\quad \dots$

- We have to compute on-shell 2-point functions with local operator insertions ($\Delta^2 = 0$).
- The operator can be resummed into a propagator like term:

$$\sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\Delta.k}.$$

- The calculation can now follow standard techniques:
  - Integration-By-Parts reduction to master integrals.
  - Calculation of the master integrals via differential equations in the resummation variable t.
  - We find the Mellin-space expression by symbolically computing the N-th derivative.
- For the calculations we make use of the packages `Sigma` [C. Schneider '07] and `HarmonicSums` [J. Ablinger et al. '10] .

The Missing Operator Matrix Element $\Gamma_{\gamma e}$

$$\begin{aligned}
\Gamma^{(1)}_{\gamma e}(N) = {} & \frac{P_8}{27(N-4)(N-3)(N-2)(N-1)N^4(N+1)^4} + \left(\frac{2P_7}{9(N-4)(N-3)(N-2)(N-1)N^3(N+1)^3} + \frac{2\left(N^2+N+2\right)}{(N-1)N(N+1)}S_2\right)S_1 \\
& + \frac{P_3}{3(N-2)(N-1)N(N+1)^2}{S_1}^2 + \frac{2\left(N^2+N+2\right)}{3(N-1)N(N+1)}{S_1}^3 + \frac{P_6}{3(N-2)(N-1)N^2(N+1)^2}S_2 + \frac{4\left(N^2+N+2\right)}{3(N-1)N(N+1)}S_3 \\
& + \frac{3\cdot 2^{6+N}}{(N-2)(N+1)^2}S_{1,1}\left(\frac{1}{2},1\right) + \frac{2^{6-N}P_5}{3(N-3)(N-2)(N-1)^2N^2}\left(S_2(2) + S_1S_1(2) - S_{1,1}(1,2) - S_{1,1}(2,1)\right) \\
& - \frac{32\left(N^2+N+2\right)}{(N-1)N(N+1)}\left[S_1(2)S_{1,1}\left(\frac{1}{2},1\right) + S_{1,2}\left(\frac{1}{2},2\right) - S_{1,1,1}\left(\frac{1}{2},1,2\right) - S_{1,1,1}\left(\frac{1}{2},2,1\right) - \frac{\zeta_2}{2}S_1(2)\right] \\
& - \frac{48\left(N^2+N+2\right)}{(N-1)N(N+1)}S_{2,1} + \frac{4P_4}{(N-2)(N-1)N^2(N+1)^2}\zeta_2
\end{aligned}$$

harmonic sums:

$$S_{a,\vec{b}} = S_{a,\vec{b}}(N) = \sum_{i=1}^{N}\frac{\mathrm{sgn}(a)^i}{i^a}S_{\vec{b}}(i)$$

generalized harmonic sums:

$$S_{a,\vec{b}}(c,\vec{d}) = S_{a,\vec{b}}(c,\vec{d};N) = \sum_{i=1}^{N}\frac{(\mathrm{sgn}(a)\cdot c)^i}{i^a}S_{\vec{b}}(\vec{d};i)$$

The Missing Operator Matrix Element $\Gamma_{\gamma e}$

Analytic Mellin-inversion with `HarmonicSums`:

