UNIVERSITY OF LIVERPOOL

Gearing up for Future Colliders

# General overview on Bhabha scattering

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**(he/him)**

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## luminometry

- experiments measures rates $N$, theorists calculate cross sections $\sigma$
- luminosity $\mathcal{L}$ connects these $\sigma = N/\mathcal{L}$
- can be calculated from beam parameters but usually measured in standard candles
- $\sigma = N \times \sigma_{\mathrm{s.c.}}/N_{\mathrm{s.c.}}$
- $\sigma_{\mathrm{s.c.}}$ needs to be large & not contaminated by signal
- usually $ee \to ee$, $ee \to \gamma\gamma$

## why do we care?

### luminometry

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### low energy

- backgrounds to $e^+N \to e^+N$ for $\sim$ nuclear physics
- displaced vertex searches in $ee \to A' \to ee$, ...

... at FCC-ee

- absolute $\delta\mathcal{L} = 10^{-4}$; relative between scan points $10^{-5}$

  cf. FCC Feasibility Study Report, Vol. 1

- about $10\times$ better than LEP
- theory improvements needed!

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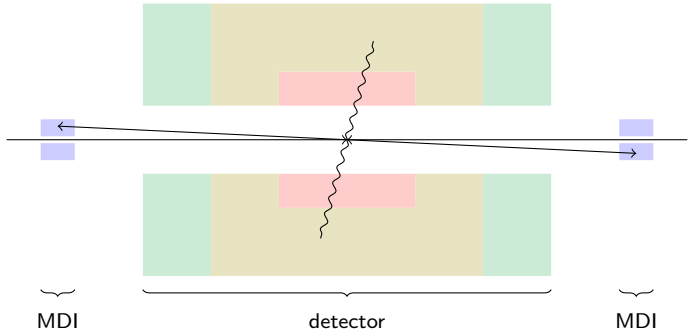
... at Belle II

- measured $\delta\mathcal{L} = 0.6\%$, theory is about a third of that

... at low energy

- JLab's Ce$^+$BAF has $\sim 10^{12}$ positrons @ $\sqrt{s} \sim 100\,\mathrm{MeV}$

- Bhabha $\mathrm{d}\sigma \sim 1/\theta^4 \rightarrow$ measure at small angle $\rightarrow$ QED $\sim 10^4\times$ EW
- $ee \rightarrow \gamma\gamma$ can be measured at larger angle (different systematics)



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- NLOPS resummation
- fragmentation functions

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hadronic correction $\alpha(M_Z^2) \sim 2 \times 10^{-4}$

- running from $\alpha_{\rm OS}$ to $\alpha(M_Z^2)$

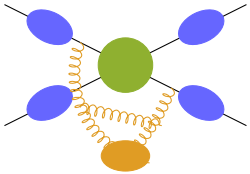
time-line of fixed order results

- 70s & 80s: NLO was done
- 2000: two-loop with $m_e = 0$
- later 00s: partial two-loop with $m_e > 0$, NLO QED for $ee \to ee\gamma$, conversion from $m_e = 0 \to \ m_e > 0$
- early 10s: $n_f$ contributions, incl. HVP, using dispersive methods
- 2021: first NNLO with $0 \lesssim m_e$ (cf. Sara's talk)
- 2023: two-loop for $ee \to ee\gamma$ with $m_e = 0$, via $q\bar{q} \to q\bar{q}\gamma$
- 2024: two-loop with $m_e \neq 0$
- 2026: three-loop with $m_e = 0$
- future: N$^3$LO is on the horizon

$\Rightarrow$ we'll hear more later

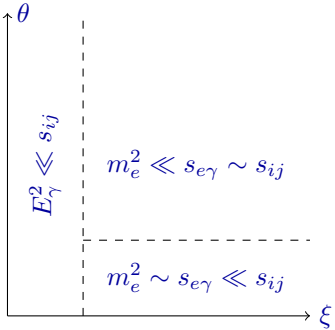
$s \sim 10^{10} m_e^2$, $|t| \sim 10^7 m_e^2$

