Classical vs. quantum spin description in perfect spin hydrodynamics

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Classical spin description

- Phase space extended to include the spin four-vector
- Classical kinetic theory
- Internal angular momentum of a particle defined Mathisson's way

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta, \quad s^\mu = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \tag{1}$$

- Equilibrium distribution functions
  Fermi–Dirac $f^{\pm}_{\rm eq}(x,p,s) = \dfrac{1}{\exp\left(\mp\xi(x)+p\cdot\beta(x)-\frac{1}{2}\omega(x):s\right)+1}$
  or Boltzmann $f^{\pm}_{0,\rm eq}(x,p,s) = \exp\left(\pm\,\xi(x) - p\cdot\beta(x) + \frac{1}{2}\omega(x):s\right)$,
  where $A:B \equiv A_{\alpha\beta}B^{\alpha\beta}$, $\xi \equiv \mu/T$ and $\beta^\mu \equiv u^\mu/T$,
  with μ denoting the chemical potential and u^μ the flow four-vector.
- The conserved currents and tensors

$$N^\mu_{\rm eq} = \int {\rm d}P\,{\rm d}S\, p^\mu \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right] \tag{2}$$

$$T^{\mu\nu}_{\rm eq} = \int {\rm d}P\,{\rm d}S\, p^\mu p^\nu \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right] \tag{3}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = \int {\rm d}P\,{\rm d}S\; p^\lambda s^{\mu\nu} \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right] \tag{4}$$

$\mathcal{N}^\mu_{\rm eq} = -\int {\rm d}P\,{\rm d}S\; p^\mu \left[\ln\left(1-f^+_{\rm eq}(x,p,s)\right) + \ln\left(f^-_{\rm eq}(x,p,s)\right)\right]$

- Integration measures ${\rm d}P = \frac{{\rm d}^3p}{(2\pi)^3 E_p}$, ${\rm d}S = \frac{m}{\pi\mathfrak{s}}{\rm d}^4s\,\delta(s\cdot s + \mathfrak{s}^2)\delta(p\cdot s)$

- The exponential functions in the integrals can be expanded in power series. For instance, up to the second order in ω,

$$N^{\mu}_{\rm eq} = (n_0 + n_2^k + n_2^\omega)u^\mu + n_t t^\mu, \tag{5}$$

$$T^{\mu\nu}_{\rm eq} = (\varepsilon_0 + \varepsilon_2^k + \varepsilon_2^\omega)u^\mu u^\nu - (P_0 + P_2^k + P_2^\omega)\Delta^{\mu\nu} + P_{k\omega}(k^\mu k^\nu + \omega^\mu\omega^\nu) + P_t(t^\mu u^\nu + t^\nu u^\mu), \tag{6}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = u^\lambda\left[A\left(k^\mu u^\nu - k^\nu u^\mu\right) + A_1 t^{\mu\nu}\right] + \frac{A}{2}\left(t^{\lambda\mu}u^\nu - t^{\lambda\nu}u^\mu + \Delta^{\lambda\mu}k^\nu - \Delta^{\lambda\nu}k^\mu\right), \tag{7}$$

using a general parametrization of $\omega_{\alpha\beta}$

$$\omega_{\alpha\beta} = k_\alpha u_\beta - k_\beta u_\alpha + t_{\alpha\beta}, \tag{8}$$

with $k\cdot u = \omega\cdot u = 0$, $t_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}u^\gamma\omega^\delta$, $t^\mu = t^{\mu\nu}k_\nu = \epsilon^{\mu\nu\alpha\beta}k_\nu u_\alpha\omega_\beta$.

- The terms in blue motivate the tensor form of thermodynamic relations,

$$S^{\mu}_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^{\mu}_{\rm eq} + \mathcal{N}^{\mu}_{\rm eq}, \tag{9}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \tag{10}$$

$$d\mathcal{N}^{\mu}_{\rm eq} = N^{\mu}_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_\lambda + \frac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta}. \tag{11}$$

ZD, W. Florkowski, M. Hontarenko, (2024), Phys. Rev. D, **110**(9), 096018.

