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Thermodynamics of rotating fermions

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Outline

1. Introduction
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Rigid rotation

- States under rigid rotation are described by the following Killing vector:

$$\beta^\mu \partial_\mu = \beta_0(\partial_t + \Omega_0 \partial_\varphi). \tag{1}$$

- u^μ has unit 4-norm and is parallel to β^μ:

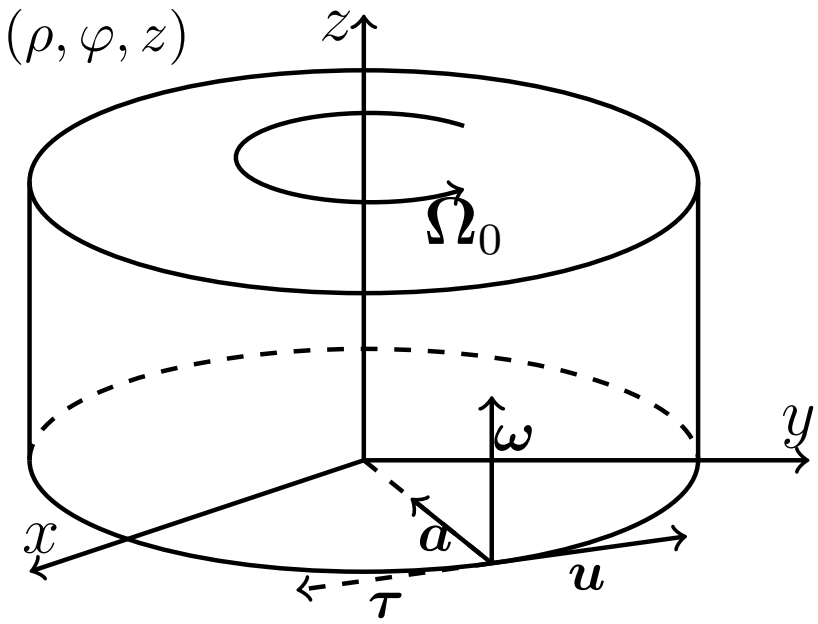
$$u^\mu \partial_\mu = \frac{\beta^\mu \partial_\mu}{\sqrt{\beta_\nu \beta^\nu}} = \Gamma(\partial_t + \Omega_0 \partial_\varphi), \qquad \Gamma = (1 - \rho^2 \Omega_0^2)^{-1/2}, \tag{2}$$

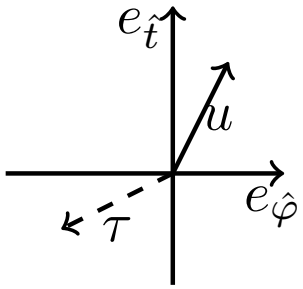
  with $\Gamma \equiv$ Lorentz factor and $\rho \equiv$ distance to the rotation axis.
- The local temperature T represents the inverse norm of β^μ:

$$T = 1/\sqrt{\beta_\mu \beta^\mu} = \Gamma T_0, \tag{3}$$

  with $T_0 \equiv$ the temperature on the rotation axis.
- In global equilibrium, $\mu/T = \text{const.} \Rightarrow \mu = \Gamma \mu_0$.
- The state diverges at the light cylinder, where $\rho \Omega_0 = 1$.

Kinematic frame for rigid rotation





A "kinematic" orthogonal tetrad is given by: [Becattini, Grossi, PRD 2015]

