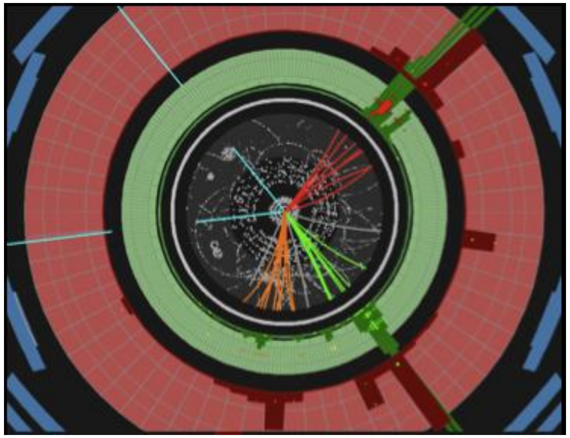
QCD at Colliders

Lecture 3
Modern QCD amplitude organization



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of High Energy Physics

SLAC NATIONAL ACCELERATOR LABORATORY

Recall factorization formula

$$\sigma^{pp\to X}(s;\alpha_s,\mu_R,\mu_F) = \sum_{a,b}\int_0^1 dx_1 \int_0^1 dx_2\ f_a(x_1,\alpha_s,\mu_F)\, f_b(x_2,\alpha_s,\mu_F) \times \hat{\sigma}^{ab\to X}(sx_1x_2;\alpha_s,\mu_R,\mu_F)$$

To improve theoretical predictions, need to:

1) Evolve PDFs as accurately as possible:

$$P_{ij}^{\mathrm{LO}}(z) + \alpha_s P_{ij}^{\mathrm{NLO}}(z) + \alpha_s^2 P_{ij}^{\mathrm{NNLO}}(z)$$

1973 1977 2004 (Moch, Vermaseren , Vogt)

2) Compute higher-order corrections to partonic cross section $\hat{\sigma}^{ab\to X}$ for as many processes as possible

Short-Distance Cross Section in Perturbative QCD

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO NLO NNLO

Leading-order (LO) predictions only qualitative:

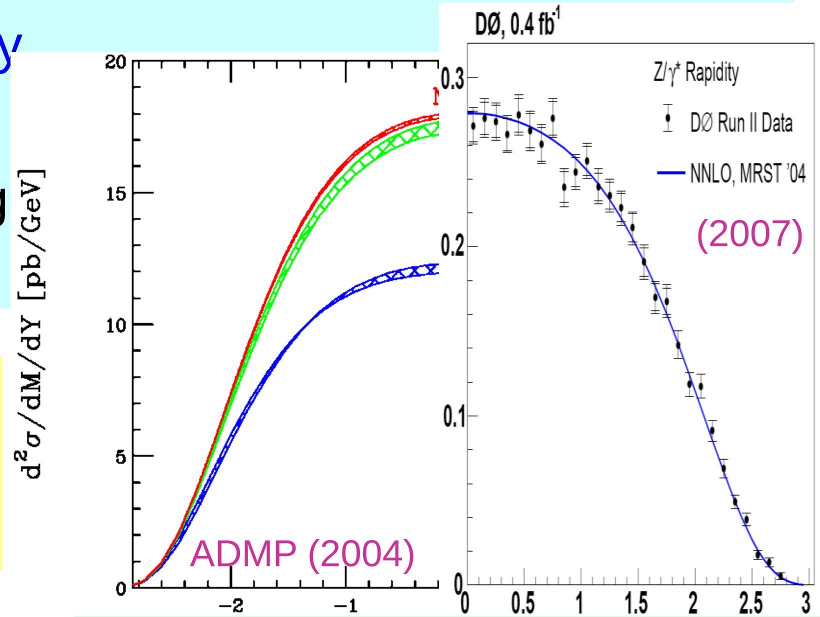
Expansion in $\alpha_s(\mu)$ behaves poorly

- Estimate "error" bands by varying

$$\mu_R = \mu_F = \mu$$

Example: *Z* production at Tevatron as function of rapidity Y (~polar angle)

