NNLO Electroweak Corrections to ZH Production at Future Lepton Colliders: the Bosonic Part

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in collaboration with Ayres Freitas
based on on-going project and arXiv:2606.22500

Gearing up for Future Colliders — Workshop on Radiative Corrections at Future Lepton Colliders

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European Research Council
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Motivation

- Future lepton colliders aim at sub-percent precision for $\sigma(ZH)$.
  2018: 1.2% at ILC, 0.4% at FCC-ee, and 0.5% at CEPC
  2025: 0.6% at ILC, 0.31% at FCC-ee, and 0.25% at CEPC.
- To turn this measurement into a precise Higgs coupling determination, the theoretical prediction must have a comparable or smaller uncertainty.
- Theoretical prediction on the hard cross section at 2-loop:
  EW+QCD: 0.4-1.3%
  EW+EW fermionic: 0.3-0.7%
  EW+EW bosonic: estimation **0.1-0.3%**

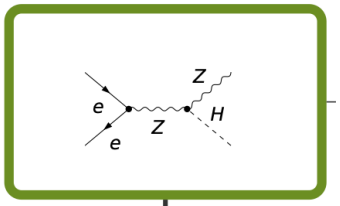
  this talk

Q. F. Sun, F. Feng, Y. Jia and W. L. Sang, Phys. Rev. D 96, no.5, 051301 (2017)
Y. Gong, Z. Li, X. Xu, L. L. Yang and X. Zh Phys. Rev. D 95, no.9, 093003 (2017)
A. Freitas and Q. Song, Phys. Rev. Lett. 130, no.3, 031801 (2023)
A. Freitas, Q. Song and K. Xie, Phys. Rev. D 108, no.5, 053006 (2023)

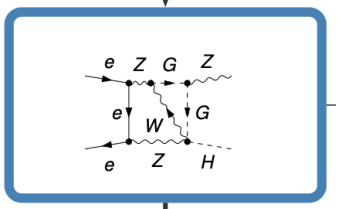
H. Baer *et al.* [ILC], [arXiv:1306.6352 [hep-ph]]
A. Abada *et al.* [FCC], Eur. Phys. J. ST **228**, no.2, 261-623 (2019); [CEPC Study Group] [arXiv:1811.10545 [hep-ex]]

H. Abramowicz *et al.* [Linear Collider Vision], arXiv:2503.19983 [hep-ex]
M. Selvaggi *et al.* [FCC], doi:10.17181/n78xk-qcv56;
X. Ai *et al.* Chin. Phys. C **49**, 123108 (2025)

Calculation: Overview



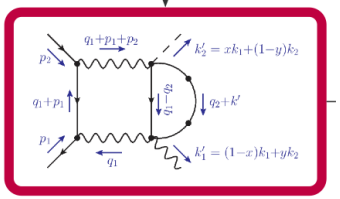
Target Process



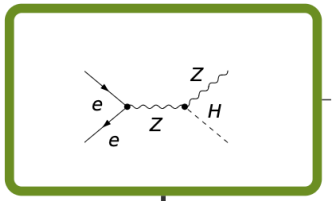
Generate Diagrams

$$A_i = \sum_k C_{i,k}\, I_{i,k}$$

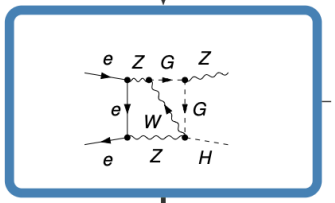
Compute Amplitude



compute tensor integrals up to $\mathcal{O}(\epsilon^0)$ on numerical points



Target Process



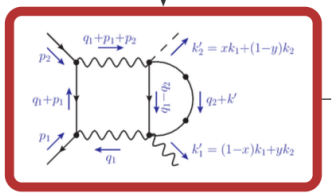
Generate Diagrams

$$A_i = \sum_k C_{i,k}\, I_k$$

Compute Amplitude

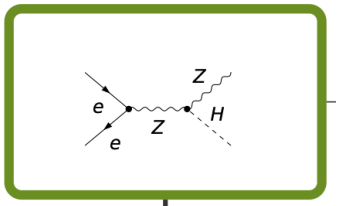
$$I_k = \sum_n C_{k,n}\, M_n$$

Reduce Integrals

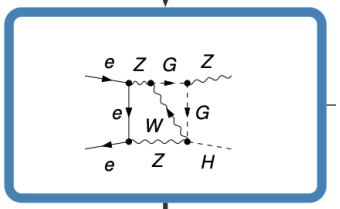


Compute Integrals

Calculation: Overview



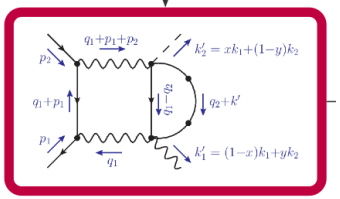
Target Process



Generate Diagrams

$$A_i = \sum_k C_{i,k}\, I_{i,k}$$

Compute Amplitude



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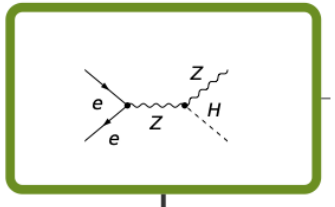
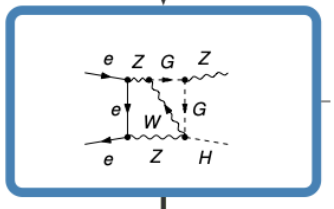