$$\begin{aligned}
\text{Velocity}: \quad & u^\mu \partial_\mu = \Gamma(\partial_t + \Omega_0 \partial_\varphi), && \Gamma = (1-\rho^2\Omega_0^2)^{-1/2},\\
\text{Acceleration}: \quad & a^\mu = u^\nu \partial_\nu u^\mu = -\rho\Omega_0^2\Gamma^2\delta^\mu_\rho, && a^2 = -\mathbf{a}^2 = -\Omega_0^2\Gamma^2(\Gamma^2-1),\\
\text{Vorticity}: \quad & \omega^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta = \Gamma^2\Omega\delta^\mu_z, && \omega^2 = -\boldsymbol{\omega}^2 = -\Omega_0^2\Gamma^4,\\
\text{Fourth vector:}^1 \quad & \tau^\mu \partial_\mu = \varepsilon^{\mu\nu\alpha\beta}u_\nu a_\alpha \omega_\beta \partial_\mu = -\Omega_0^3\Gamma^5(\rho^2\Omega_0\partial_t + \partial_\varphi), && \tau^2 = -\boldsymbol{\tau}^2 = -\Omega_0^4\Gamma^6(\Gamma^2-1).
\end{aligned}$$

The vorticity tensor is then

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu}u^\mu\omega^\nu = \Omega_0\Gamma(g_{\alpha x}g_{\beta y} - g_{\alpha y}g_{\beta x}). \tag{4}$$

[1] Aka *another vector*.

[Buzzegoli, Becattini, JHEP**12**(2018)002]

Thermodynamics of static systems

- In the case of a static system ($\Omega_0 = 0$), we have

$$\hat{\rho} = e^{-\beta_0(\widehat{H} - \mu_0 \widehat{Q})}, \tag{5}$$

  where $\widehat{H} \equiv$ Hamiltonian and $\widehat{Q} \equiv$ charge operators.
- The total thermodynamic potential Φ is obtained as:

$$\Phi = -T_0 \ln Z = \phi V, \qquad Z = \mathrm{tr}(\hat{\rho}), \tag{6}$$

- The grand potential (density) satisfies:

$$P = -\frac{d\Phi}{dV} = -\phi, \quad Q = -\frac{\partial \phi}{\partial \mu_0}, \quad s = -\frac{\partial \phi}{\partial T_0}, \tag{7}$$

  while $\epsilon = sT_0 + \phi + \mu_0 Q \equiv$ Energy density.

Rotating system: Grand potential

- In the rotating system,

$$\hat{\rho} = e^{-\beta_0(\widehat{H}-\mu_0\widehat{Q}-\Omega_0\widehat{M}^z)}, \tag{8}$$

with $\widehat{M}^z \equiv$ total angular momentum.

- The rotating system is inhomogeneous and ill defined as $\rho \to \Omega^{-1} \Rightarrow P \neq -\Phi/V$!
- In the rotating case, we introduce the grand canonical potential as

$$\Phi = \int_V d^3x\phi(x) = \mathcal{E} - T_0\mathcal{S} - \mu_0\mathcal{Q} - \Omega_0\mathcal{M}^z, \tag{9}$$

where we demand

$$\mathcal{S} = -\frac{\partial\Phi}{\partial T_0}, \quad \mathcal{Q} = -\frac{\partial\Phi}{\partial\mu_0}, \quad \mathcal{M}^z = -\frac{\partial\Phi}{\partial\Omega_0}, \tag{10}$$

thus ensuring the Gibbs-Duhem relation:

$$d\Phi = -\mathcal{S}dT_0 - \mathcal{Q}d\mu_0 - PdV - \boldsymbol{\mathcal{M}}\cdot d\boldsymbol{\Omega}_0. \tag{11}$$

Rotating system: Grand potential

- We can express $\mathcal{E}$ in terms of Φ as

$$\mathcal{E} = \Phi - T_0 \left(\frac{\partial \Phi}{\partial T_0}\right)_{\mu_0,\Omega_0} - \mu_0 \left(\frac{\partial \Phi}{\partial \mu_0}\right)_{T_0,\Omega_0} - \Omega_0 \left(\frac{\partial \Phi}{\partial \Omega_0}\right)_{T_0,\mu_0}$$
$$= \left[\frac{\partial}{\partial \beta_0}\left(\beta_0 \Phi[T_0, (\beta_0\mu_0)T_0, (\beta_0\Omega_0)T_0]\right)\right]_{\beta_0\mu_0,\beta_0\Omega_0}. \tag{12}$$