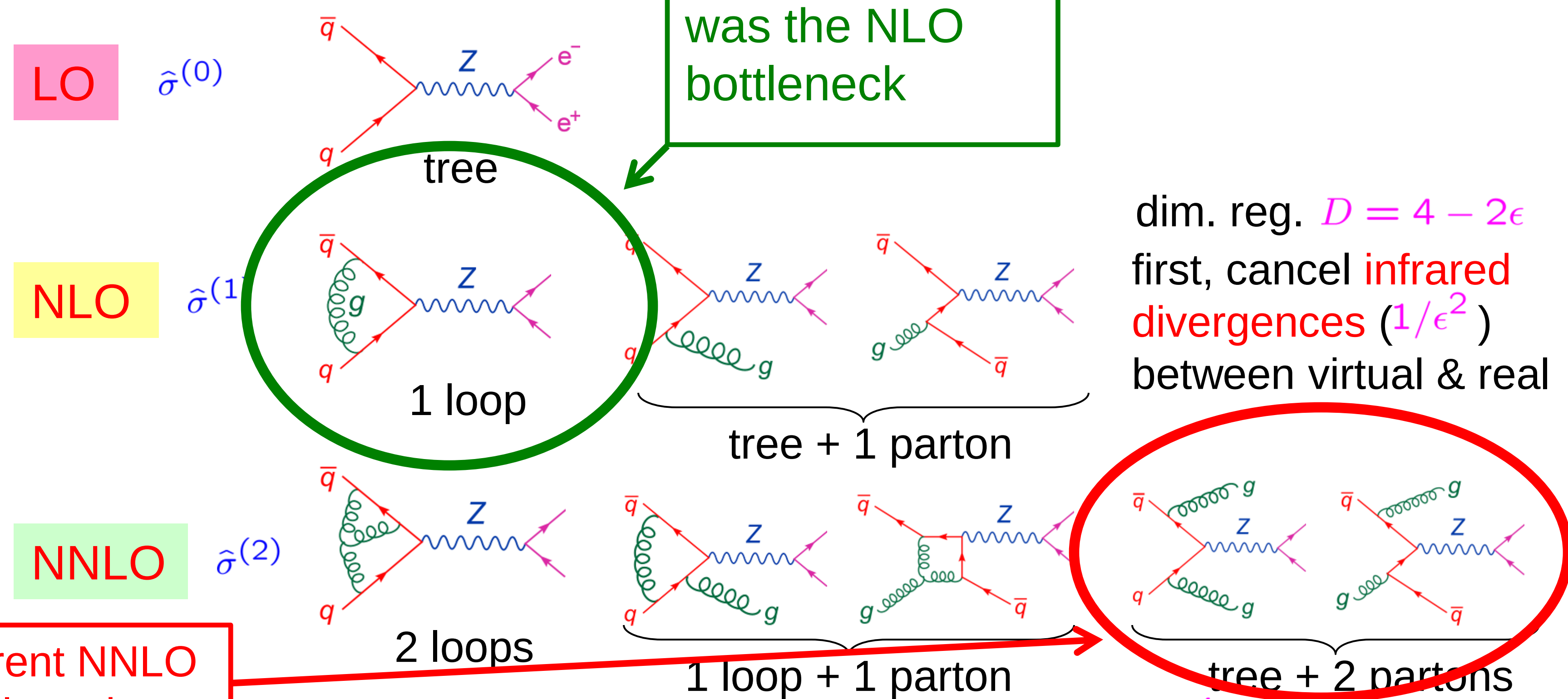
50% shift, LO → NLO



by NNLO, a precision observable

QCD corrections in a nut-shell

"Trivial" example: Z production at hadron colliders



was the NLO bottleneck

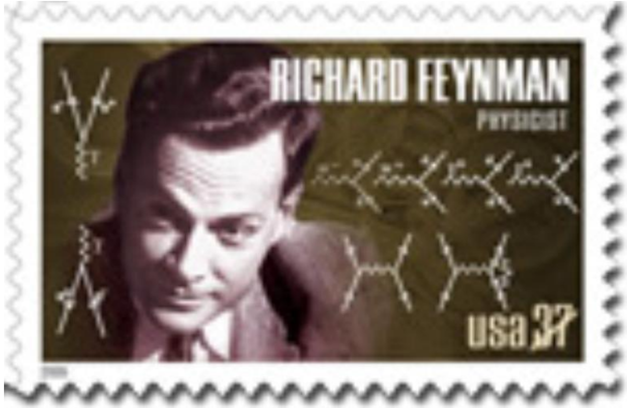
dim. reg. $D = 4 - 2\epsilon$

first, cancel infrared divergences ($1/\epsilon^2$) between virtual & real

current NNLO bottleneck

intricate ($1/\epsilon^4$) IR cancellations

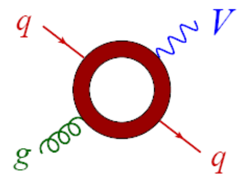
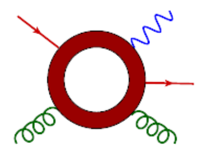
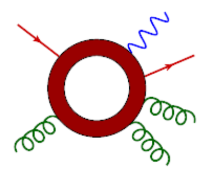
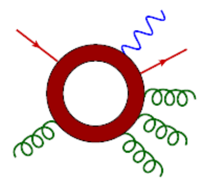
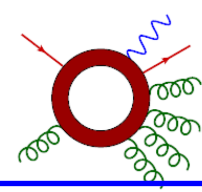
Beyond Feynman Diagrams



- **Feynman diagrams** are very general and **powerful**