$$\begin{aligned}
\Gamma^{(1)}_{\gamma e}(z) = {} & \frac{P_9}{135z^3} - \frac{320-335z+231z^2}{15z}\mathrm{H}_0 + \frac{12+23z}{6}\mathrm{H}_0^2 + \frac{2-z}{3}\mathrm{H}_0^3 + 32(2-z)\left(\frac{(2-z)^2}{3z^2} - \mathrm{H}_0\right)\left(\tilde{\mathrm{H}}_{-1}\tilde{\mathrm{H}}_0 - \tilde{\mathrm{H}}_{0,-1}\right) \\
& - 8(2-z)\mathrm{H}_{0,0,1} - \frac{96-190z+118z^2-41z^3}{3z^2}\mathrm{H}_1^2 - 32(2-z)\left(\tilde{\mathrm{H}}_{-1}\tilde{\mathrm{H}}_0 - \tilde{\mathrm{H}}_{0,-1}\right)\tilde{\mathrm{H}}_1 \\
& - \left(\frac{2\left(32-48z+36z^2-13z^3\right)}{3z^2} + 4(2-z)\mathrm{H}_0\right)\mathrm{H}_{0,1} - \left(\frac{2P_{10}}{45z^4} - \frac{2\left(32-48z+12z^2+7z^3\right)}{3z^2}\mathrm{H}_0\right)\mathrm{H}_1 \\
& + \frac{2\left(2-2z+z^2\right)}{z}\left(\frac{\mathrm{H}_1^3}{3} + 8\mathrm{H}_1\mathrm{H}_{0,1} + 16\tilde{\mathrm{H}}_0\tilde{\mathrm{H}}_{0,-1} - 32\tilde{\mathrm{H}}_{0,0,-1} - 16\mathrm{H}_{0,1,1} + 8\tilde{\mathrm{H}}_0\zeta_2\right) + \left(\frac{4\left(32-48z+24z^2-3z^3\right)}{3z^2}\right. \\
& \left. - 8(2-z)\left(\mathrm{H}_0 + 2\tilde{\mathrm{H}}_1\right)\right)\zeta_2 + \frac{8\left(12-10z+5z^2\right)}{z}\zeta_3
\end{aligned}$$

harmonic polylogarithms of argument z and $1-z$ ($\tilde{\mathrm{H}}(z) = \mathrm{H}(1-z)$):

$$H_{a,\vec{b}} = H_{a,\vec{b}}(z) = \int_0^1 \mathrm{d}\tau f_a(\tau) H_{\vec{b}}(\tau), \quad \text{with} \quad f_0(\tau) = \frac{1}{\tau},\ f_1(\tau) = \frac{1}{1-\tau},\ f_{-1}(\tau) = \frac{1}{1+\tau}$$

Results for the Forward-Backward Asymmetry

Application to the Forward-Backward Asymmetry A_{FB}

- The forward-backward asymmetry is defined by:

$$A_{FB}(s) = \frac{\sigma_F(s) - \sigma_B(s)}{\sigma_F(s) + \sigma_B(s)},$$

with

$$\sigma_F(s) = 2\pi \int_0^1 \mathrm{d}\cos(\theta) \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}, \qquad \sigma_B(s) = 2\pi \int_{-1}^0 \mathrm{d}\cos(\theta) \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega},$$

and θ the angle between the incoming e^- and outgoing μ^-.

- The technique of radiators can also be used for A_{FB}:
  [Böhm et al. (LEP Physics Workshop 1989, p.203–234)]

$$A_{FB}(s) = \frac{1}{\sigma_F(s) + \sigma_B(s)} \int_{z_0}^1 \mathrm{d}z \frac{4z}{(1+z)^2} H_{FB}(z) \sigma_{FB}^{(0)}(s')$$

- Due to the angle dependence the radiators are not the same as in the total cross-section.

Application to the Forward-Backward Asymmetry A_{FB}

- At leading logarithmic (LL) accuracy the radiators are given by:

$$H_{FB}^{LL} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{(1+z)^2}{(x_1+x_2)^2} \Gamma_{ee}^{LL}(x_1)\Gamma_{ee}^{LL}(x_2)\delta(z-x_1x_2).$$

- Due to the angular dependence the integral does not factorize with the Mellin transform.
- At subleading logarithmic accuracy the integral will likely become more involved due to additional angle dependence of the cross-sections.
- At LL the integrals can be solved analytically in Mellin and momentum fraction space.

Application to the Forward-Backward Asymmetry A_{FB}

- A direct integration in terms of iterated integrals at argument z is complicated by singularities at $z \to 0$ and $z \to 1$.
- A direct computation of the Mellin transform

$$\mathcal{M}[\tilde{H}_{FB}^{LL}(z)](N) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \, x_1^N x_2^N \left(\frac{(1+x_1x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{ee}^{LL}(x_1)\Gamma_{ee}^{LL}(x_2)$$

  is complicated by the symbolic powers of N.
- We chose to calculate the generating function:

$$\mathcal{G}[\tilde{H}_{FB}^{LL}(z)](t) = \sum_{N=0}^{\infty} t^N \mathcal{M}[H_{FB}^{LL}(z)](N) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \, \frac{1}{1 - t\, x_1 x_2} \left(\frac{(1+x_1x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{ee}^{LL}(x_1)\Gamma_{ee}^{LL}(x_2)$$

- We use `Sigma` and `HarmonicSums` for the integration, taking the Nth moment and inverse Mellin transform.