Quantum spin description

- Start from the Wigner function

$$W^{\pm}(x,k) = \frac{1}{4m}\int \mathrm{d}P\,\delta^{(4)}(k \mp p)(\not{p} \pm m)X_s^{\pm}(\not{p} \pm m), \tag{12}$$

- Use the new equilibrium spin density

$$X_s^{\pm}(x,p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu + \gamma_5 \not{a}\right]. \tag{13}$$

- The conserved currents and tensors are

$$N^\mu(x) = \sum_{r=1}^{2}\int \mathrm{d}P\, p^\mu \left[f_{rr}^{+}(x,p) - f_{rr}^{-}(x,p)\right], \tag{14}$$

$$T^{\mu\nu}(x) = \sum_{r=1}^{2}\int \mathrm{d}P\, p^\mu p^\nu \left[f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p)\right], \tag{15}$$

$$S^{\lambda,\mu\nu}(x) = \frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\, p^\lambda \sigma_{sr}^{+\mu\nu}(p) f_{rs}^{+}(x,p) + \sigma_{sr}^{-\mu\nu}(p) f_{rs}^{-}(x,p), \tag{16}$$

$\mathcal{N}^\mu(x) = \sum_{r=1}^{2}\int \mathrm{d}P\, p^\mu \left[f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p)\right]$,
where

$\sigma_{sr}^{+\mu\nu}(p) = 1/(2m)\,\bar{u}_s(p)\sigma^{\mu\nu}u_r(p)$ and $\sigma_{sr}^{-\mu\nu}(p) = 1/(2m)\,\bar{v}_r(p)\sigma^{\mu\nu}v_s(p)$

S. Bhadury, ZD, W. Florkowski, S.K. Kar, V. Mykhaylova, arXiv:2505.02657 [hep-ph].

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Generating functions

- Classical

$$n_{\rm cl} = 2\int {\rm d}P \cosh\xi \exp\left(-p\cdot\beta\right) \int {\rm d}S \exp\left(\frac{1}{2}\omega : s\right) \tag{17}$$

$$N^{\mu}_{\rm eq} = -\frac{\partial^2 n_{\rm cl}}{\partial\beta_\mu \partial\xi} = \int {\rm d}P\, {\rm d}S\, p^\mu \left[f^{+}_{0,{\rm eq}}(x,p,s) - f^{-}_{0,{\rm eq}}(x,p,s)\right], \tag{18}$$

$$T^{\mu\nu}_{\rm eq} = \frac{\partial^2 n_{\rm cl}}{\partial\beta_\mu \partial\beta_\nu} = \int {\rm d}P\, {\rm d}S\, p^\mu p^\nu \left[f^{+}_{0,{\rm eq}}(x,p,s) + f^{-}_{0,{\rm eq}}(x,p,s)\right], \tag{19}$$

$$S^{\lambda,\mu\nu}_{\rm eq} = -\frac{\partial^2 n_{\rm cl}}{\partial\beta_\lambda \partial\omega_{\mu\nu}} = \int {\rm d}P\, {\rm d}S\; p^\lambda\, s^{\mu\nu} \left[f^{+}_{0,{\rm eq}}(x,p,s) + f^{-}_{0,{\rm eq}}(x,p,s)\right]. \tag{20}$$

- Quantum

$$n_{\rm qt}(x) = 4\int {\rm d}P \exp\left(-p\cdot\beta\right) \cosh\xi\; \cosh\sqrt{-a^2}, \tag{21}$$

$$N^{\mu}(x) = -\frac{\partial^2 n_{\rm qt}}{\partial\beta_\mu \partial\xi} = 2\int {\rm d}P\, p^\mu \left[f^{+}_{0,{\rm eq}}(x,p) - f^{-}_{0,{\rm eq}}(x,p)\right] \cosh\sqrt{-a^2}, \tag{22}$$

$$T^{\mu\nu}(x) = \frac{\partial^2 n_{\rm qt}}{\partial\beta_\mu \partial\beta_\nu} = 2\int {\rm d}P\, p^\mu p^\nu \left[f^{+}_{0,{\rm eq}}(x,p) + f^{-}_{0,{\rm eq}}(x,p)\right] \cosh\sqrt{-a^2}, \tag{23}$$