- Then, Φ can be obtained from $\mathcal{E}$ by ensemble integration:

$$\Phi = \frac{1}{\beta_0}\int d\beta_0 (\mathcal{E})_{\beta_0\mu_0,\beta_0\Omega_0}. \tag{13}$$

- The most natural candidate for the total energy is

$$\mathcal{E} = \int d^3x \Theta^{tt} \quad \Rightarrow \quad \phi(x) = \frac{1}{\beta_0}\int d\beta_0 (\Theta^{tt})_{\beta_0\mu_0,\beta_0\Omega_0}, \tag{14}$$

with $\Theta^{\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\psi$ energy-momentum tensor.

Rotating system: Grand potential

VEA, JHEP**08**(2020)016

- Θ^{tt} is given by $\left[\sum_j = \sum_{\sigma_j=\pm 1} \sum_{\lambda_j=\pm\frac{1}{2}} \sum_{m_j=-\infty}^{\infty} \int_M^{\infty} dE_j \int_{-p_j}^{p_j} dk_j\right]$

$$\Theta^{tt} = \frac{1}{8\pi^2} \sum_j f_j \Theta_j^{tt}, \quad \Theta_j^{tt} = E_j \left(J_j^+ + \frac{2\lambda_j k_j}{p_j} J_j^- \right), \tag{15}$$

with $J_j^{\pm} \equiv J^2_{m_j-\frac{1}{2}}(q_j\rho) \pm J^2_{m_j+\frac{1}{2}}(q_j\rho)$ and $f_j = [e^{\beta_0(E_j-\Omega_0 m_j-\sigma_j\mu_0)}+1]^{-1}$.

- Performing the β_0 integration leads to $\left[F_j = \ln(1+e^{-\beta_0(E_j-\Omega_0 m_j-\sigma_j\mu_0)})\right]$

$$\phi(x) = -\frac{1}{8\pi^2\beta_0} \sum_j E_j^{-1} \Theta_j^{tt} F_j. \tag{16}$$

- The expression for $\phi(x)$ can be further processed using integration by parts:

$$\begin{aligned}\phi(x) &= -\frac{1}{8\pi^2} \sum_j f_j \left(\frac{k_j^2}{E_j} J_j^+ + \frac{2\lambda_j p_j k_j}{E_j} J^- \right) \\ &= -\Theta^{zz}.\end{aligned} \tag{17}$$

- The (relativistically non-covariant) thermodynamic pressure is $P_{\rm GP} = \Theta^{zz}$.

Rotating system: Thermodynamic consistency

- The Grand Canonical pressure $P_{\rm GP}$ is related to the time components of the relevant currents:

$$\frac{\partial P_{\rm GP}}{\partial \mu_0} = J_V^t, \quad \frac{\partial P_{\rm GP}}{\partial \Omega_0} = \rho^2 \Theta^{t\varphi} + \frac{1}{2} J_A^z = M_C^{t,xy},$$
$$\frac{\partial P_{\rm GP}}{\partial T_0} = \frac{1}{T_0}\left(\Theta^{tt} + P_{\rm GP} - \rho^2 \Omega_0 \Theta_j^{t\varphi} - \mu_0 J_V^t - \frac{1}{2}\Omega_0 J_A^z\right), \tag{18}$$

  with $M_C^{t,xy} = x\Theta^{ty} - y\Theta^{tx} + S_C^{t,xy}$ and $S_C^{t,xy} = -\frac{1}{2}\varepsilon^{txyz} J_{A;z}$ chosen automatically in the canonical pseudogauge.

