Black Hole Quasi-Normal modes

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What are BH QNMs

Particles and fields in the vicinity of a BH slightly change the background space-time of a system. That is, we can consider these as a perturbation.

At the classical level perturbations can be described by damped characteristic modes, called QNMs.

The complex frequencies of such oscillations do not depend on the manner of excitation but only on the parameters of the BH and the field under consideration. Therefore, they are usually called the "fingerprints" of a BH.

In order to calculate QNMs, we impose the QNM b.c. for the wave equations, i.e. we require that at the BH horizon we have only purely in-going waves,

$$R(r_* \to -\infty) \propto \exp(-i\tilde{\omega} r_*) \ ,$$

while we should have only purely out-going waves at spatial infinity, i.e.

$$R(r_* \to \infty) \propto \exp(i\Omega r_*) \ ,$$

where QNMs are solutions of the master equations discussed later, satisfying the above b.c.

How do they relate to Emission spectra

In the same way, as QNMs are an essential classical characteristic of a BH, the thermal Hawking radiation is its essential quantum feature that carries information about the dynamics of evaporation of the BH.

For calculations of the emission rates of particles due to Hawking radiation, one needs first to solve the problem of classical scattering in order to obtain the grey-body factors. This implies the posing of classical scattering b.c.

At the event horizon, this again means imposing the condition of purely in-going wave, while, at spatial infinity ($r \to \infty$), we have a different condition,

$$R(r) \simeq Z_{in} \exp(-i\Omega r_*) + Z_{out} \exp(i\Omega r_*) \ ,$$

where Z_{in} and Z_{out} are integration constants which correspond to the in-going and out-going waves, respectively. Thus, we would like to know which portion of particles will be able to pass through the barrier of the effective potential.

If the coefficients Z_{in} and Z_{out} are calculated, one can find the absorption probability

$$|A_{l,m}| = 1 - |Z_{out}/Z_{in}|^2$$

The emission rates for the energy, charge and angular momentum are proportional to the grey-body factors, and are expressed as

$$-\frac{d}{dt}\begin{pmatrix} M \\ Q \\ J \end{pmatrix} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int |A_{l,m}|^2 \frac{1}{\exp(\tilde{\omega}/T_H) - 1} \frac{d\omega}{2\pi} \begin{pmatrix} e \\ \omega \\ m \end{pmatrix}$$

Here, we perform the summation over all possible values of the quantum numbers l and m.

Thus, we could say that the QN spectrum and Hawking radiation are, respectively, classical and quantum "fingerprints" of a BH.

QNMs and Hawking radiation also have one technical point in common: analysis of QNMs as well as of the Hawking radiation (in semi-classical approximations) begin from the linear perturbations of the fields under consideration whose dynamics should be reduced to a single wave-like equation, called the master equation.

The Master Equations

Perturbations of a BH space-time can be performed in two ways:
- by adding fields to the BH space-time or
- by perturbing the BH metric (the background) itself.

In the linear approximation (where a field does not backreact on the background), the first type of perturbation is reduced to the propagation of fields in the background of a BH, which is, in many cases, a general covariant equation of motion of the corresponding field.

The covariant form of the equation of motion is quite different for fields of different spin s in curved backgrounds. Thus, for a scalar field Φ of mass μ in the background of the metric $g_{\mu\nu}$, the e.o.m. is the general covariant Klein-Gordon equation

$$(\nabla^{\nu}\nabla_{\nu} - \mu^2)\Psi = 0 \ ,$$

where ∇_{ν} is the covariant derivative. The above equation can be written explicitly as follows:

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\left(g^{\mu\nu}\sqrt{-g}\partial_{\mu}\Psi\right) - \mu^2\Psi = 0 \quad (s = 0).$$