Target Process



Generate Diagrams

Identify isomorphic Topo and Ancesters

$$A_i = \sum_k C_{i,k}\, I_k$$

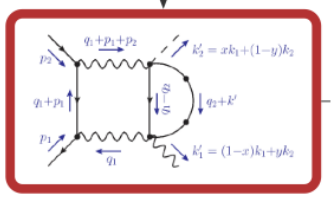
Compute Amplitude

Singularity-Free Basis

$$I_k = \sum_n C_{k,n}\, M_n$$

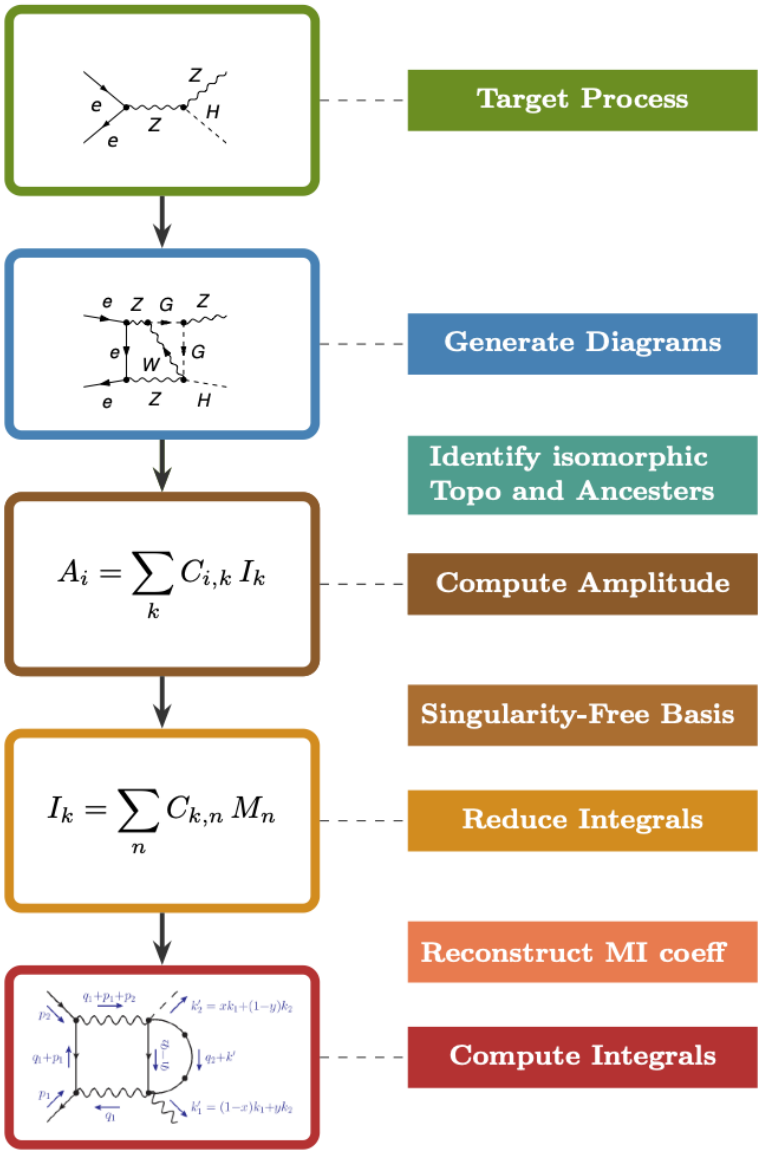
Reduce Integrals

Reconstruct MI coeff



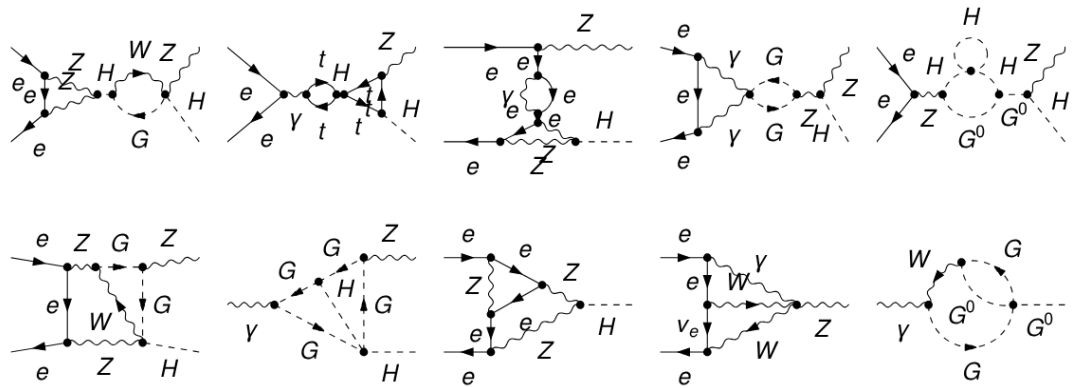
Compute Integrals

Calculation Details: Minimal Topologies I



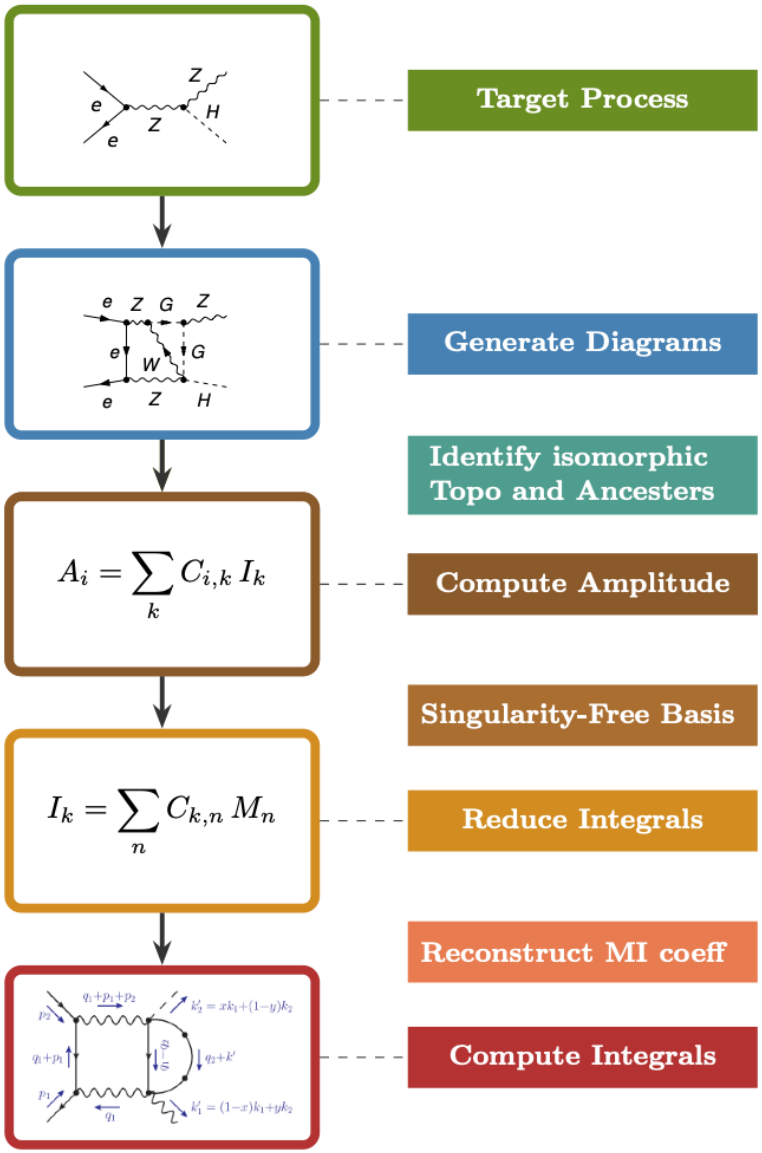
FeynArts, QGRAF, number of diagrams at $\mathcal{O}(\alpha^2) = 23185(16044 \text{ boson} + 7141 \text{ fermi})$ 12683 non-factorizable diagrams = 1125 topologies.

categorization	number of diagrams	
	factorizable	non-factorizable
eeZH	505=487+18	2515=2191+324
VZH	4398=1528+2870	9456=8004+1452
eeH	90=90+0	670=532+138
eeV	252=166+86	628=460+168