$$S^{\lambda,\mu\nu}(x) = -\frac{\partial^2 n_{\rm qt}}{\partial\beta_\lambda \partial\omega_{\mu\nu}} = \frac{1}{m}\int {\rm d}P\, p^\lambda \left[f^{+}_{0,{\rm eq}}(x,p) + f^{-}_{0,{\rm eq}}(x,p)\right] \frac{\sinh\sqrt{-a^2}}{\sqrt{-a^2}} \epsilon^{\mu\nu}{}_{\rho\sigma} a^\rho p^\sigma, \tag{24}$$

where

$$a_\mu(x,p) \equiv -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^\nu = -\frac{1}{4m}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}p^\nu. \tag{25}$$

Application range

- The integrals in the generating functions must converge.
- The exponent must be negative for every direction of p as $|p| \to \infty$

$$n_{\mathrm{cl}} = \int \mathrm{d}P\mathrm{d}S \cosh\xi \exp\left(-p\cdot\beta + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) \tag{26}$$

$$n_{\mathrm{qt}} = 2\int \mathrm{d}P \cosh\xi \exp\left(-p\cdot\beta\right) \cosh\sqrt{-\frac{1}{4m^2}\tilde{\omega}_{\alpha\beta}(x)p^{\beta}\tilde{\omega}^{\alpha}{}_{\mu}(x)p^{\mu}} \tag{27}$$

- The spin polarization tensor $\omega_{\mu\nu}$ can be written using its electriclike and magneticlike components, $\boldsymbol{e} = (e^1, e^2, e^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$, as

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}, \tag{28}$$

- The dual tensor $\tilde{\omega}_{\mu\nu}$ can be obtained from it via $\boldsymbol{e} \to \boldsymbol{b}$ and $\boldsymbol{b} \to -\boldsymbol{e}$.

Application range – classical

- We write $n_{\rm cl} = \int \mathrm{d}P\mathrm{d}S \cosh\xi \exp\left(-p'\cdot\beta + \frac{1}{2}\omega_{\alpha\beta}s'^{\alpha\beta}\right)$ in the local rest frame of the fluid element (LRF)
- $u'^{\mu} = (1,0,0,0)$ and the transformed quantities are denoted by primes
- The three-vectors $\boldsymbol{e}$ and $\boldsymbol{b}$ transform as

$$\boldsymbol{e}' = \gamma\left(\boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{e}), \tag{29}$$

$$\boldsymbol{b}' = \gamma\left(\boldsymbol{b} - \boldsymbol{v}\times\boldsymbol{e}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{b}) \tag{30}$$

- The integral over the spin space has a closed form in the LRF, and

$$n_{\rm cl} = 2\int \mathrm{d}P' \exp\left(-\frac{E_{p'}}{T}\right)\frac{\sinh(\mathfrak{s}|\boldsymbol{b}'_*|)}{\mathfrak{s}\,|\boldsymbol{b}'_*|}, \tag{31}$$

where

$$\boldsymbol{b}'_* = \frac{1}{m}\left(E_{p'}\,\boldsymbol{b}' - \boldsymbol{p}'\times\boldsymbol{e}' - \frac{\boldsymbol{p}'\cdot\boldsymbol{b}'}{E_{p'}+m}\boldsymbol{p}'\right) \tag{32}$$

- Thus, convergence is decided by the sign of the exponent

$$|\boldsymbol{p}'|\left[-\frac{1}{T} + \frac{\mathfrak{s}}{m}\sqrt{\boldsymbol{b}'^2 - (\hat{\boldsymbol{p}}'\cdot\boldsymbol{b}')^2 + (\hat{\boldsymbol{p}}'\times\boldsymbol{e}')^2 - 2\hat{\boldsymbol{p}}'\cdot(\boldsymbol{e}'\times\boldsymbol{b}')}\right], \tag{33}$$

where hats denote unit vectors

Application range – quantum

- Analogously to the classical case, we write n_{qt} in the LRF

$$n_{\mathrm{qt}} = 2\int \mathrm{d}P' \cosh\xi \exp\left(-p'\cdot\beta\right) \cosh\sqrt{-a'^2}. \tag{34}$$

- The integral converges if for large $|\boldsymbol{p}'|$ the expression

$$-\frac{p'}{T} + \sqrt{-a'^2} \tag{35}$$

  is negative for every direction of $\boldsymbol{p}'$.