- However, for many applications, **on-shell methods** based on analyticity are a much more efficient way to get the same answer.
- They also give new insight into structure and properties of scattering amplitudes, not only in QCD

Just one QCD loop can be a challenge

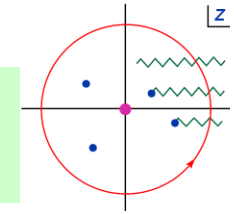
***pp* → *W* + *n* jets** (just amplitudes with most gluons)

# of jets		# 1-loop Feynman diagrams
1		11
2		110
3		1,253
4		16,648
5		256,265

Current limit with Feynman diagrams

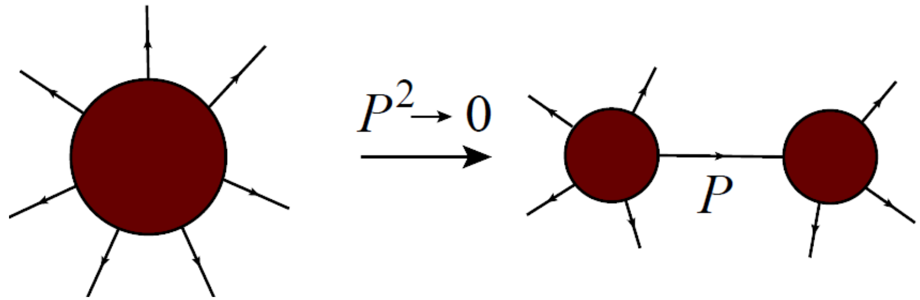
Current limit with on-shell methods

The Analytic *S*-Matrix



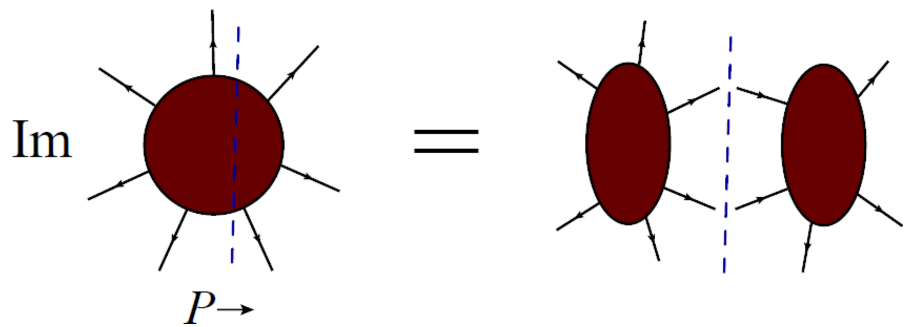
Bootstrap program for strong interactions: Reconstruct scattering amplitudes **directly** from **analytic properties: "on-shell" information**