self-energy	2663=1090+1573	2008=1496+512
total	7908(3361+4547)	15277(12683+2594)



[T. Hahn, hep-ph/0012260; J. Küblbeck and M. Böhm and A. Denner, Comput. Phys. Commun. **60**, 165-180 (1990)]

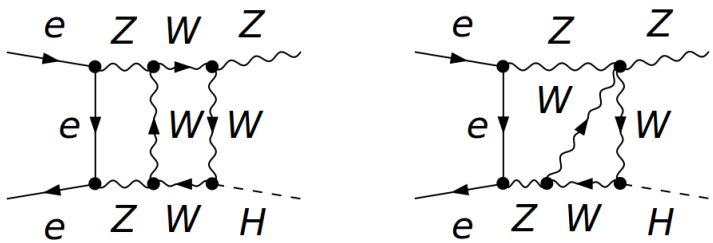
Calculation Details: Minimal Topologies II



- identifying isomorphism between topologies with same number propagators ($\sim$ 10min)

$$1125 = \{17, 141, 319, 407, 241\}$$
$$\rightarrow 759 = \{10, 48, 166, 321, 214\}$$

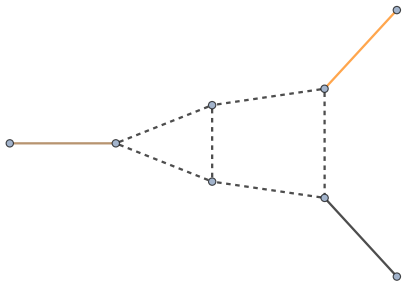
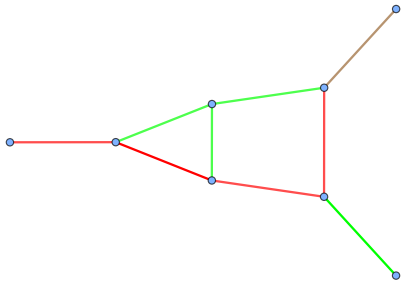
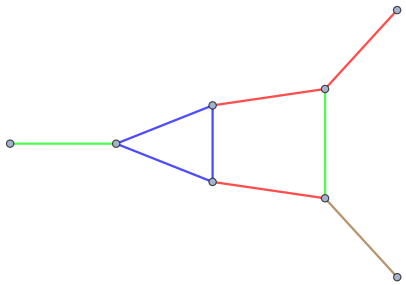
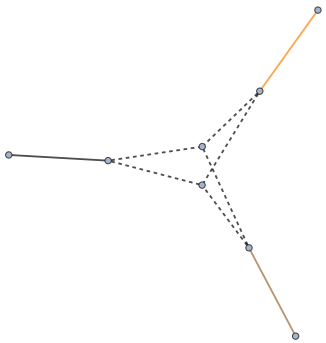
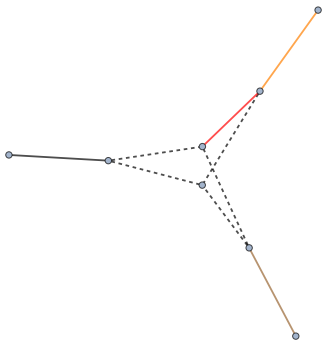
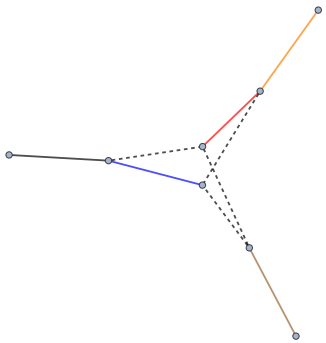
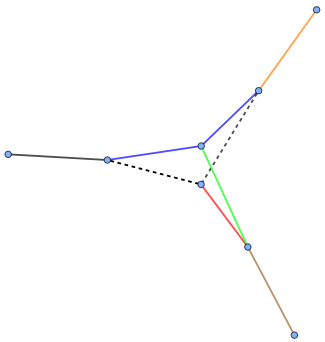
- classifying subsectors according to their corresponding top sectors ($\sim$ 10min)