- It can be verified that

$$-a'^2 = \frac{|\boldsymbol{p}'|}{4m^2}\left[\boldsymbol{b}'^2 - (\hat{\boldsymbol{p}}'\cdot\boldsymbol{b}')^2 + (\hat{\boldsymbol{p}}'\times\boldsymbol{e}')^2 - 2\hat{\boldsymbol{p}}'\cdot(\boldsymbol{e}'\times\boldsymbol{b}')\right] \tag{36}$$

- Therefore, in the limit, the exponent is exactly the same as in the classical case but with $\mathfrak{s}$ replaced by $1/2$

$$|\boldsymbol{p}'|\left[-\frac{1}{T} + \frac{1/2}{m}\sqrt{\boldsymbol{b}'^2 - (\hat{\boldsymbol{p}}'\cdot\boldsymbol{b}')^2 + (\hat{\boldsymbol{p}}'\times\boldsymbol{e}')^2 - 2\hat{\boldsymbol{p}}'\cdot(\boldsymbol{e}'\times\boldsymbol{b}')}\right] \tag{37}$$

- Maximization of the square root leads to applicability criteria of perfect spin hydrodynamics
- We also derived simplified criteria to be used when less information about the system is known,
  e.g., just the magnitude of $\boldsymbol{v}$ and $\omega_{\max} \equiv \max(|e_i|, |b_i|)$, $i = 1, 2, 3$

Variables	Classical	Quantum
$\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{v}$	$\mathfrak{F}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2\lvert\boldsymbol{e}' \times \boldsymbol{b}'\rvert} < \frac{m}{T}$	$\frac{1}{2}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2\lvert\boldsymbol{e}' \times \boldsymbol{b}'\rvert} < \frac{m}{T}$
$\boldsymbol{e}, \boldsymbol{b}, v$	$\mathfrak{F}\sqrt{\boldsymbol{b}^2 + \boldsymbol{e}^2 + 2\lvert\boldsymbol{e} \times \boldsymbol{b}\rvert}\sqrt{\frac{1+v}{1-v}} < \frac{m}{T}$	$\frac{1}{2}\sqrt{\boldsymbol{b}^2 + \boldsymbol{e}^2 + 2\lvert\boldsymbol{e} \times \boldsymbol{b}\rvert}\sqrt{\frac{1+v}{1-v}} < \frac{m}{T}$
$\omega_{\max}, v$	$\mathfrak{F}\,\omega_{\max}(2+\sqrt{2})\sqrt{\frac{1+v}{1-v}} < \frac{m}{T}$	$\frac{1}{2}\omega_{\max}(2+\sqrt{2})\sqrt{\frac{1+v}{1-v}} < \frac{m}{T}$

- Primes denote fields transformed to fluid element local reference frame.
- Nonperturbative character of the theory:

$$f^{\pm}_{\mathrm{eq}}(x,p,s) = \exp\left(\pm\xi - p\cdot\beta + \frac{1}{2}\,\omega : s\right) = \exp\left(\pm\xi - p\cdot\beta\right)\left(1 + \frac{1}{2}\,\omega : s + \cdots\right) \tag{38}$$

ZD, W. Florkowski, V. Mykhaylova, (2025), Phys. Rev. D, **112**(5), L051901.