- Poles



Landau; Cutkosky; Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; … (1960s)

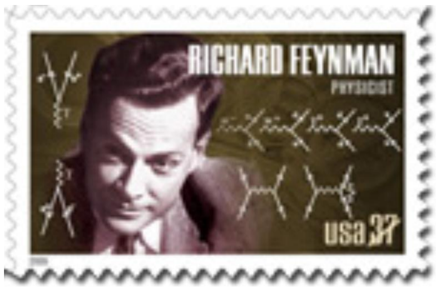
- Branch cuts

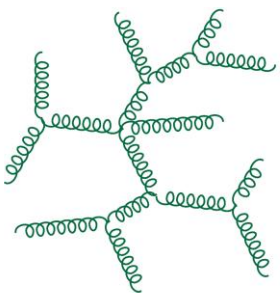


Analyticity fell out of favor in 1970s with the rise of QCD & Feynman rules

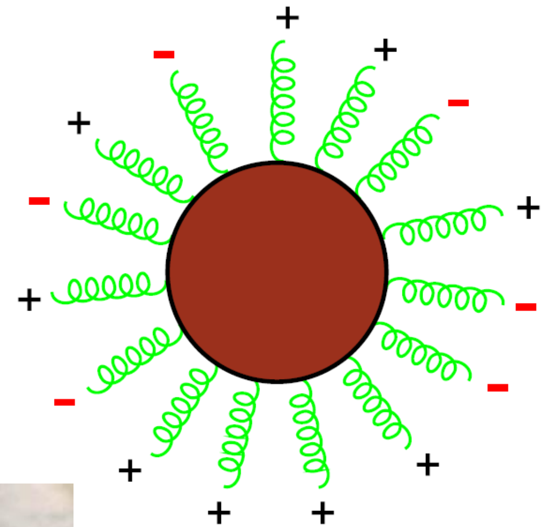
Now **resurrected** for computing loop amplitudes in **perturbative QCD** as **alternative to Feynman diagrams! Perturbative information now assists analyticity. Works for many other theories too.**