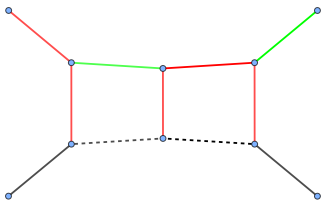
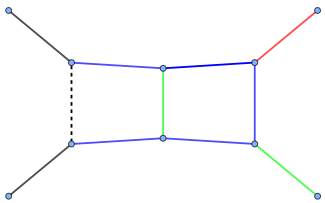
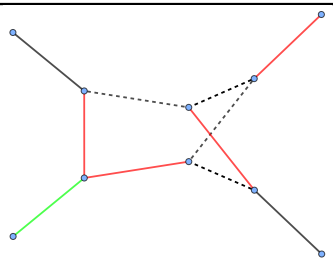
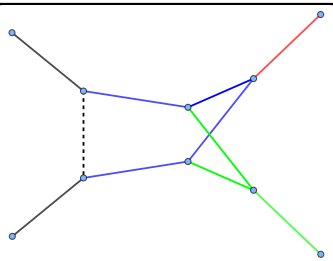


$$759 = \{10, 48, 166, 321, 214\}$$
$$\rightarrow 278 = \{0, 0, 3, 61, 214\}$$

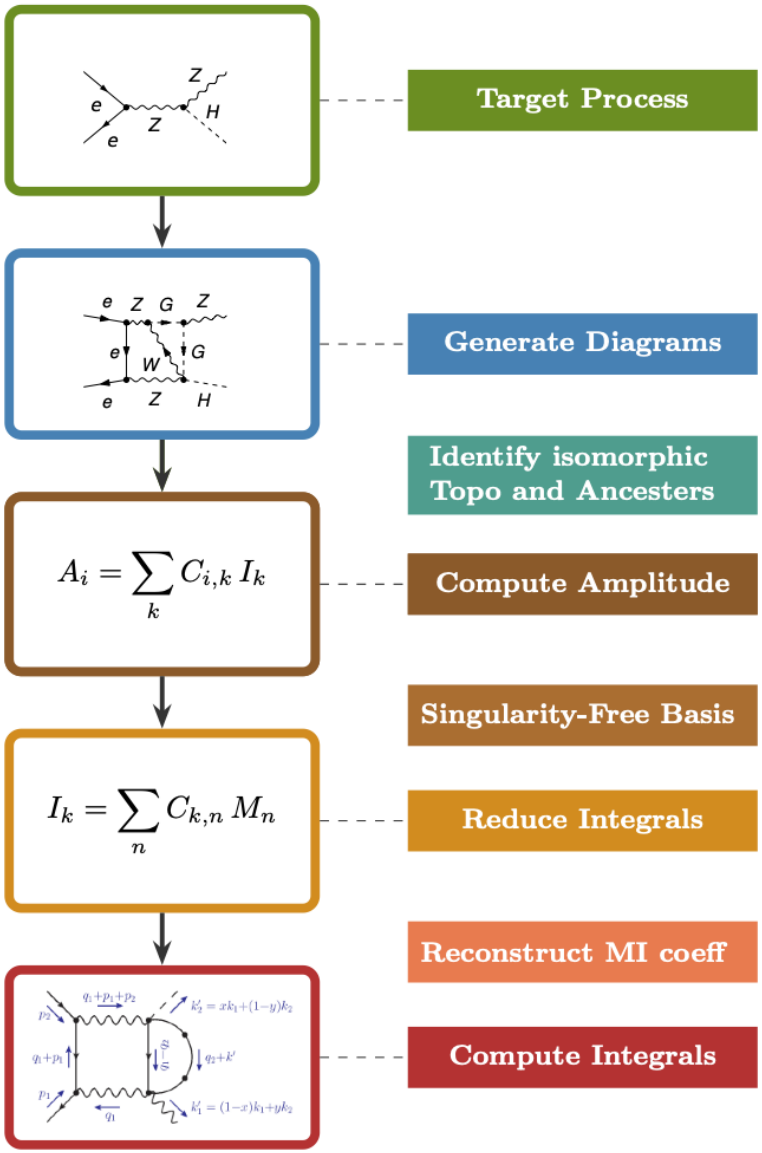
Calculation Details: Minimal Topologies III

278 topologies = 203 planar + 73 non-planar: N-pt M-mass: N ext. leg, M kinematic var.

3-pt 2-mass	3-pt 3-mass	3-pt 4-mass	3-pt 5-mass
			
			

4-pt 4-mass	4-pt 5-mass	4-pt 4-mass	4-pt 5-mass
			

Calculation Details: IBP Reduction I



- express the squared amplitude in FIs of minimal top-sectors

$$\sum_{i=1}^{12683} M_i^{(2)} |M^{(0)}|^* = \sum_{j=1}^{47190} c_j I_j$$

$c_j(s, t, m_z, m_w, m_h, D)$ are kept symbolic, and the size of all squared amplitudes is ~ 20 Mb.

- reduce 47190 Feynman integrals into singularity-free basis/$\epsilon-$finite basis

Calculation Details: IBP Reduction II

- IBP reduction methods typically yield Feynman integral bases where the reduction coefficients singular as the dimensional regulator $\epsilon \to 0$.

$$I[1,1,1,1,1,0,0] = -\frac{2\epsilon - 1}{\epsilon s} I[0,1,1,1,1,0,0]$$

- The $\epsilon-$finite bases or singularity-free basis leads to reduction coefficients which are finite as $\epsilon \to 0$.

$$I[0,1,1,1,1,0,0] = -\frac{\epsilon s}{2\epsilon - 1} I[1,1,1,1,1,0,0]$$

- Such a basis simplifies and stabilizes the numerical evaluation of the master integrals.
- With $\epsilon-$finite bases or singularity-free basis, integrals at $\mathcal{O}(\epsilon^0)$ are needed.