For massive Dirac fields in a curved background $g_{\mu\nu}$, the e.o.m. reads

$$(\gamma^a e_a{}^\mu(\partial_\mu + \Gamma_\mu) + \mu)\Phi = 0 \ , \quad (s = \pm 1/2)$$

where μ is the mass of the Dirac field, and $e_a{}^\mu$ is the tetrad field, defined by the metric $g_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b \ , \quad g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu \ ,$$

$$e_a{}^\mu e_\mu{}^b = \delta_a^b, \quad e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu \ ,$$

where η_{ab} is the Minkowskian metric, γ^a are the Dirac matrices: $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and Γ_μ is the spin connection

$$\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b] \, g_{\nu\lambda} \, e_a{}^\nu \, \nabla_\mu e_b{}^\lambda \ .$$

For massive vector perturbations we have the general covariant generalization of the Proca equations. For a vector potential A_μ, one has

$$\nabla^\nu F_{\mu\nu} - \mu^2 A_\mu = 0, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

In a curved space-time these equations read

$$\frac{1}{\sqrt{-g}}\partial_\nu(F_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu}\sqrt{-g}) - \mu^2 A^\mu = 0 \ . \qquad (s=1)$$

When $\mu = 0$ in the above form of the Proca equation, we obtain the Maxwell equation

$$\partial_\nu((\partial_\alpha A_\sigma - \partial_\sigma A_\alpha) g^{\alpha\mu} g^{\sigma\nu}\sqrt{-g}) = 0.$$

There may be various generalizations of the massive scalar, spinor, and vector fields considered above. Thus, if we study perturbations of massive charged particles in scalar electrodynamics in a curved charged background, we have to deal with a complex scalar field

$$(D^{\nu} D_{\nu} - \mu^2)\Psi = 0 \ ,$$

where $D_{\nu} = \nabla_{\nu} - ieA_{\nu}$ is an "extended" covariant derivative, and e is the charge of the particle. Finally, we find that the e.o.m. of the charged scalar field in a curved space-time reads

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\left(g^{\mu\nu}\sqrt{-g}(\partial_{\mu}\Psi - ieA_{\mu}\Psi)\right) - ieA^{\nu}\partial_{\nu}\Psi - (\mu^2 + e^2 A^{\nu} A_{\nu})\Psi = 0$$

In a similar fashion, the massive charged Dirac particle is described by the e.o.m. with an extended derivative $\partial_\mu \to \partial_\mu - ieA_\mu$,

$$(\gamma^a e_a{}^\nu(\partial_\nu + \Gamma_\nu - ieA_\nu) + \mu)\Phi = 0$$

Another type of perturbation, metric perturbations, can be written in the linear approximation in the form

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu} \ ,$$

$$\delta R_{\mu\nu} = \kappa\ \delta\left(T_{\mu\nu} - \frac{1}{D-2} T g_{\mu\nu}\right) + \frac{2\Lambda}{D-2}\delta g_{\mu\nu}$$

Linear approximation means that the terms of order $\sim \delta g^2_{\mu\nu}$ and higher are neglected. The unperturbed space-time given by the metric $g^0_{\mu\nu}$ is called the background.

Separation of variables

The first step toward the analysis of BH perturbation equations is their reduction to a 2d wavelike form with decoupled angular variables. Once the variables are decoupled, an equation for the radial and time variables usually has the Schrödinger-like form for stationary backgrounds,

$$-\frac{d^2 R}{dr_*^2} + V(r,\omega)R = \omega^2 R \ ,$$

and can be treated by a number of sophisticated and well developed numerical, analytical, and semi-analytical methods.

As a simple example, for the massless scalar field on the Schwarzschild background

$(g^0_{\mu\nu} : g_{tt} = -g^{-1}_{rr} = 1 - 2M/r$, $g_{\theta\theta} = g_{\phi\phi} \sin^{-2}\theta = r^2)$,

and after using a new variable $dr_* = \dfrac{dr}{1 - 2M/r}$,

where the coordinate r_* maps the semi-infinite region from the horizon to infinity into the $(-\infty, +\infty)$ region (these coordinates are known as tortoise coordinate).