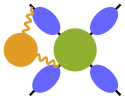
- expansions in $m_e$ are perfectly fine
- observable $\mathrm{d}\sigma/\mathrm{d}\theta$ or even $\sigma$ is not collinear safe $\rightarrow$ need $\log(m_e^2/Q^2)$
- connection to QCD: massification
- based on SCET factorisation & method of regions
- $\mathcal{A}(m) = \mathcal{S} \times \sqrt{Z} \times \sqrt{Z} \times \sqrt{Z} \times \sqrt{Z} \times \mathcal{A}(0) + \mathcal{O}(m) \supset \{1/\epsilon^2, L^2\}$
- soft: process-dependent $S = 1 +$ fermion loops
  $\rightarrow$ compute separately to combine with hadron loops
- collinear: universal $Z$, known up to $\alpha^3$, converts $1/\epsilon \rightarrow \log(m^2/Q^2)$



NTS expansion

cf. Sara



massification

jettification

following yesterday's discussion

- can we use massification for real corrections? maybe
- can we use massification for small-angle Bhabha scattering where $|t| \ll s$?

please note: this is the result of messing with Mathematatica for a few minutes. NOT a thorough investigation!

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- at two-loop fully massive is known, so we can check
- relative to massified result with counting parameter $m_e \sim \lambda$
- $$\Re\left(\mathcal{A}^{(i)} \times \mathcal{A}^{(0)}\right) \propto \begin{cases} 1 + 8.9 \times 10^{-9}\lambda^2 + 7.7 \times 10^{-16}\lambda^4 - 6.2 \times 10^{-23}\lambda^6 & i = 1 \\ 1 + 1.8 \times 10^{-8}\lambda^2 + 5.5 \times 10^{-15}\lambda^4 - 4.1 \times 10^{-22}\lambda^6 & i = 2 \end{cases}$$

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Krakow-Knoxville-Waco Extended Universe cf. Zbigniew's and Alan's talks

- for small $|t|$, soft logarithms matter a lot!
- use YFS theorem to build a Monte Carlo. (C)EEX at $|\mathcal{A}|^2$ ($\mathcal{A}$) level
- difficult-ish for Bhabha because of interference diagrams
- written as $L = \log(|t|/m_e^2)$, leads to 'pragmatic counting' $\sim$ LL counting
- $\alpha^2\Big|_{\text{prag.}} = \{\alpha L, \alpha^2 L^2\} \sim \text{NLO} + \text{LL}$
  $\alpha^3\Big|_{\text{prag.}} = \{\alpha^2 L, \alpha^3 L^3\} \sim \text{NNLO} + \text{LL}$
- higher-order matching is possible to any fixed-order result, e.g. $\text{N}^3\text{LO}$

Parton Showers, structure function, PDFs, ... cf. talks by Giovanni, Kay, Marek

- basic idea: electron structure function + DGLAP running
- PDFs for ISR, incl. up to NLL
- 'classical' LL parton shower, i.e. solve DGLAP numerically
- use eikonal to include soft effects and generate transverse momenta for photons
- matching at NLO possible, NNLO?
- NLL QED parton shower?

$\sigma_{\text{s.c}} \propto \alpha^2 \Rightarrow \ \delta\sigma_{\text{s.c.}}/\sigma_{\text{s.c.}} \sim 2 \times \delta\alpha(M_Z^2)$

- direct measurement via Bhabha scattering defeats the point
- direct measurements in $A_{FB}^{\mu\mu}(s) \propto 1 + \mathcal{C} \times \alpha(s) \dfrac{s - M_Z^2}{2s}$
- theory predictions from running, i.e. $\alpha(s) = \dfrac{\alpha(0)}{1 - \Delta\alpha_{\text{lep}}(s) - \Delta\alpha_{\text{had}}(s)}$
- $\Delta\alpha_{\text{had}}(s) =$ $= -\dfrac{\alpha s}{3\pi} \displaystyle\int_{4m_\pi^2}^{\infty} \mathrm{d}s' \dfrac{R(s')K_\alpha(s')}{s'}$
- HVP is a major problem