- Then, the energy density is recovered as expected:

$$\Theta^{tt} = T_0 \frac{\partial P_{\rm GP}}{\partial T_0} - P_{\rm GP} + \mu_0 \frac{\partial P_{\rm GP}}{\partial \mu_0} + \Omega_0 \frac{\partial P_{\rm GP}}{\partial \Omega_0}. \tag{19}$$

Rotating system: Restoring Lorentz covariance

- The grand canonical pressure is not a Lorentz scalar: $P_{\rm GP} = \Theta^{zz}$.
- The thermodynamic quantities arising from $P_{\rm GP}$ are also not Lorentz scalars (Θ^{tt}, J^t_V, $M^{t,xy}_C$,...).
- A first step towards restoring Lorentz covariance is to write

$$\Theta^{tt} - \rho^2\Omega_0\Theta^{t\varphi}_j = \frac{1}{\Gamma}\Theta^{t\mu}u_\mu, \quad u^\mu\partial_\mu = \Gamma(\partial_t + \Omega_0\partial_\varphi). \tag{20}$$

- Using $\frac{1}{2}\Omega_0 J^z_A = \Omega_0 S^{t,xy}_C$, and the vorticity tensor

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu}u^\mu\omega^\nu = \Omega_0\Gamma(g_{\alpha x}g_{\beta y} - g_{\alpha y}g_{\beta x}), \tag{21}$$

we can write

$$\frac{\partial P_{\rm GP}}{\partial T_0} = \frac{1}{T}\left(\Theta^{t\mu}u_\mu - \tilde{\phi}^t - \mu J^t_V - \frac{1}{2}S^{t,\alpha\beta}_C\omega_{\alpha\beta}\right) \equiv s^t, \tag{22}$$

with $T = \Gamma T_0$, $\mu = \Gamma\mu_0$ and $\tilde{\phi}^t = \Gamma\phi$.

Grand potential current

- It is clear that $P_{\rm GP} = -\phi^t$ is just the time component of a four vector, which we construct below:

$$\phi^\mu = \frac{1}{\beta_0}\int d\beta_0 (\Theta^{\mu t})_{\beta_0\mu_0,\beta_0\Omega_0} = -\frac{1}{8\pi^2\beta_0}\sum_j E_j^{-1}\Theta_j^{\mu t}F_j. \tag{23}$$

- This current satisfies

$$-\frac{\partial\phi^\mu}{\partial\mu_0} = J_V^\mu, \quad -\frac{\partial\phi^\mu}{\partial\Omega_0} = M_C^{\mu,xy} = \rho^2\Theta_C^{\mu\varphi} + S_C^{\mu,xy},$$
$$-\frac{\partial\phi^\mu}{\partial T_0} = \frac{1}{T_0}(\Theta^{\mu t} - \phi^\mu - \mu_0 J_V^\mu - \rho^2\Omega\Theta^{\mu\varphi} + \tfrac{1}{2}\Omega_0\varepsilon^{\mu xy\nu}J_{A;\nu}). \tag{24}$$

- Using $\Theta^{\mu\nu}u_\nu = \Gamma(\Theta^{\mu t} - \rho^2\Omega_0\Theta^{\mu\varphi})$ and $\omega_{\alpha\beta}S_C^{\mu,\alpha\beta} = -\Gamma\Omega_0\varepsilon^{\mu xy\nu}J_{A;\nu}$ gives

$$s^\mu = \frac{1}{T}\left(\Theta^{\mu\nu}u_\nu - \tilde{\phi}^\mu - \mu J_V^\mu - \frac{1}{2}S_C^{\mu,\alpha\beta}\omega_{\alpha\beta}\right), \tag{25}$$

with $\tilde{\phi}^\mu = \Gamma\phi^\mu$.

Pressure as a Lorentz scalar

- Contracting s^μ with u_μ leads to

$$s = \frac{1}{T}\left(\epsilon + P - \mu Q_V - \frac{1}{2} S_C^{\alpha\beta}\omega_{\alpha\beta}\right), \tag{26}$$

with the usual definitions:

$$\epsilon = u_\mu \Theta^{\mu\nu} u_\nu, \quad Q_V = u_\mu J_V^\mu, \quad S_C^{\alpha\beta} = u_\mu S_C^{\mu,\alpha\beta}, \tag{27}$$

while the thermodynamic pressure reads now $P = -\tilde{\phi}^\mu u_\mu \neq \Theta^{zz}$.