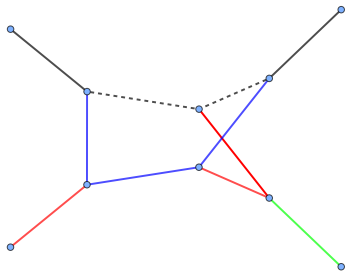
K. G. Chetyrkin, M. Faisst, C. Sturm and M. Tentyukov, Nucl. Phys. B **742**, 208-229 (2006)
S. De Angelis, D. A. Kosower, R. Ma, Z. Wu and Y. Zhang, Phys. Rev. D **113**, no.5, 056013 (2026);

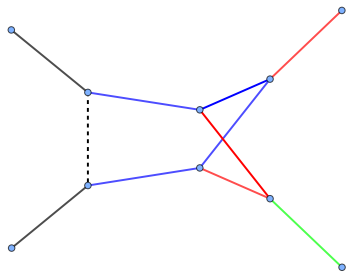
Calculation Details: IBP Reduction III

- We build a new algorithm for finding the singularity-free basis.
- The algorithm starts with IBP relations with symbolic D dependence instead of the reduction result. Kira, FIRE ✖; NeatIBP ✔

A. V. Smirnov and M. Zeng, Comput. Phys. Commun. **302**, 109261 (2024); P. Maierhöfer, J. Usovitsch and P. Uwer, Comput. Phys. Commun. **230**, 99-112 (2018); Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang, Comput. Phys. Commun. **295**, 108999 (2024); $\cdots$

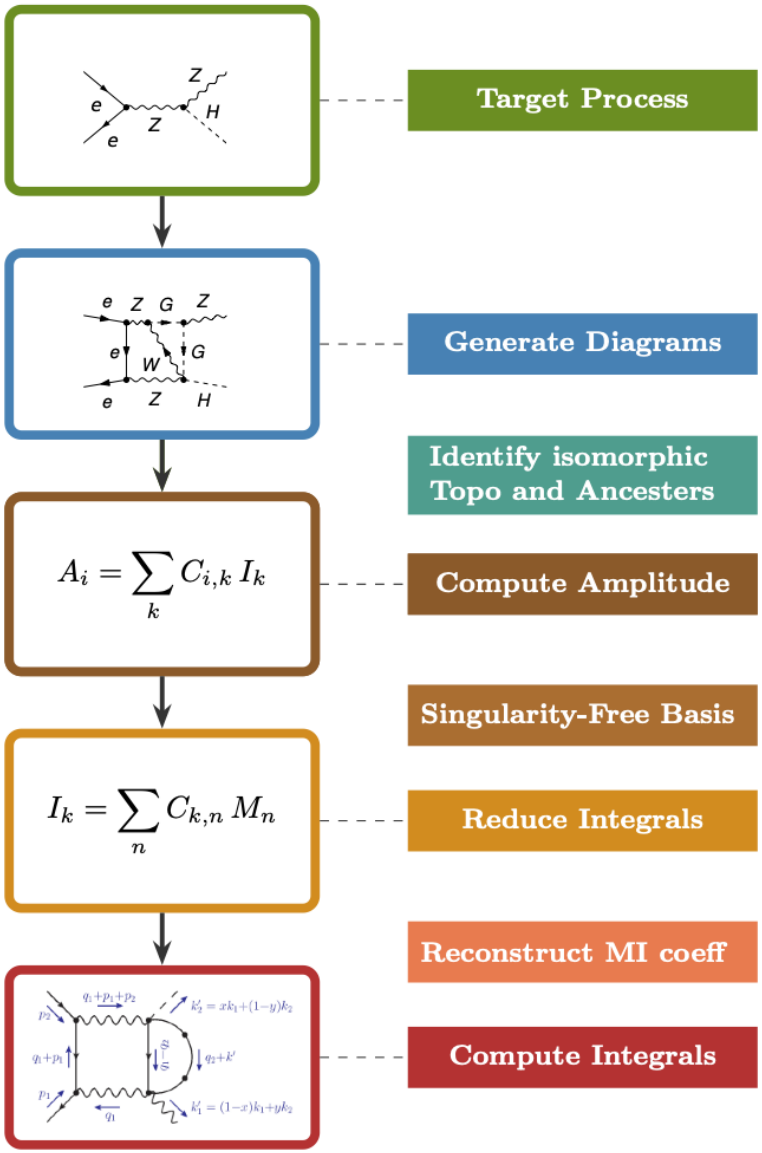
- $\sim 2 \times 200$h(32-core 64GB machine) for generating symbolic IBP relations for 278 topologies. For most topologies, takes < 1h, but $\mathcal{O}(5)$ topologies take ≥ 15h. (the largest system contains ~ 15000 FIs, although only ~ 200 FIs need to be reduced)





- Our algorithm took ~ 2h for finding singularity-free basis for 278 topologies (depend on linear solve).