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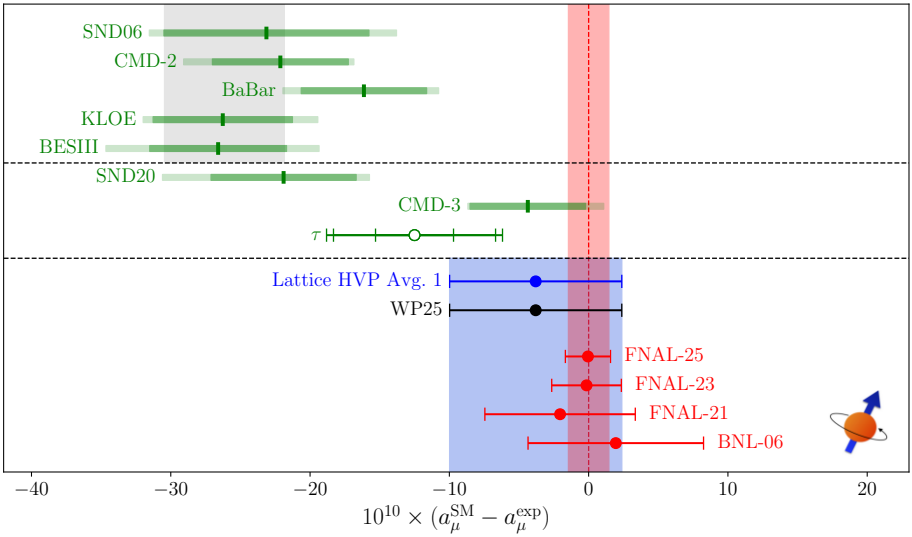
- HVP is a major problem
- ... just as in $(g-2)_\mu$!


SND06
CMD-2
BaBar
KLOE
BESIII
SND20
CMD-3
$\tau$
Lattice HVP Avg. 1
WP25
FNAL-25
FNAL-23
FNAL-21
BNL-06
−40
−30
−20
−10
0
10
20
$10^{10} \times (a_\mu^{\mathrm{SM}} - a_\mu^{\mathrm{exp}})$

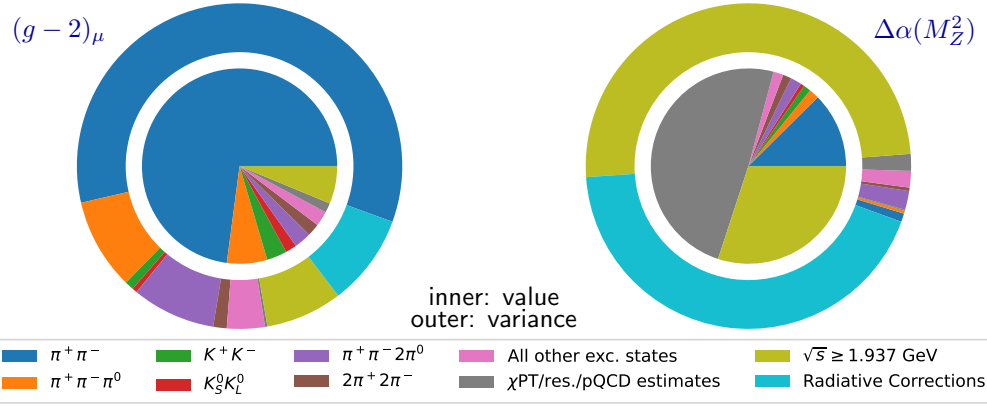
## interlude: status $(g-2)_\mu$ at WP25

- $a_\mu^{\mathrm{HVP}} = \dfrac{\alpha^2}{3\pi^2}\displaystyle\int_{4m_\pi^2}^{\infty} \mathrm{d}s \dfrac{R(s)K_\mu(s)}{s}$
- WP25 now uses exclusively* the lattice QCD value for HVP
- data is presently judged too inconsistent to be useful anywhere*
- lattice hybrids were proposed to aggressively push the error but not included in world average (yet)
- $\tau \to \nu +$ hadrons still unclear

## differences between $\alpha(M_Z^2)$ and $(g-2)_\mu$

$\alpha(M_Z^2)$ is not $(g-2)_\mu$!

- kernel functions differ, $K_\mu(s) \sim 1/s$, $K_\alpha \sim 1$
- $K_\mu$ favours low-energy $\rightarrow$ optiminal for lattice & 70% from $2\pi$ channel
- $K_\alpha$ favours higher energy, 55% from inclusive data above $1.937\,\mathrm{GeV}$



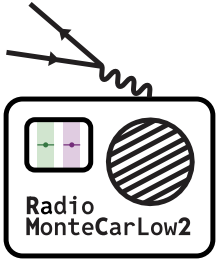
## where does this leave us?