- The pressure P can easily be differentiated w.r.t. T_0, μ_0:

$$\frac{1}{\Gamma}\frac{\partial P}{\partial T_0} = \frac{\partial P}{\partial T} = s, \quad \frac{1}{\Gamma}\frac{\partial P}{\partial \mu_0} = \frac{\partial P}{\partial \mu} = Q_V. \tag{28}$$

- The derivative w.r.t. Ω_0 is subtle, as both Γ and u^μ depend on Ω_0:

$$\frac{\partial P}{\partial \Omega_0} = -\frac{\partial(\Gamma u_\mu)}{\partial \Omega_0}\phi^\mu - \Gamma u_\mu \frac{\partial \phi^\mu}{\partial \Omega_0}. \tag{29}$$

- To achieve thermodynamic consistency, $\partial P/\partial\omega_{\alpha\beta} = S_C^{\alpha\beta}$ and for this, one must reexpress P w.r.t. the local parameters: T, μ, $\omega_{\alpha\beta}$.

From global to local state

- So far, we considered thermodynamic derivatives with respect to the global ensemble parameters: T_0, μ_0 and Ω_0.
- We aim to regard $\tilde{\phi}^\mu = \Gamma\phi^\mu \equiv \tilde{\phi}^\mu(T, \mu, \omega_{xy})$.
- The transition $(T_0, \mu_0) \to (T, \mu)$ is straightforward, since:

$$\frac{\partial\tilde{\phi}^\alpha}{\partial T} = \frac{\partial\phi^\alpha}{\partial T_0} = -s^\alpha, \quad \frac{\partial\tilde{\phi}^\alpha}{\partial\mu} = \frac{\partial\phi^\alpha}{\partial\mu_0} = -J_V^\alpha. \tag{30}$$

- However, $\Omega_0 \to \omega_{xy}$ is less clear, since

$$\frac{\partial\phi^\mu}{\partial\Omega_0} = \Gamma^2\frac{\partial\tilde{\phi}^\mu}{\partial\omega_{xy}} - \rho^2\Omega_0\Gamma(\tilde{\phi}^\mu - Ts^\mu - \mu J_V^\mu). \tag{31}$$

- A little rearrangement reveals that

$$-\frac{\partial\tilde{\phi}^\mu}{\partial\omega_{xy}} = S_C^{\mu,xy} + \Theta^{\mu\nu}\tilde{\tau}_\nu, \quad \tilde{\tau}^\mu\partial_\mu = -\rho^2\Omega\partial_t - \partial_\varphi. \tag{32}$$

- The RHS displays the desired spin term, plus an undesirable orbital term.

Spin potential / vorticity duality

- **Idea:** In equilibrium, $\Omega_{\alpha\beta} = \omega_{\alpha\beta}$. [Becattini, PRL **108** (2012) 244502; Weickgenannt et al, PRL **127** (2021) 052301]
- $\omega_{\alpha\beta}$ is kinematic in nature, while $\Omega_{\alpha\beta}$ is thermodynamic.
- We wish to disentangle vortical = kinematic + thermodynamic, s.t.:

$$-\frac{\partial\tilde{\phi}^{\mu}}{\partial\omega_{xy}} = \Theta^{\mu\nu}\tilde{\tau}_{\nu}, \quad -\frac{\partial\tilde{\phi}^{\mu}}{\partial\Omega_{xy}} = S_{C}^{\mu,xy}. \tag{33}$$

- The kinematic part is comprised of u^{μ}, a^{μ}, ω^{μ} and $\tau^{\mu} = \varepsilon^{\mu\nu\alpha\beta}u_{\nu}a_{\alpha}\omega_{\beta}$ and

$$\omega_{\alpha\beta} = a_{\alpha}u_{\beta} - a_{\beta}u_{\alpha} - \varepsilon_{\alpha\beta\mu\nu}u^{\mu}\omega^{\nu}. \tag{34}$$

- Similarly, we introduce

$$\begin{aligned}\Omega_{\alpha\beta} &= \kappa_{\alpha}u_{\beta} - \kappa_{\beta}u_{\alpha} - \varepsilon_{\alpha\beta\mu\nu}u^{\mu}\Omega^{\nu},\\ \kappa^{\mu} &= \Omega^{\mu\nu}u_{\nu}, \quad \Omega^{\mu} = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_{\nu}\Omega_{\alpha\beta}.\end{aligned} \tag{35}$$

- To implement this programme, we consider massless (free) fermions.