The wave function can be written as

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} Y_\ell(\theta, \phi) R(r)/r \ ,$$

produces a potential

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right) ,$$

where $s = 0$, and ℓ is the multipole quantum number, which arises from the separation of angular variables by expansion into spherical harmonics

$$\Delta_{\theta,\phi} Y_\ell(\theta,\phi) = -\ell(\ell+1) Y_\ell(\theta,\phi) ,$$

exactly in the same way as happens for the hydrogen atom in QM when dealing with the Schrödinger equation.

When $s = 1$ the effective potential corresponds to the Maxwell field. When $s = 2$ we obtain the effective potential of the gravitational perturbations of the axial type, which was derived by Regge and Wheeler

The separation of variables, however, is not always so easy. The variables in perturbation equations cannot be decoupled for perturbations of an arbitrary metric. For this to happen, the metric must possess sufficient symmetry, expressed in the existence of the Killing vectors, Killing tensors, and Killing-Yano tensors.

The choice of appropriate coordinates is crucial for separation of variables.

Methods for QNM calculations

Remembering that $\Psi \sim e^{-i\omega t}$, we write the QNM frequencies in the following form:

$$\omega = \omega_{Re} - i\omega_{Im} \ .$$

Here ω_{Re} is the real oscillation frequency of the mode and ω_{Im} is proportional to its damping rate. Positive ω_{Im} means that Ψ is damped, negative ω_{Im} means an instability.

Note also that for Kerr BHs, as well as for other astrophysical or string theory motivated cases, QNMs form a countable set of discrete frequencies

QNMs calculated in the linear approximation are in good agreement with those obtained by the fully nonlinear integration of the Einstein equations, at least at sufficiently late time.

Mashoon method

We start from the usual wavelike equation, with an effective potential, which depends on some parameter α:

$$\frac{d^2\Psi}{dr_*^2} + (\omega^2 - V(r_*, \alpha))\Psi = 0 \; .$$

Because of "symmetric" b.c. for the QNM problem at both infinities $r_* \to \pm\infty$, it is reasonable to consider transformations $r_* \to -ir_*$ and $p \to p'$, such that the potential is invariant under these transformations

$$V(r_*, \alpha) = V(-ir_*, \alpha') \; .$$

The wave function Ψ and the QN frequency ω transform as

$$\Psi(r_*, \alpha) = \Phi(-ir_*, \alpha') \; , \quad \omega(\alpha) = \Omega(\alpha')$$

The wave equation for Φ and the b.c. will read

$$\frac{d^2\Phi}{dr_*^2} + (-\Omega^2 + V)\Phi = 0, \quad \Phi \sim e^{\mp\Omega r_*}, \quad r_* \to \mp\infty \,.$$

These boundary conditions correspond to a vanishing wave function at the boundaries, such that the QNM problem is now reduced to the bound states problem for an inverse potential $V \to -V$.

This potential can be approximated by the Pöschl-Teller potential,

$$V_{PT} = \frac{V_0}{\cosh^2 \alpha(r_* - r_*^0)} \,.$$

Here V_0 is the height of the effective potential and $-2V_0\alpha^2$ is the curvature of the potential at its maximum.

The bound states of the Pöschl-Teller potential are well known, where the QNMs ω can be obtained from the inverse transformation $\alpha' = i\alpha$,

$$\omega = \pm\sqrt{V_0 - \frac{1}{4}\alpha^2} - i\alpha\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

Technically one has to fit a given black hole potential to the inverted Pöschl-Teller potential.

Note that this method gives quite accurate results for the regime of high multipole numbers ℓ, i.e. for the eikonal (geometrical optics) approximation.

