Stability of Non-Relativistic Fluids: Thermodynamic Conditions

R. Somogyfoki[1,2,3], P. Ván[2,3]

[1] *Inst. of Physics, Fac. of Nat. Sci., Eötvös Loránd University, Budapest*
[2] *Inst. for Particle and Nuclear Physics, Wigner Res. Centre for Phys., Budapest*
[3] *Montavid Thermod. Res. Group, Budapest*
somogyfoki.reka@wigner.hu

Thermodynamics

as a stability theory

Thermodynamics is presumably a stability theory, in a dynamic sense [1]. Various extended and generalized heat conduction theories are analysed from this point of view. Local theories of heat conduction and dissipative fluids that are constructed in the framework of non-equilibrium thermodynamics are expected to fulfil this requirement. Therefore, their homogeneous equilibrium should be linearly stable. In general flow-frames it is a challenge in both nonrelativistic and special relativistic theories of dissipative fluids.

Method

- Small perturbation of equilibrium;
- Linearization;
- Damped wave, decreasing amplitudes;
- Application of Routh–Hurwitz criteria;
- E.g. ballistic case stability formula is:

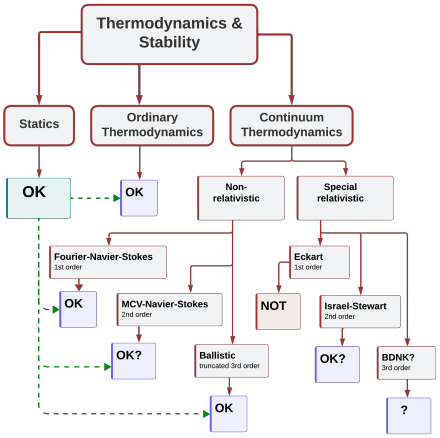
$$0=\rho c_v m_1 m_2 \Gamma^3+\rho c_v\Big[m_1\Big(k_1+nk^2\Big)+m_2\Big(l_1+k_2k^2\Big)\Big]\Gamma^2+$$
$$+\left[\frac{k^2 m_2}{T_0^2}-\rho c_v\Big(k_{12}^2k^2+(l_1+k_2k^2)(k_1+nk^2)\Big)\right]\Gamma+\frac{k^2}{T_0^2}(k_1+nk^2)$$

Expectation: there are no conditions beyond the second law.

Where to go next?

Further challenges

- Higher order fluids. Role of flow-frames.
- Extended heat conduction [4].
- Jeans instability. Thermogravity.
- Open systems. Pattern stability and formation.
- Nonlinear stability.
- Stability of inhomogeneous thermodynamics equilibrium: homogenisation.



Linear stability of various theories of special and Galilean relativistic fluids

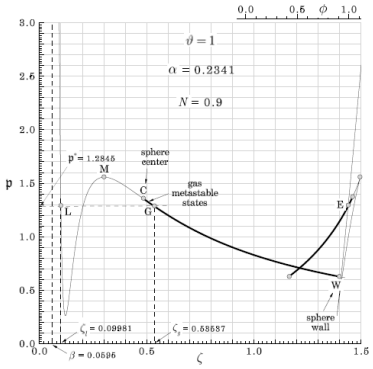
Ballistic heat conduction SoE

$$\rho c_v \partial_t T = -\partial_x q,$$
$$m_1\partial_t q + l_1 q - k_2\partial_{xx} q = -\frac{1}{T^2}\partial_x T - k_{12}\partial_x Q,$$
$$m_2\partial_t Q + k_1 Q - n\partial_{xx} Q = k_{12}\partial_x q,$$

where basic fields are temperature T, heat flux q, and flux of the heat flux Q (a second order tensorial internal variable) incorporating higher order effects in heat transport. m_1, m_2 positive constants, l_1, k_1, k_2, n, k_{12} material coefficients, and l_1 derived from the Fourier heat conduction coefficient.

Conditions: $m_1, m_2, \rho, c_v, l_1, k_1, k_2 > 0$
$k_1k_2 - k_{12}^2 > 0$

Inhomogeneous thd. equilibrium



Self-gravitating isothermal van der Waals' gas sphere [3] fig. 18. Pressure vs. specific volume, (p, ζ), and pressure vs. chemical potential, (p, Φ) states. Thick: center to boundary.

Stability of various fluids

Dissipative fluids are subjects of the Second Law of Thermodynamics by construction. Therefore asymptotic stability of homogeneous thermodynamic equilibrium of the nonlinear partial differential equations of fluid mechanics is the ultimate benchmark of that conjecture. The linear stability follows as a necessary condition.

The equations of fluid mechanics exhibit a hierarchy in both relativistic and non-relativistic cases. In the special relativistic case, the first-order Eckart theory is unstable, and the instability is connected to the dissipative energy transport. The higher order extensions still pose a fundamental challenge in this regard. In the non-relativistic case, the stability of Fourier-Navier-Stokes equations follows from thermodynamics, [2]. The stability of higher order theories is partially extended even to the third-order level.

Fourier–Navier–Stokes SoE

The basic equations for the mass, momentum and energy balances:

$$\dot\rho + \rho\partial_i v^i + \partial_i j^i = 0,$$
$$\dot p^i + p^i\partial_k v^k + \partial_k(P^{ki} + j^k v^i/\rho) = 0^i,$$
$$\dot e + e\partial_k v^k + \partial_i q^i + P^{ik}\partial_i v_k = 0.$$

The constitutive functions are

$$j^i = -\chi\partial^i\frac{\mu}{T} + \chi_{12}\partial^i\frac{1}{T},$$
$$q^i = -\chi_{21}\partial^i\frac{\mu}{T} + \lambda\partial^i\frac{1}{T},$$
$$P^{ij} = p\delta^{ij} - \eta_v\partial_k v^k\delta^{ij} - \eta\Big(\partial^i v^j + \partial^j v^i - \frac{2}{3}\partial_k v^k\delta^{ij}\Big).$$

Here, the mass, momentum and internal energy densities ρ, p^i and e respectively, are the basic fields. P^{ij} is the pressure tensor, v^i is the velocity field, q^i is the energy flux and j^i is the particle flux.

The linear stability follows if the specific entropy is concave and the transport coefficients are nonnegative.

References

[1] Matolcsi, T. Ordinary thermodynamics, Budapest: Akadémiai Kiadó, 2004.

[2] Ván, P., Generic stability of dissipative non-relativistic and relativistic fluids, *Journal of Statistical Mechanics: Theory and Experiment*, 02054, 2009.

[3] Giordano, D. et al. Fluid statics of a self-gravitating isothermal sphere of van der Waals' gas. arXiv:2311.12150, 2023.

[4] Kovács, R. Heat equations beyond Fourier: from heat waves to thermal metamaterials. Physics Reports, 1048, p.1-75., 2024.

[5] Hiscock, W. A., Lindblom, L. Generic instabilities in first-order dissipative relativistic fluid theories. Physical Review D, 31.4: 725., 1985.

Research supported by the National Research, Development and Innovation Office (FK 134277).