Application to A_{FB} – Results

- In Mellin space we additionally encounter cyclotomic harmonic sums.
- In momentum fraction space we encounter cyclotomic harmonic polylogarithms, i.e. we have to introduce the additional letters:

$$f_{\{4,0\}}(\tau) = \frac{1}{1+\tau^2}, \qquad f_{\{4,1\}}(\tau) = \frac{\tau}{1+\tau^2}.$$

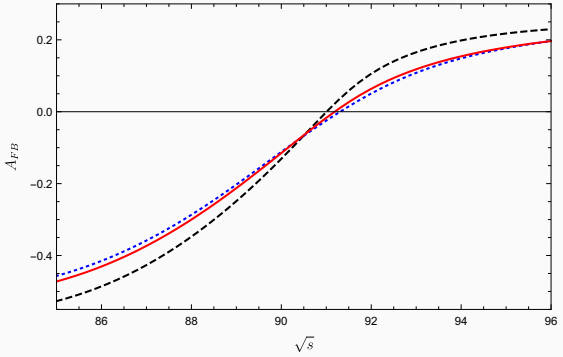
For example: $\left(S_{\vec{w}} \equiv S_{\vec{w}}(N)\right)$

$$H_{FB}^{(2),LL}(N) = \frac{8\left(3N^2+3N-1\right)P_1}{(N-1)N^2(N+1)^2(N+2)(2N-1)(2N+3)} - \frac{32\left(4N^2+4N-1\right)(-1)^N}{(2N-1)(2N+1)(2N+3)}[S_{-1}+\ln(2)],$$
$$H_{FB}^{(3),LL}(N) = -(-1)^N \frac{256(4N^2+4N-1)}{(2N-1)(2N+1)(2N+3)}\left[S_{-1,1} - \frac{1}{2}\ln^2(2) + \sum_{i=1}^{N}\frac{\ln(2)+S_{-1}(i)}{1+2i}\right] + \dots \quad .$$

For example: $\left(H_{\vec{w}} \equiv H_{\vec{w}}(\sqrt{z})\right)$

$$H_{FB}^{(2),LL}(z) = \frac{2(1-z)(1+z)^2}{z} + 2\pi\frac{(1-z)^2}{\sqrt{z}} - 8(1+z)H_0 - 8(1-z)^2\frac{H_{\{4,0\}}}{\sqrt{z}},$$
$$H_{FB}^{(3),LL}(z) = \frac{64(1-z)^2}{\sqrt{z}}\left[H_{1,\{4,0\}} - H_{-1,\{4,0\}} - H_{\{4,0\},\{4,1\}} + \frac{1}{2}H_{0,\{4,0\}}\right] + \dots$$

Application to A_{FB} – Numerical Results

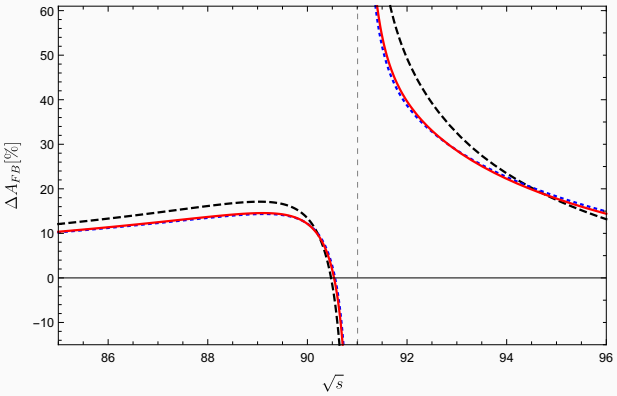


A_{FB} evaluated at $s_- = (87.9\,\mathrm{GeV})^2$, M_Z^2 and $s_+ = (94.3\,\mathrm{GeV})^2$ for the cut $z > 4m_\tau^2/s$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	−0.3564803	0.0225199	0.2052045
$+\mathcal{O}(\alpha L^1)$	−0.2945381	−0.0094232	0.1579347
$+\mathcal{O}(\alpha L^0)$	−0.2994478	−0.0079610	0.1611962
$+\mathcal{O}(\alpha^2 L^2)$	−0.3088363	0.0014514	0.1616887
$+\mathcal{O}(\alpha^3 L^3)$	−0.3080578	0.0000198	0.1627252
$+\mathcal{O}(\alpha^4 L^4)$	−0.3080976	0.0001587	0.1625835
$+\mathcal{O}(\alpha^5 L^5)$	−0.3080960	0.0001495	0.1625911
$+\mathcal{O}(\alpha^6 L^6)$	−0.3080960	0.0001499	0.1625911