Granularity vs. Plasticity





+ …

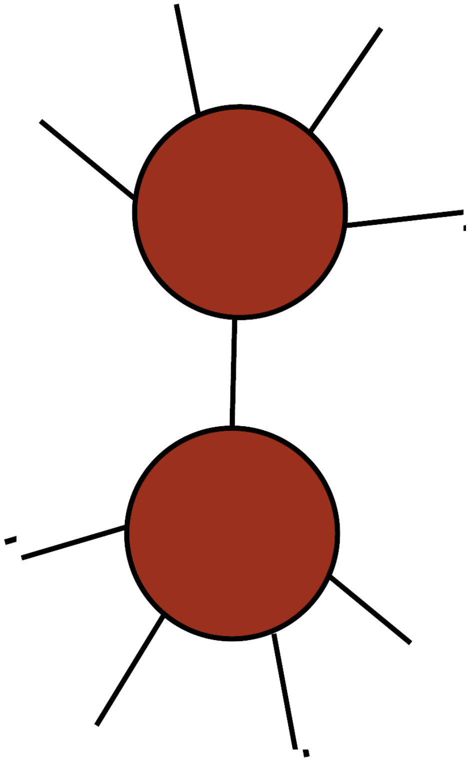


Recycling “Plastic” Amplitudes

Amplitudes fall apart into simpler ones in special limits – pole information

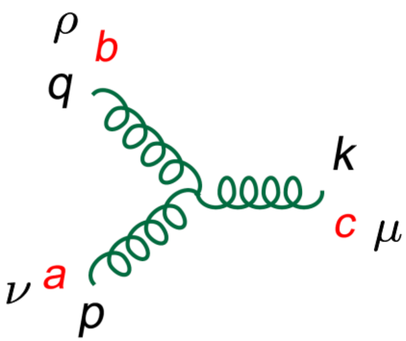
Trees recycled into trees

BCFW recursion relations



How to organize gauge theory amplitudes

- Avoid tangled algebra of color and Lorentz indices generated by Feynman rules



$$= i g_s f^{abc} [\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho]$$

structure constants

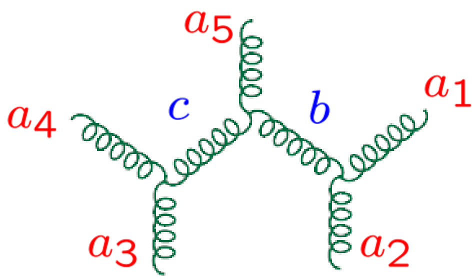
- Take advantage of physical properties of amplitudes

- Basic tools:
  - dual (trace-based) color decompositions
  - spinor helicity formalism

Color

Book by Cvitanovic

Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:



$$\propto \; f^{a_1 a_2 b}\, f^{a_3 a_4 c}\, f^{b c a_5}$$

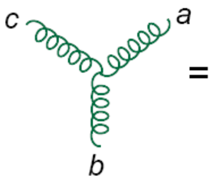
We can write every n-gluon tree graph color factor as a sum of traces of matrices T^a in the fundamental (defining) representation of $SU(N_c)$:

$$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) \quad \text{+ all non-cyclic permutations}$$

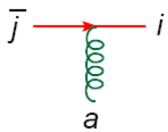
Use definition: $[T^a, T^b] = i\, f^{abc}\, T^c$

+ normalization: $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ $\rightarrow$ $\boxed{f^{abc} = -i\, \mathrm{Tr}([T^a, T^b]\, T^c)}$

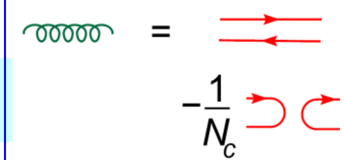
Color in pictures

Insert 

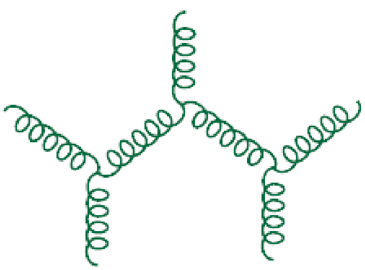
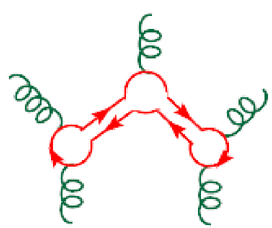
$= f^{abc} = Tr([T^a, T^b]\ T^c) =$ 

where 

$= (T^a)_i^{\bar{j}}$ is color factor for *qqg* vertex

and 

into typical string of f^{abc} structure constants for a Feynman diagram:

 $=$  + *permutations* $= \mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ + permutations

- Always single traces (at tree level)
- $\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ comes only from those planar diagrams

with cyclic ordering of external legs fixed to 1,2,…,n

Trace-based (dual) color decomposition

Similarly $q\bar{q}gg\cdots g$ amplitudes $\Rightarrow (T^{a_1}T^{a_2}\cdots T^{a_n})_i^{\bar{j}}$ + permutations

In summary, for the n-gluon trees, the color decomposition is

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g_s^{n-2}\,\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\,A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) + \text{non-cyclic perm's}$$

momenta

color

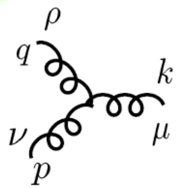
helicities $h_i = \pm 1$

color-ordered subamplitude only depends on momenta. Compute separately for each cyclicly inequivalent helicity configuration $(h_1, h_2, \ldots, h_n)$

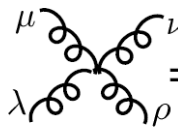
- Because $A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ comes from planar diagrams with cyclic ordering of external legs fixed to $1,2,\ldots,n$, it only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

Color-ordered Feynman rules

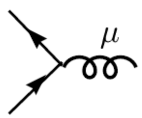
In Feynman gauge, use these “color-stripped” rules