Massless fermions: exact results

- In the case of massless fermions, we have the following beta-frame decomposition of the Belinfante energy-momentum tensor $T^{\mu\nu} = \Theta^{(\mu\nu)}$:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P_{\text{eff}} \Delta^{\mu\nu} + \pi^{\mu\nu} + \sigma^\tau_\varepsilon (\tau^\mu u^\nu + \tau^\nu u^\mu),$$
$$J^\mu_V = Q_V u^\mu + \sigma^\tau_V \tau^\mu, \quad J^\mu_A = \sigma^\omega_A \omega^\mu, \tag{36}$$

- We'll focus just on the blue terms, given as:

$$\frac{1}{3}\epsilon = P_{\text{eff}} = P_{\text{cl}} - \frac{3\omega^2 + a^2}{12}\sigma^\omega_{A;\text{cl}} + \frac{\omega^4 + \frac{46}{45}\omega^2 a^2 - \frac{17}{15}a^4}{192\pi^2},$$
$$\sigma^\omega_A = \sigma^\omega_{A;\text{cl}} - \frac{\omega^2 + 3a^2}{24\pi^2}, \quad Q_V = Q_{V;\text{cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}, \tag{37}$$

with $\omega^2 = \omega_\mu \omega^\mu = -\Omega^2\Gamma^4$ and $a^2 = a_\mu a^\mu = -\Omega^2\Gamma^2(\Gamma^2 - 1)$ and the classical contributions:

$$P_{\text{cl}} = \frac{7\pi^2 T^4}{180} + \frac{\mu_V^2 T^2}{6} + \frac{\mu_V^4}{12\pi^2}, \tag{38}$$

$$Q_{V;\text{cl}} = \frac{\mu_V T^2}{3} + \frac{\mu_V^3}{3\pi^2}, \quad \sigma^\omega_{A;\text{cl}} = \frac{T^2}{6} + \frac{\mu_V^2}{2\pi^2}, \tag{39}$$

Thermodynamic potential current

- To compute $\phi^\mu = \beta_0^{-1} \int d\beta_0 (\Theta^{\mu t})_{\beta_0\mu_0, \beta_0\Omega_0}$, we require $\Theta^{\mu\nu}$, satisfying

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\lambda (S_C^{\lambda,\mu\nu} + S_C^{\mu,\nu\lambda} - S_C^{\nu,\lambda\mu}), \tag{40}$$

  which leads to $\Theta^{t\varphi} = T^{t\varphi} + \frac{1}{4\rho}\partial_\rho J_A^z$ and $\Theta^{\varphi t} = T^{\varphi t} - \frac{1}{4\rho}\partial_\rho J_A^z$.

- Performing the integration $\tilde{\phi}^\mu = \beta_0^{-1}\Gamma \int d\beta_0 (\Theta^{\mu t})_{\beta_0\mu_0, \beta_0\Omega_0}$, we arrive at

$$\begin{aligned}
\tilde{\phi}^t &= -\Gamma \left[P_{\rm cl} - \frac{3\omega^2 + a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{122}{15}\omega^2 a^2 - \frac{17}{15}a^4}{192\pi^2} \right], \\
\tilde{\phi}^\varphi &= -\Omega\Gamma \left[P_{\rm cl} - \frac{\omega^2 + 3a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + 22\omega^2 a^2 + 17a^4}{960\pi^2} \right].
\end{aligned} \tag{41}$$