- KNTW expects $\approx 2\times$ improvement due to improved FSR, eaten up by $2\pi$ discrepancies
- more data / better data could improve further, e.g. MUonE

$\Rightarrow$ need to solve lattice vs. data

- one avenue, RadioMonteCarLow2 `radiomontecarlow2.gitlab.io`

- theory Monte Carlos, esp. for $ee \to \pi\pi\gamma$ seem a bit sus..
- all rad. ret. measurements use PHOKHARA
- compare and improve Monte Carlo codes & try to understand what is important etc.
- input from high energy community most welcome!
- some shortcomings have been identified but no definite fault has been found!



a few more talks & plenty of time for discussion

10:30 *Event Generators for Bhabha scattering*, Carlo Carloni Calame

14:00 **TH Colloquium** *The discovery potential of FCC-ee*, Gino Isidori

15:30 *Feynman integrals and prospects*, Pierpaolo Mastrolia

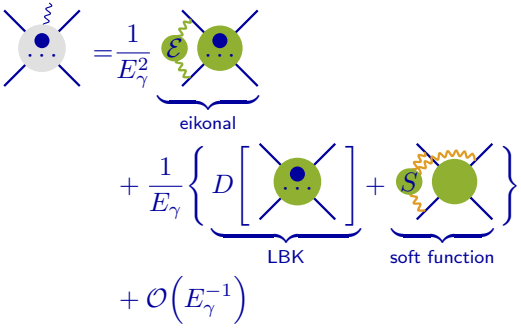
16:15 *Next-to-soft Stabilisation*, Sara Gündogdu

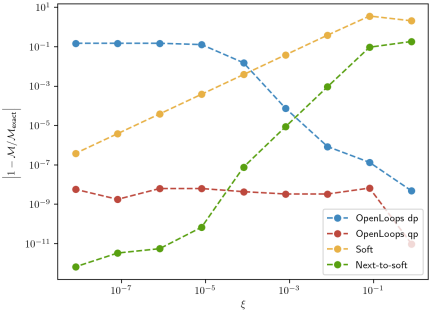
17:00 discussion

things to consider

- where can we go fixed-order? $N^3LO$? further?
- resummation: who's doing what? (C)EEX, PS, SF, NTS, ...?
- much overlap with yesterday? what can be stolen from each other?
- we will have more codes for FCC than we had at LEP, let's make use of this!

# NTS stabilisation



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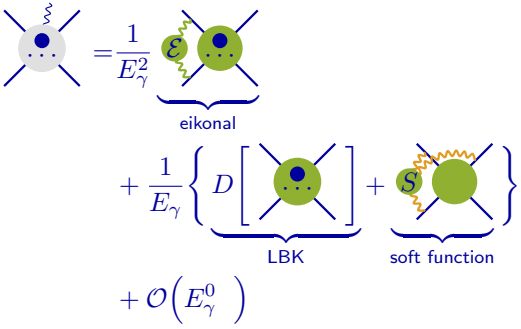
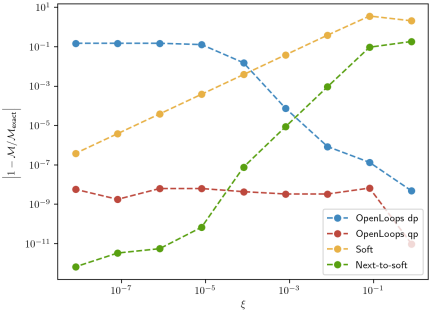


## HVP lattice (in 2025)

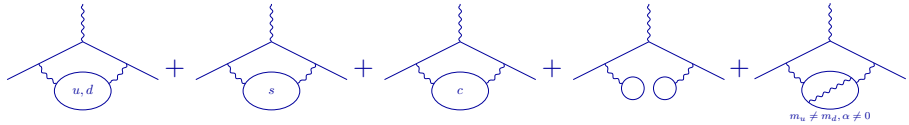
[Blum, Bruno, Cè, Clarke, Della Morte, DeTar, El-Khadra, Frezzotti, Gagliardi, Gérardin, Giusti, Gottlieb, Gülpers, Kuberski, Lahert, Lehner, Lellouch, Marinković, Meyer, Neil, Parrino, Portelli, Sitison, Tsang, Van de Water, Wang, Wittig, WP25.3]