A_{FB} and its initial state QED corrections as a function of $\sqrt{s}$. Black (dashed) the Born approximation, blue (dotted) the $O(\alpha)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

Application to A_{FB} – Numerical Results



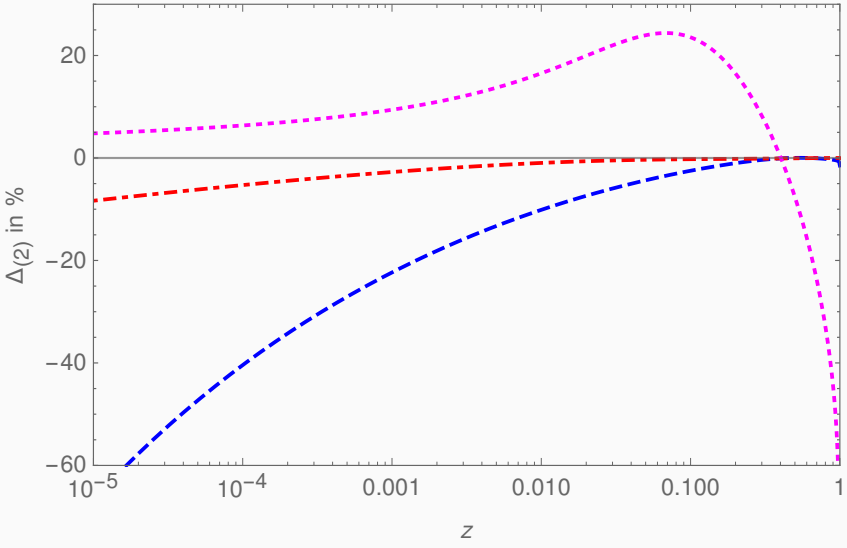
$$\Delta A_{FB} = 1 - \frac{A_{FB}^{(l)}}{A_{FB}^{(0)}},$$

where (l) denotes the order of LL ISR-corrections considered

ΔA_{FB} in % as a function of $\sqrt{s}$. Black (dashed) the $O(\alpha)$ improved approximation, blue (dotted) the $O(\alpha^2 L^2)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

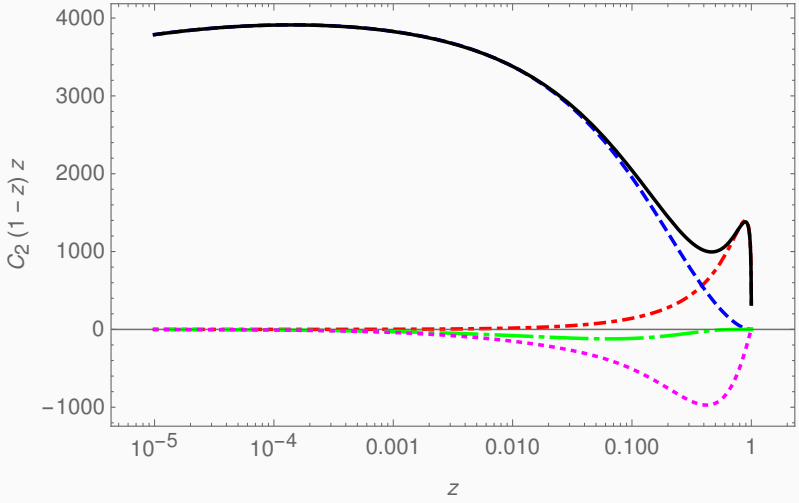
The traditional calculation

Numerical Illustration



- Relative deviation of the non-singlet (red), pure-singlet (blue) and interference (magenta) contribution in %.

Numerical Illustration



- Illustration of the Wilson coefficients for the non-singlet (red), pure-singlet (blue), interference $\times 10$ (magenta) and all others $\times 100$ (green) multiplied with the factor $z(1-z)$. The black line represents the whole contribution to initial state fermion-pair radiation.