Entropy, charge and dynamic pressure

- Since $\tilde{\phi}^\rho = \tilde{\phi}^\varphi = 0$, we can decompose $\tilde{\phi}^\mu$ w.r.t. u^μ and τ^μ:

$$\tilde{\phi}^\mu = -Pu^\mu - \sigma^\tau_\phi \tau^\mu, \tag{42}$$

where

$$P = P_{\rm cl} - \frac{\omega^2 + a^2}{4}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{134}{15}\omega^2 a^2 + \frac{17}{5}a^4}{192\pi^2},$$
$$\sigma^\tau_\phi = \frac{1}{6}\sigma^\omega_{A;{\rm cl}} - \frac{3\omega^2 + 17a^2}{720\pi^2}. \tag{43}$$

- The entropy and charge densities read

$$s = \frac{\partial P}{\partial T} = s_{\rm cl} - \frac{T(\omega^2 + a^2)}{12}, \quad Q_V = \frac{\partial P}{\partial \mu} = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}. \tag{44}$$

- Since $P \neq P_{\rm eff}$, the system develops a dynamic pressure:

$$\Pi = P_{\rm eff} - P = \frac{a^2}{6}\sigma^\omega_{A;{\rm cl}} - \frac{a^2(89\omega^2 + 51a^2)}{2160\pi^2}. \tag{45}$$

Spin tensor

P. Singha et al, arXiv:2508.20237

- The spin tensor is known for systems under rotation ($\Omega_{\mu\nu} = \omega_{\mu\nu}$):

$$\Omega_{\mu\nu} = \omega_{\mu\nu}: \qquad S_C^{\alpha\beta} = -\frac{1}{2} u_\mu \varepsilon^{\mu\alpha\beta\nu} J_{A;\nu} = \frac{1}{2}(\omega^{\alpha\beta} + u^\alpha a^\beta - u^\beta a^\alpha)\sigma_A^\omega. \quad (46)$$

- In the absence of rotation but finite spin potential ($\omega_{\mu\nu} = 0$, $\Omega_{\mu\nu} = -\frac{1}{2}\varepsilon_{0\mu\nu\lambda}\mu^\lambda$), the energy dispersion reads:

$$E_{\mathbf{p}}^{(s)} = \sqrt{E_p^2 + s^2\mu_\Sigma^2 - 2s\mu_\Sigma E_{p_z}}, \quad (47)$$

with $s = \pm 1/2$, $E_p = \sqrt{\mathbf{p}^2 + m^2}$ and $E_{p_z} = \sqrt{p_z^2 + m^2}$.

- Using $\mathcal{L}_q = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \mu_{\mu\nu} S_C^{0,\mu\nu}$ gives $\phi = \phi_{\rm ZP} + \phi_\beta$, with

$$\phi_{\rm ZP} = -\sum_s \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}^{(s)}, \quad \phi_\beta = -\frac{2}{\beta}\sum_s \frac{d^3p}{(2\pi)^3} \ln(1 + e^{-\beta E_{\mathbf{p}}^{(s)}}). \quad (48)$$

- For massless fermions, $\phi_\beta = -7\pi^2 T_0^4/180$ is speen-independent!

Spin tensor

- Since $S^{\alpha\beta} = 0$ at finite $\Omega^{\mu\nu}$ but vanishing $\omega^{\alpha\beta}$, the tensor structure of $S^{\alpha\beta}$ is still given by the vorticity tensor: [Speen]

$$S_C^{\alpha\beta} = \frac{1}{2}(\omega^{\alpha\beta} + u^\alpha a^\beta - u^\beta a^\alpha)\sigma_A^\omega \equiv \frac{\partial P_\Omega}{\partial \Omega_{\alpha\beta}}, \tag{49}$$

  which can be integrated to yield

$$P_\Omega = P_{\rm cl} - \frac{\omega^2 + a^2}{4}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{134}{15}\omega^2 a^2 + \frac{17}{5}a^4}{192\pi^2} - \frac{1}{2}\sigma_A^\omega(\omega \cdot \delta\Omega), \tag{50}$$

  where we introduced $\delta\Omega^\mu = \Omega^\mu - \omega^\mu$.