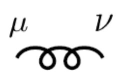
$$= \frac{i}{\sqrt{2}}\Big(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho\Big)$$



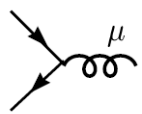
$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$



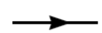
$$= \frac{i}{\sqrt{2}}\gamma_\mu$$



$$= -i\frac{\eta_{\mu\nu}}{p^2}$$



$$= -\frac{i}{\sqrt{2}}\gamma_\mu$$



$$= \frac{i}{\not{p}}$$

in the (1,2,…,n)-ordered planar diagrams, to compute

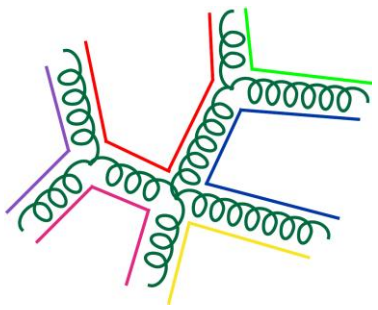
$$A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \qquad A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, 3^{h_3}, \ldots, n^{h_n})$$

$$gg\cdots g \qquad q\bar{q}gg\cdots g$$

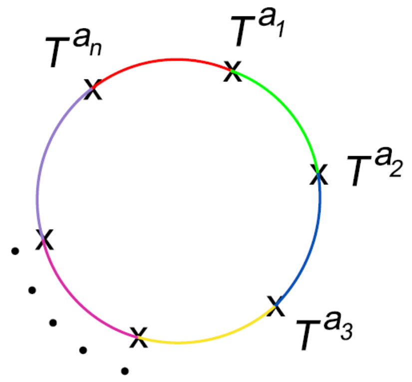
etc.

Aside: Strings and Color

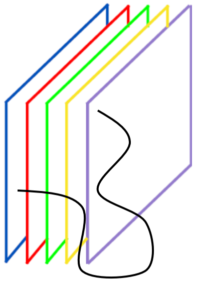
• The "trace" color basis for QCD is also called the "dual" basis because it first arose, as Chan-Paton factors in the description of SU(N) symmetric dual models (string theories)
– initially it was describing flavor!
• A modern string theorist would say that a string end moves from one of N D-branes to another by emitting a green-antiblue gluon
• Also related to 't Hooft double-line formalism



't Hooft



dual



D-brane

Color sums

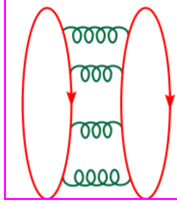
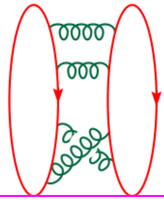
In the end, we want to sum/average over final/initial colors (as well as helicities):

$$d\sigma^{\text{tree}} \propto \sum_{a_i} \sum_{h_i} |\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\})|^2$$

Insert:

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g_s^{n-2}\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) + \text{non-cyclic perm's}$$

and do the color sums diagrammatically:

 $= N_c^n$  $= N_c^n \times \frac{1}{N_c^2}$

Exercise: Convince yourself of this!

to get:

$$\boxed{d\sigma^{\text{tree}} \propto N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |A_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}), \ldots, \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{-2})}$$

→ Up to $1/N_c^2$ suppressed effects, squared subamplitudes have definite color flow – important for development of parton shower

Spinor helicity formalism

Scattering amplitudes for massless plane waves of definite momentum:
Lorentz 4-vectors k_i^μ $\qquad$ $k_i^2=0$

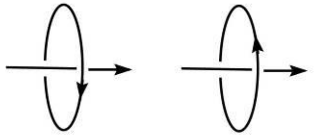
Natural to use Lorentz-invariant products (invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for elementary particles with **spin** (*e.g.* all observed ones!)
there is a better way:

Take "square root" of 4-vectors k_i^μ (spin 1)
use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ $\qquad$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

q,g,γ, all have 2 helicity states, $\quad h = \pm$



Massless Dirac spinors

- Positive and negative energy solutions to the massless Dirac equation, $\not{k}\, u(k) = 0, \quad \not{k}\, v(k) = 0$

  are identical up to normalization.
- Chirality/helicity eigenstates are

$$u_\pm(k) = \frac{1}{2}(1 \pm \gamma_5) u(k), \quad v_\pm(k) = \frac{1}{2}(1 \pm \gamma_5) v(k)$$

- Explicitly, in the Dirac representation,

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \\ \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \end{bmatrix}, \quad u_-(k) = v_+(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^-} e^{-i\varphi_k} \\ -\sqrt{k^+} \\ -\sqrt{k^-} e^{-i\varphi_k} \\ \sqrt{k^+} \end{bmatrix}$$

$$e^{\pm i\varphi_k} \equiv \frac{k^1 \pm i k^2}{\sqrt{(k^1)^2 + (k^2)^2}}$$

$$k^\pm = k^0 \pm k^3$$

Spinor products

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products: $\bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$

$$\bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_j)_{\dot\beta} = [i\, j]$$

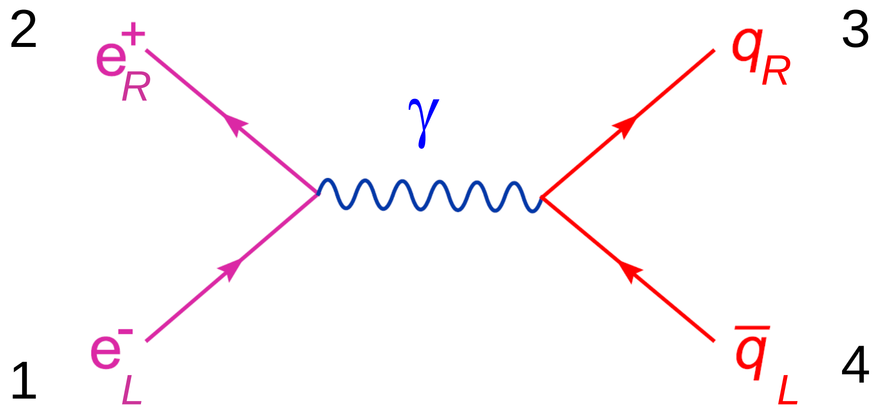
Identity $k_i^\mu(\sigma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = u_+(k_i)\bar{u}_+(k_i) = (\lambda_i)_\alpha(\tilde\lambda_i)_{\dot\alpha}$

$\Rightarrow$ These are **complex square roots** of Lorentz products (for real k_i):

$$\langle i\, j\rangle[j\, i] = \frac{1}{2}\mathrm{Tr}\left[\not{k}_i\ \not{k}_j\right] = 2k_i \cdot k_j = s_{ij}$$

$$\boxed{\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}}$$

Most famous (simplest) Feynman diagram



add helicity information,
numeric labels

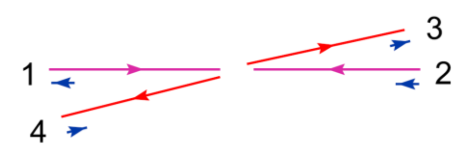
$$\mathcal{A}_4 = 2ie^2 Q_e Q_q \delta_{i_3}^{\bar{\imath}_4} A_4$$

$$\begin{aligned}
A_4 &= \frac{1}{2\,s_{12}} \overline{v_-}(k_2)\gamma^\mu u_-(k_1)\ \overline{u_+}(k_3)\gamma_\mu v_+(k_4) \\
&= \frac{1}{2\,s_{12}} (\sigma^\mu)_{\alpha\dot\alpha}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}\ (\sigma_\mu)^{\dot\beta\beta}(\tilde\lambda_3)_{\dot\beta}(\lambda_4)_\beta \\
&= \frac{1}{s_{12}} (\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_4)_\alpha(\tilde\lambda_3)_{\dot\alpha}
\end{aligned}$$