Calculation Details: IBP Reduction IV



- 278 topologies have 23120 MIs (47190 FIs).
- Identifying isomorphism should reduce the number of MIs down to < 7675 according to X. Chen, X. Guan, C. Q. He, Z. Li, X. Liu and Y. Q. Ma, [arXiv:2209.14953 [hep-ph]].
- convert IBP relations to reduction result
- For phenomenology study, reduction on floating-point kinematics and symbolic D:
  - symbolic reduction + floating point evaluation ✖ computationally heavy
  - numerical reduction over finite-fields + reconstruction reduction coefficients with FireFly ⚠ computationally heavy

Calculation Details: IBP Reduction V

- Direct symbolic reduction suffers from severe expression swell and, for frontier problems, the simplification of intermediate coefficients can dominate the run time
  Bring intermediate coefficients into GCD form to avoid fake zeros
- For complicated topologies, symbolic IBP reduction can take $\mathcal{O}(100)$Gb memory and $\mathcal{O}(1)$days

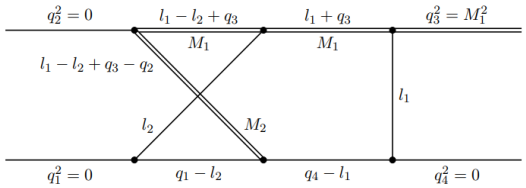


Table 5: Runtime and memory usage for `Kira 1.2` and `Kira` with with $r_{\max} = 7$. We use the option `select_mandatory_recursively`, as m_1, s, m_2, t, d, and set m_1 to one. The runtime only includes the time spent on the forward elimination and back substitution or the time for the reconstruction, respectively. These tests were run on a computer with two Intel Xeon Gold 6138 and 768 GiB of RAM.

	`Kira 1.2`		`Kira` with `FireFly`		
$s_{\max}$	Runtime	Memory usage	Runtime	CPU time for `pyRed`	Memory usage
1	4 min	7.6 GiB	3 min	99 %	0.9 GiB
2	1 h 53 min	33 GiB	1 h 42 min	97 %	3.3 GiB
3	18 h 28 min	102 GiB	18 h 34 min	91 %	18 GiB

A. Smirnov and M. Zeng, [arXiv:2409.19099 [hep-ph]];
K. Mokrov, A. Smirnov and M. Zeng, [arXiv:2304.13418 [hep-ph]].
J. Klappert and F. Lange, Comput. Phys. Commun. **247**, 106951 (2020)

Reconstructing Reduction Coefficients I

How to reconstruct IBP reduction coefficients efficiently with low memory usage? **Q. Song**, [arXiv:2606.22500 [hep-ph]].

- The IBP relations from which the reduction starts usually have simple polynomial coefficients, even when the final reduction coefficients are complicated.

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mm} \end{pmatrix} \begin{pmatrix} J_1 \\ \vdots \\ J_m \end{pmatrix} \Rightarrow \begin{pmatrix} J_1 \\ \vdots \\ J_m \end{pmatrix} = \overline{M}_{m\times n} \begin{pmatrix} J_{m-n+1} \\ \vdots \\ J_m \end{pmatrix}$$

- This suggests that the complexity is not intrinsic to the IBP relations themselves, but is generated during the reduction process. This naturally raises the question: can we avoid generating these complicated coefficients by choosing the elimination strategy more carefully?

Reconstructing Reduction Coefficients II

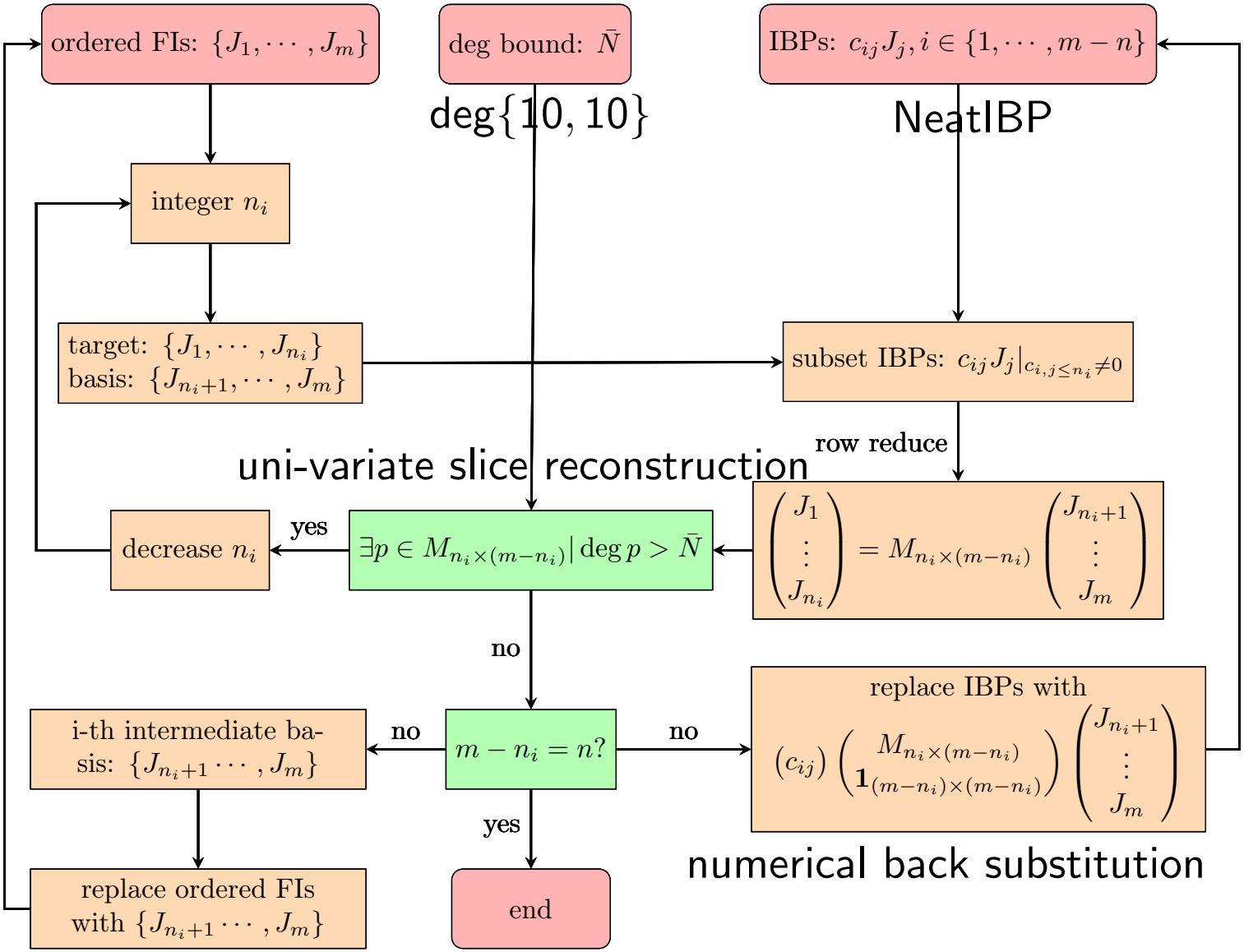
- The basic idea is to iteratively express complicated Feynman integrals in terms of simpler Feynman integrals (intermediate bases) by row reducing a subset of IBP relations, until remaining Feynman integrals are the master integrals.