- calculation similar to HLbL: $a_\mu^{\mathrm{HVP}} \sim \int \langle 0|j_{\mathrm{em}}^\mu(x) j_{\mathrm{em}}^\nu(0)|0\rangle$
- significant improvements since WP20: finite volume corrections, lattice scale uncertainty, full inclusion of IB, many more lattice actions are implemented
- 17 papers by 8 lattice collaborations, incl. 3 almost complete [BMW, RBC/UKQCD, Mainz]
- everything is thoroughly checked!
- introduction of window observables really helped!

# HVP lattice contributions

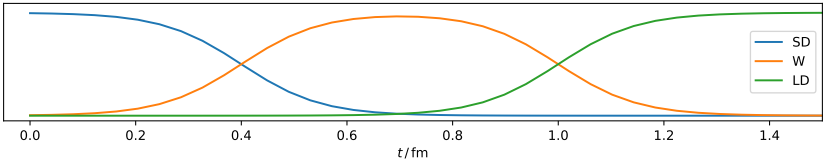
Section 3.6.1

$$a_\mu^{\mathrm{HVP}} = \underbrace{\quad}_{\sum_f a_\mu^{\mathrm{HVP}}(f,\mathrm{iso})}$$




$$= \sum_f a_\mu^{\mathrm{SD}}(f,\mathrm{iso}) + \sum_f a_\mu^{\mathrm{W}}(f,\mathrm{iso}) + \sum_f a_\mu^{\mathrm{LD}}(f,\mathrm{iso}) + \sum_f \delta a_\mu(\mathrm{IB})$$

- split by contribution and short-distance, intermediate, long-distance window



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$$a_\mu^{\mathrm{HVP}} = \underbrace{\quad + \quad + \quad + \quad}_{\sum_f a_\mu^{\mathrm{HVP}}(f,\mathrm{iso})} + \quad (m_u \neq m_d, \alpha \neq 0)$$

$$= \sum_f \left\langle a_\mu^{\mathrm{SD}}(f,\mathrm{iso}) \right\rangle + \sum_f \left\langle a_\mu^{\mathrm{W}}(f,\mathrm{iso}) \right\rangle + \sum_f \left\langle a_\mu^{\mathrm{LD}}(f,\mathrm{iso}) \right\rangle + \left\langle \sum_f \delta a_\mu(\mathrm{IB}) \right\rangle$$

- average all isospin-symmetric results for each window and add
- $a_\mu^{\mathrm{LD}}(\mathrm{iso})$ is tricky and was only computed by a few groups
- IB effects first added then averaged

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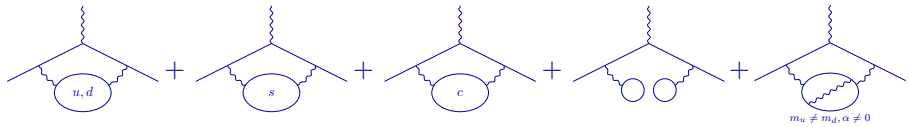


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$$= \underbrace{69.06(22)}_{(3.21)} + \underbrace{236.16(42)}_{(3.24)} + \underbrace{407.9(5.0)}_{(3.28)} + \underbrace{0.1(3.4)}_{(3.30)}$$

- average all isospin-symmetric results for each window and add
- $a_\mu^{\mathrm{LD}}(\mathrm{iso})$ is tricky and was only computed by a few groups
- IB effects first added then averaged
- value and error dominated by LD, light quark channel

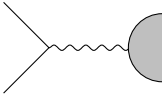
[Aliberti, Benton, Boito, Bruno, Carloni Calame, Cirigliano, Colangelo, Cotrozzi, Cottini, Davier, Denig, Druzhinin, Golterman, Hoecker, Hoferichter, Hoid, Holz, Ignatov, Keshavarzi, Kubis, Kupich, Laporta, Leplumey, Liu, Logashenko, López Castro, Lutz, Malaescu, Maltman, Miranda, Müller, Nesterenko, Nomura, Passera, Peris, Piccinini, Pilato, Polat, Roig, Ruiz de Elvira, Signer, Stoffer, Teubner, Toledo, Ulrich, Venanzoni, Wright, Zaid, Zhang, WP25.2]