- The local quantities read

$$\begin{aligned} s_\Omega &= s_{\rm cl} - \frac{T(\omega^2 + a^2)}{12} - \frac{T}{6}\omega \cdot \delta\Omega, \\ Q_V^\Omega &= Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2} - \frac{\mu}{2\pi^2}\omega \cdot \delta\Omega. \end{aligned} \tag{51}$$

Local energy density

- The energy density $\epsilon = sT - P + \mu Q_V + \frac{1}{2}\Omega_{\alpha\beta} S_C^{\alpha\beta}$ reads

$$\epsilon_\Omega = \epsilon_{\rm cl} - \frac{3\omega^2 + a^2}{4}\sigma^\omega_{A;cl} + \frac{\omega^4 + \frac{46}{45}\omega^2 a^2 - \frac{17}{15}a^4}{64\pi^2}. \tag{52}$$

- The dynamic pressure $\Pi_\Omega = \frac{1}{3}\epsilon_\Omega - P_\Omega$ gets modified to

$$\Pi_\Omega = \frac{a^2}{6}\sigma^\omega_{A;{\rm cl}} - \frac{a^2(89\omega^2 + 51a^2)}{2160\pi^2} + \frac{1}{2}\sigma^\omega_A(\omega \cdot \delta\Omega). \tag{53}$$

- Using $S_C^{\mu,\alpha\beta} = -\frac{1}{2}\sigma^\omega_A \varepsilon^{\mu\alpha\beta\lambda}\omega_\lambda = -\partial\tilde{\phi}^\mu_\Omega/\partial\Omega_{\alpha\beta}$, the thermodynamic four-current can be obtained as:

$$\tilde{\phi}^\mu_\Omega = -(P + \delta P)u^\mu - \sigma^\tau_\phi \tau^\mu - \delta\sigma^\tau_\phi \delta\tau^\mu, \tag{54}$$

where

$$\delta P = -\frac{1}{2}\sigma^\omega_A \omega \cdot \delta\Omega, \quad \delta\tau^\mu = \varepsilon^{\mu\nu\alpha\beta} u_\nu (\kappa_\alpha - a_\alpha)\omega_\beta, \quad \delta\sigma^\tau_\phi = \frac{1}{2}\sigma^\omega_A. \tag{55}$$

Bonus: Unruh effect

- Consider now a fluid undergoing accelerated motion (no vorticity):

$$\epsilon = \epsilon_{\rm cl} - \frac{a^2}{4}\sigma^{\omega}_{A;{\rm cl}} - \frac{17a^4}{960\pi^2}. \tag{56}$$

- The solution $\epsilon = 0$ gives

$$\mathbf{a}^2 = \frac{120\pi^2}{17}\left(\sigma^{\omega}_{A;{\rm cl}} + \sqrt{(\sigma^{\omega}_{A;{\rm cl}})^2 + \frac{17}{15\pi^2}\epsilon_{\rm cl}}\right). \tag{57}$$

- When $\mu = 0$, we have $\sigma^{\omega}_{A;{\rm cl}} = T^2/6$ and $\epsilon_{\rm cl} = 7\pi^2 T^4/60$, such that

$$\mathbf{a}^2 = (2\pi T)^2. \tag{58}$$

- The above relation confirms our thermodynamic relations are compatible with the Unruh effect:

$$T > T_U = \frac{|\mathbf{a}|}{2\pi}. \tag{59}$$

- When $T < T_U$, we have $\epsilon < 0$ and consequently, $P_{\rm eff} = P + \Pi < 0 \Rightarrow$ instability.

Conclusion

- We explored the thermodynamics of a quantum fluid exhibiting vorticity and acceleration.
- In the rigidly rotating ensemble, we constructed the thermodynamic potential and its four-current within a fictitious cylinder of radius $R < \Omega^{-1}$.
- Locally, kinematic (vorticity tensor) and thermodynamic (spin potential) vortical contributions must be disentangled.
- In the frame of massless fermions, we obtained the thermodynamic pressure of a fluid at finite spin potential and under finite vorticity.

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