Note that there are many techniques to solve for the QN frequencies, we shall now present only two of these:

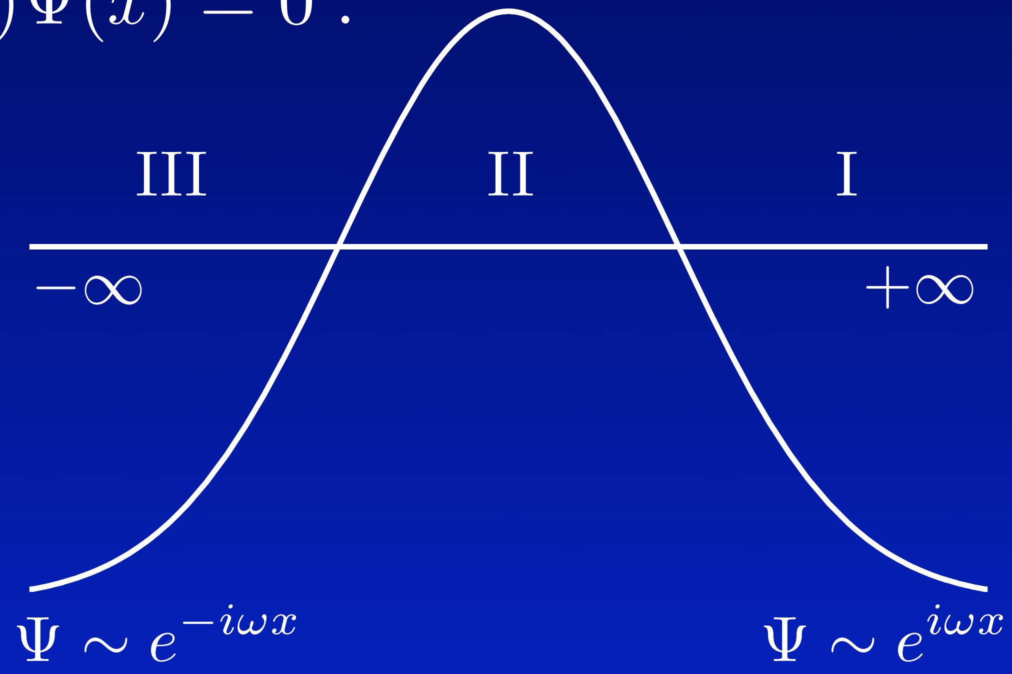
WKB method

The method is based on matching asymptotic WKB solutions at spatial infinity and the event horizon with a Taylor expansion near the top of the potential barrier through the two turning points.

We rewrite the wave equation as

$$\frac{d^2\Psi}{dx^2} + Q(x)\Psi(x) = 0 \; .$$

where we used $x = r_*$ and $Q(x) = \omega^2 - V$.



The asymptotic WKB expansion

$$\Psi \sim exp\left(\sum_{n=0}^{\infty} \frac{S_n(x)\epsilon^n}{\epsilon}\right) \ .$$

$$\Rightarrow S_0(x) = \pm i \int^x Q(\eta)^{1/2} d\eta \ , \text{ and } S_1(x) = -\frac{1}{4} \ln Q(x) \ldots$$

The two choices of sign above correspond to either incoming or outgoing waves at either of the infinities

Therefore we have four solutions: Ψ_+^I, Ψ_-^I, Ψ_+^{III} and Ψ_-^{III} respectively for plus and minus signs in S_0 in regions I and III, with general solution

$$\Psi \sim Z_{in}^I \Psi_-^I + Z_{out}^I \Psi_+^I, \quad region \quad I \ ,$$

$$\Psi \sim Z_{in}^{III} \Psi_+^{III} + Z_{out}^{III} \Psi_-^{III}, \quad region \quad III \ .$$

The amplitudes at $+\infty$ are connected with the amplitudes at $-\infty$ through the linear matrix relation

$$\begin{pmatrix} Z_{out}^{III} \\ Z_{in}^{III} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} Z_{out}^{I} \\ Z_{in}^{I} \end{pmatrix}.$$

Now we need to match both WKB solutions with a solution in region II through the two turning points $Q(x) = 0$.