$$\begin{pmatrix} J_1 \\ \vdots \\ J_m \end{pmatrix} = M_{m\times(m-n_1)} \begin{pmatrix} J_{m-n_1+1} \\ \vdots \\ J_m \end{pmatrix} = M_{m\times(m-n_1-n_2)} \begin{pmatrix} J_{m-n_1-n_2+1} \\ \vdots \\ J_m \end{pmatrix}$$

$$= \underbrace{M_{m\times(m-n_1)} \times \cdots \times M_{m\times n}}_{k \text{ times}} \begin{pmatrix} J_{m-n+1} \\ \vdots \\ J_m \end{pmatrix}$$

- The choice of intermediate bases directly impacts the entries in the transformation matrices.

Reconstructing Reduction Coefficients III



Algorithm for finding intermediate bases such that the transformation matrices have polynomial degree below $\bar{N}$

Reconstructing Reduction Coefficients IV

The FIs are ordered according to their reconstruction complexity:

$$\begin{pmatrix} G[1,1,1,1,1,1,-3] \\ G[1,1,1,1,1,0,-3] \\ \vdots \\ G[0,1,0,0,1,1,0] \\ \text{Master Integrals} \end{pmatrix} \begin{matrix} \to \deg\{35,36\} \\ \to \deg\{32,31\} \\ \vdots \\ \to \deg\{0,0\} \end{matrix}$$

conjecture:
FIs with high degree couples more relations, eleminating them earlier can keep the system cleaner.

$$G[1,1,1,1,1,1,-3] = \frac{\deg 35}{\deg 36} G[1,0,0,1,0,0,0] + \frac{\deg 24}{\deg 23} G[0,1,0,1,1,0,0] \cdots$$

$$G[1,1,1,1,1,0,-3] = \frac{\deg 32}{\deg 31} G[1,0,0,1,0,0,0] + \frac{\deg 20}{\deg 19} G[0,1,0,1,1,0,0] + \cdots$$

$$G[0,1,0,0,1,1,0] = \deg 0 G[1,1,0,0,0,1,0]$$

$\deg\{\cdots\}$ is obtained by reconstructing MI coefficients on a uni-variate slice, thus requires solving the complete system multiple times

Reconstructing Reduction Coefficients V

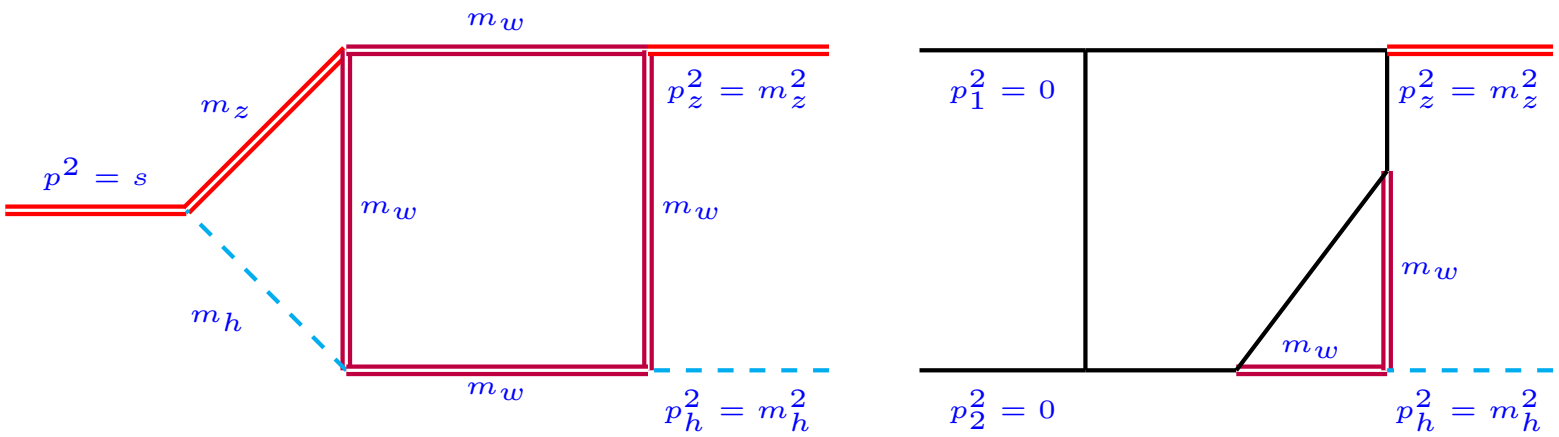
$$\begin{pmatrix} J_1 \\ \vdots \\ J_m \end{pmatrix} = \underbrace{M_{m\times(m-n_1)} \times \cdots \times M_{m\times n}}_{k \text{ times}} \begin{pmatrix} J_{m-n+1} \\ \vdots \\ J_m \end{pmatrix}$$

This representation on the RHS has two advantages:

- reduce computation cost in both IBP reduction and reconstruction
  - the size of the IBP equation system is small
  - the number of samplings required for reconstructing M is significantly fewer. It is bounded by $\bar{N}$.
- Second, the product form gives a compact analytic representation that is well suited for fast numerical evaluation, since complicated rational polynomials ($\sim$ Gb) are decomposed into simpler pieces ($\sim$ Mb).