Order-by-order comparison

- Let us expand the exponents in the generating functions in power series and compare order by order in components of $\omega_{\mu\nu}$

$$n_{\rm cl} = 2\int {\rm d}P \exp\left(-p\cdot\beta\right)\cosh\xi \int {\rm d}S \sum_{n=0}^{\infty} \frac{(\omega : s)^n}{n! 2^n} \tag{39}$$

$$n_{\rm qt} = 4\int {\rm d}P \exp\left(-p\cdot\beta\right)\cosh\xi \sum_{n=0}^{\infty} (-1)^n \frac{(-a^2)^n}{(2n)!} \tag{40}$$

- We need the following result

$$\int {\rm d}S \, (\omega : s)^{2n} = (-1)^n \frac{2\mathfrak{s}^{2n}(-a^2)^n 2^{4n}}{2n+1} \tag{41}$$

- It turns out that $n_{\rm cl}$ has the same form as $n_{\rm qt}$ at each order in ω
- The difference is a numerical factor that increases with expansion order.
- The classical and quantum spin pictures exactly agree at the second order for spin four-vector normalization $\mathfrak{s} = \sqrt{3/4}$.

- If $\mathfrak{s}$ were to absorb the numerical factor, it would be no longer a constant but

$$\mathfrak{s}_{2n} = \frac{(2n+1)^{\frac{1}{2n}}}{2} \tag{42}$$

at the order $2n$ in the components of ω.

- It would slowly decrease from $\mathfrak{s}_2 = \sqrt{3/4}$ to

$$\lim_{n\to\infty} \mathfrak{s}_n = \frac{1}{2}, \tag{43}$$

in the limit. This explains again the similarity of application ranges.

- If $\mathfrak{s} = \sqrt{3/4} = \mathrm{const.}$, then the classical and quantum spin pictures exponentially diverge from each other at higher orders.
The $2n$-th order contribution to the generating function is

$$\left(\frac{\mathfrak{s}}{\mathfrak{s}_{2n}}\right)^{2n} = \frac{3^n}{2n+1} \tag{44}$$

times larger in the classical description.

ZD, arXiv:2509.06014 [hep-ph].

Causality and stability

- We use a multi-index notation $N^{\lambda A} \equiv (N^\lambda, T^{\lambda\mu}, S^{\lambda,\mu\nu})$ to write the conservation laws in a compact form,

$$\partial_\lambda N^{\lambda A} = 0 \tag{45}$$

or

$$M^{\lambda AB}\partial_\lambda \zeta_B = 0, \tag{46}$$

where

$$\zeta_A = (\zeta, \zeta_\mu, \zeta_{\mu\nu}) = (\xi, -\beta_\mu, \omega_{\mu\nu}), \quad M^{\lambda AB} = \left(\frac{\partial N^{\lambda A}}{\partial \zeta_B}\right) = -\left(\frac{\partial^3 \chi}{\partial\zeta_B \partial\zeta_A \partial\beta_\lambda}\right). \tag{47}$$

- Equations (46) are symmetric hyperbolic if the four-vector

$$M^\lambda \equiv M^{\lambda AB} Z_A Z_B \tag{48}$$

for any nonvanishing real elements $Z_A = (Z, Z_\mu, Z_{\mu\nu})$ is future oriented and timelike. Then, spin hydrodynamics is nonlinearly causal and stable.

- We write (48) as

$$M^\lambda = \left(Z\frac{\partial}{\partial\xi} - Z_\mu \frac{\partial}{\partial\beta_\mu} + \frac{1}{2} Z_{\mu\nu}\frac{\partial}{\partial\omega_{\mu\nu}}\right)^2 \mathcal{N}^\lambda, \quad \mathcal{N}^\lambda = \frac{\partial\chi}{\partial\zeta_\lambda} = -\frac{\partial\chi}{\partial\beta_\lambda} \tag{49}$$

and prove those properties for each of the four cases: underlying particle statistics Fermi–Dirac vs. Boltzmann, classical vs. quantum spin.

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Summary

Classical vs. quantum spin in perfect spin hydrodynamics

- Both approaches to spin lead to the same thermodynamic relations.
- The applicability range of both formulations are similar.
- The theories have a nonperturbative character.
- The classical and quantum approaches lead to the same conserved currents at the second-order expansion in the spin polarization tensor $\omega_{\mu\nu}$ and differ at higher orders by multiplicative constants.
- The classical results can be transformed to agree with quantum ones through order-by-order rescaling.
- Spin hydrodynamics can be defined as a divergence-type theory, and it is nonlinearly causal and stable.

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