If the turning points are closely spaced, i.e. if $-Q(x)_{max} \ll Q(\pm\infty)$, then the solution in region II can be well approximated by the Taylor series

$$Q(x) = Q_0 + \frac{1}{2} Q_0''(x - x_0)^2 + O((x - x_0)^3) \ ,$$

where region II corresponds to $|x - x_0| < \sqrt{\frac{-2Q_0}{Q_0''}} \approx \epsilon^{1/2}$, region II. The latter relation gives also the region of validity of the WKB approximation: ϵ must be small.

The general solution in region II can be expressed in terms of parabolic cylinder functions $D_\nu(t)$,

$$\Psi = AD_\nu(t)+BD_{-\nu-1}(it)\ .\ \text{where}\ \nu+\frac{1}{2} = -iQ_0/(2Q_0'')^{1/2}\ .$$

such that we obtain the elements of the S matrix,

$$\left(\begin{array}{c} Z^{III}_{out} \\ Z^{III}_{in} \end{array}\right) = \left(\begin{array}{cc} e^{i\pi\nu} & \frac{iR^2 e^{i\pi\nu}(2\pi)^{1/2}}{\Gamma(\nu+1)} \\ \frac{R^{-2}(2\pi)^{1/2}}{\Gamma(-\nu)} & -e^{i\pi\nu} \end{array}\right) \left(\begin{array}{c} Z^{I}_{out} \\ Z^{I}_{in} \end{array}\right),$$

where $R = (\nu+1)^{(\nu+1/2)/2} e^{-(\nu+1/2)/2}$.

When expanding to higher WKB orders, the S matrix has the same general form, though with modified expression for R, which still depends only on ν.

Note that for a BH $Z_{in}^{III} = 0$, due to the QNM b.c. $Z_{in}^{I} = 0$. Both these conditions are satisfied only if $\Gamma(-\nu) = \infty$, and, consequently, ν must be an integer.

This gives us the complex QNMs labelled by an overtone number ν at the first WKB order. Later this approach was extended to the third and sixth order WKB.

In addition to solving the QNM problem, the S-matrix allows us to solve the standard scattering problem, which describes tunneling of waves and particles through the potential barrier of a BH. One can easily check that for real $Q(x)$ (for real energy of the incident wave and spherically symmetric backgrounds)

$$S_{11}^{*} = S_{22}, \quad S_{12} = S_{21}^{*}, \quad |S_{21}|^2 - |S_{11}|^2 = 1$$

and the transmission coefficient is

$$T = \frac{|Z_{out}^{III}|^2}{|Z_{in}^{I}|^2} = S_{21}^{-1} \, .$$

The reflection coefficient is $R = 1 - T$.

QNMs of mini-BHs

To conclude I'd like to mention that the QN spectra of BHs has attracted considerable interest in the following extra dimensional models:

• The **large extra dimensions** scenario, which allows for the size of the extra dimensions to be of a macroscopic order. When the size of the BH is much smaller than the size of the extra dimensions, the BH can be considered as effectively living in a D-dimensional world and, thereby, approximated by a solution of higher-dimensional Einstein equations. The simplest example of such a solution is the Tangherlini metric.

• **Randall-Sundrum models** assumed that our world is a brane in higher-dimensional warped AdS space-time. This implies a quick decay of the fields outside the brane. The warp factor, which is the parameter of the theory, can be set up in order to obtain a large size (of the order of TeV) or, if one wishes, a small size of the extra dimensions.

• **Brane-localized fields**: When the size of extra dimensions is larger than the size of the BH, one can consider the model where the SM particles are restricted to live on a $(3+1)$-brane, while gravitons propagate in the bulk. When considering the evolution of the SM fields in the background of a mini BH, one can think that the mini BH effectively behaves similar to a higher-dimensional one projected onto the brane.