H. A. Chawdhry, M. A. Lim and A. Mitov, Phys. Rev. D **99**, no.7, 076011 (2019) [arXiv:1805.09182 [hep-ph]].

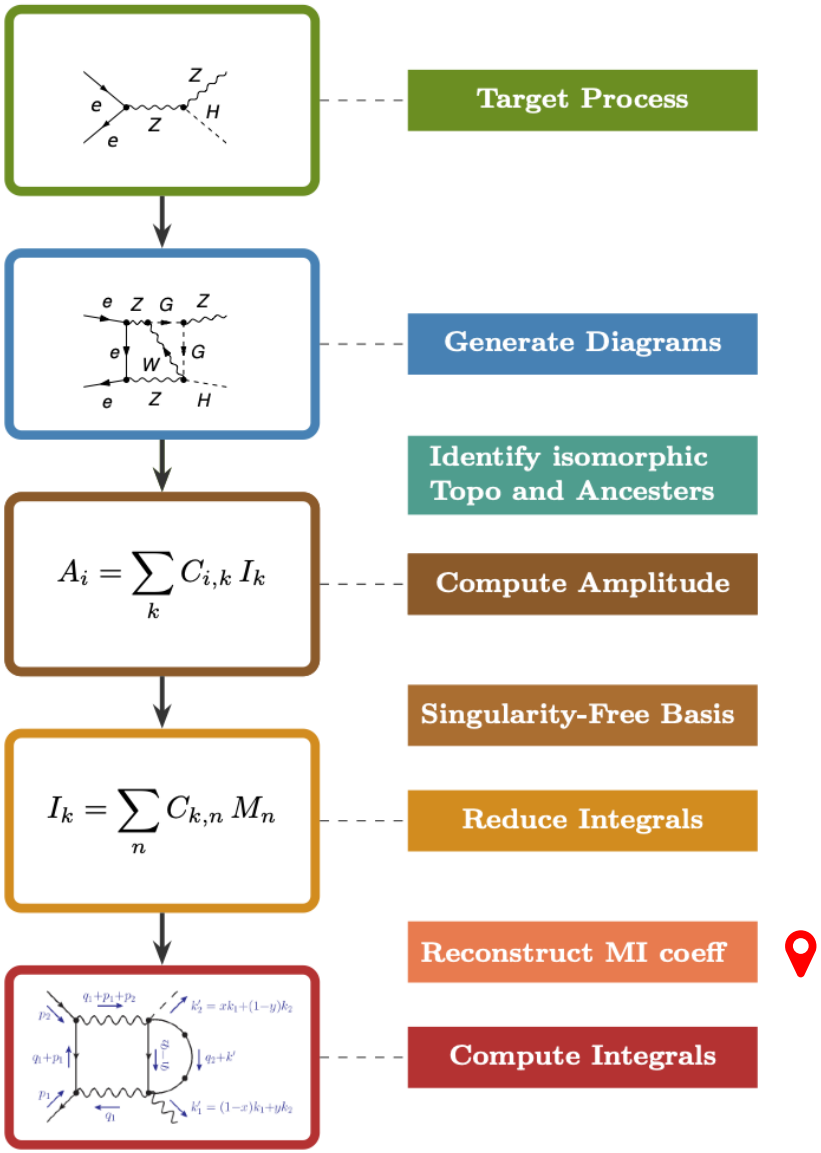
Reconstructing Reduction Coefficients VI



most complicated reduction coefficient	$\deg\{35, 36\}$	$\deg\{40, 42\}$
variable list	$\{s, m_w, m_z, m_h, D\}$	$\{s, t, m_w, m_z, m_h, D\}$
samplings in general ansatz	1,407,406	21,638,331
number of intermediate bases	11	13
most complicated entries	$\deg\{9, 8\}$	$\deg\{9, 10\}$
samplings for reconstruction	3,289	13,013
file size	1.7 Mb	2.3 Mb

Numerical evaluation on floating-point values of kinematics variables, with D kept symbolic, is fast ($\sim$ 10s)and stable. Reduction result is cross-check with Kira (a few minutes).

Calculation Details: FI Evaluation



- TODO-1: automate reconstruction

$$\sum_{i=1}^{12683} M_i^{(2)} |M^{(0)}|^* = \sum_{j=1}^{23120} d_j M_j$$

$d_j(s, t, m_z, m_w, m_h, D)$ are kept symbolic, which allows efficient evaluation at floating-point kinematics and symbolic D.

- TODO-2: compute all integrals in singularity-free bases up to $\mathcal{O}(\epsilon^0)$. Use dispersion relation techniques used in the calculation for fermionic contributions. Cross-check with AMFlow.

[X. Liu and Y. Q. Ma,
Comput. Phys. Commun. **283**, 108565 (2023)

Conclusion

- Due to the expected high precision for σ_{ZH} at future lepton colliders, bosonic contributions at NNLO EW must be included.
- With the large number of FIs, we use IBP reduction to reduce FIs in singularity-free basis and numerically compute all basis integrals.
- With singularity-free basis, all the reduction coefficients are finite at $\epsilon \to 0$, thus we only need to evaluate integrals in basis integrals up to $\mathcal{O}(\epsilon^0)$, which is expected to reduce numerical instabilities.
- We use the dispersion relation technique to compute all basis integrals.
- We also introduce a method for reconstructing reduction coefficients, which requires significantly fewer samplings and allows efficient floating-point evaluation.

Thanks for your attention!