basic idea: relate HVP to data

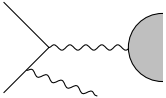
- $a_\mu^{\mathrm{HVP}} =$ $\sim \int \frac{\mathrm{d}^4 q}{(2\pi)^4}$
- dispersion relation $= \frac{1}{\pi} \int \frac{\mathrm{d}s}{s - q^2} \Im$
- optical theorem $2\Im$ $= \sum_{\text{had}} \int \mathrm{d}\Phi$
- measure either in $e^+e^- \to \gamma^* \to$ hadrons or $\tau \to \nu_\tau + W(\to \text{hadrons})$ (+ flavour corrections)

# HVP from $e^+e^-$ data

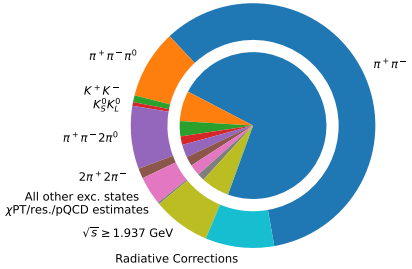
- need to measure all hadronic final states as a function of $s_h$
- kernel function favours low $s_h$ contribution, $> 70\%$ from $\pi^+\pi^-$
- two ways of measuring:
  - scan [SND, CMD]



  - radiative return [KLOE, BES, BaBar]



- three fitting groups: [KNTW, DHMZ, CHKLS]




inner: $a_\mu$, outer: $\delta a_\mu^2$, KNTW data

## KNTW

- data dynamically lustered to prevent over-fitting
- utilises full given covariance matrices/assumes full systematic correlation
- fit to avoid incurred d'Agostini bias
- full covariance matrices propagated to final result

## DHMZ

- measurements quadratic spline interpolated and averaged on a fixed binning
- central value derived using uncertainties and local correlations
- uncertainties on channels generated from 'pseudo-experiments'

## CHKLS

- select channels' spectra constrained by analyticity and unitarity

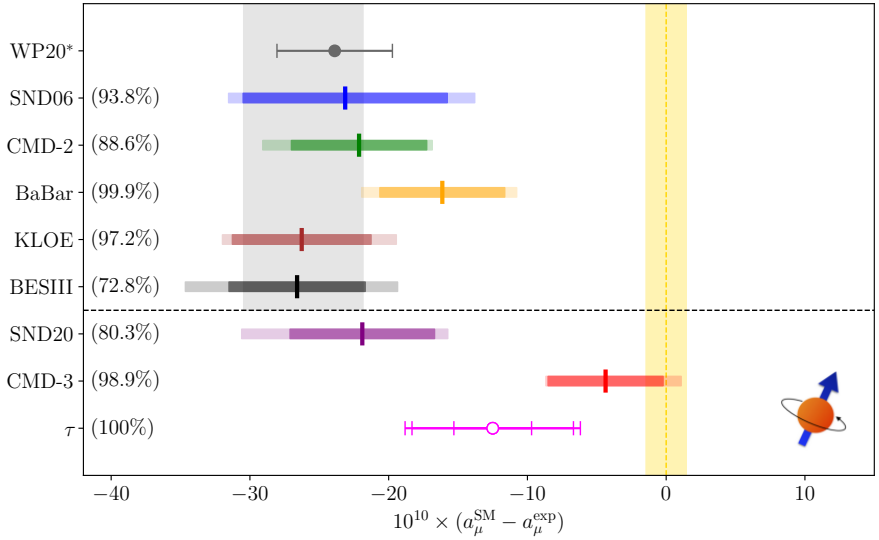
$\Rightarrow$ fit functions

- consistent w. DHMZ, KNTW despite method differing significantly

courtesy Aidan Wright, MPP2025

## values for $\pi\pi$ channel

old tensions between BaBar & KLOE, new tension between CMD-3 and everything else Figure 27



*: lattice is used for the LO HVP. The radiative corrections to the HVP, i.e. NLO HVP and NNLO HVP, have different kernels and do use data. These effect is small and errors have been conservatively estimated as the envelope of KNT and KNT+CMD3. $\delta a_\mu \lesssim$ HVP (N)NLO so even if you don't believe the data at all, it's fine