Fierz identity

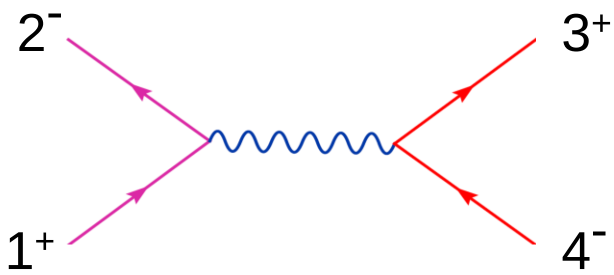
$$(\sigma^\mu)_{\alpha\dot\alpha}\,(\sigma_\mu)^{\dot\beta\beta} = 2\ \delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta}$$

$$\boxed{A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}}} = e^{i\phi}\frac{s_{13}}{s_{12}} = \frac{-e^{i\phi}}{2}(1-\cos\theta)$$



helicity suppressed
as 1 || 3 or 2 || 4

Useful to rewrite answer



Crossing symmetry more manifest
if we switch to all-outgoing helicity labels
(flip signs of incoming helicities)

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}} = \frac{\langle 2\,4\rangle\,[1\,3]}{\langle 1\,2\rangle[2\,1]}\frac{\langle 1\,3\rangle}{\langle 1\,3\rangle} = -\frac{\langle 2\,4\rangle\,[2\,4]\langle 2\,4\rangle}{\langle 1\,2\rangle[2\,4]\langle 4\,3\rangle}$$

$$\boxed{A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}}$$ "holomorphic"

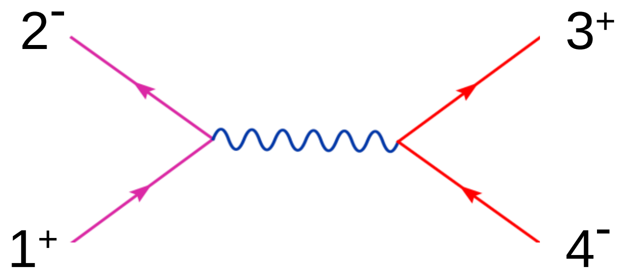
or $$\boxed{A_4 = \frac{[1\,3]^2}{[1\,2][3\,4]}}$$ "antiholomorphic"

useful identities:

$$\langle i\,j\rangle = -\langle j\,i\rangle$$
$$[i\,j] = -[j\,i]$$
$$\langle i\,i\rangle = [i\,i] = 0$$
$$\langle i\,j\rangle\,[j\,i] = s_{ij}$$
$$\sum_{j=1}^{n} \langle i\,j\rangle\,[j\,k] = 0$$
$$s_{12} = s_{34}$$
$$s_{13} = s_{24}$$
$$\langle i\,j\rangle\langle k\,l\rangle - \langle i\,k\rangle\langle j\,l\rangle = \langle i\,l\rangle\langle k\,j\rangle$$

Schouten

Symmetries for all other helicity config's

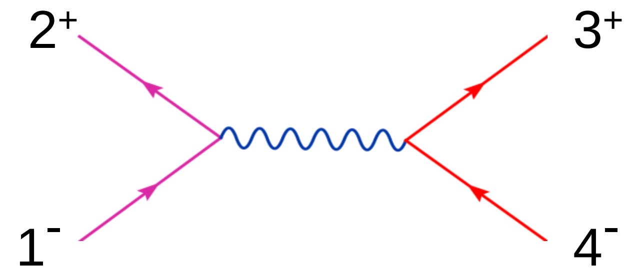


$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

C

$1 \leftrightarrow 2$

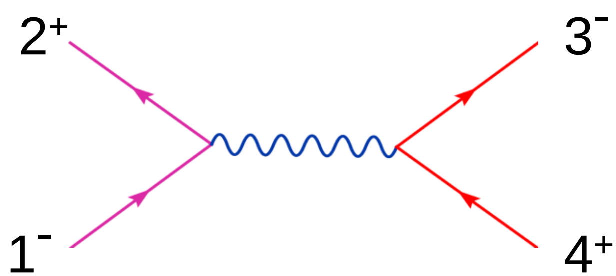
$\Rightarrow$



$$A_4 = -\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

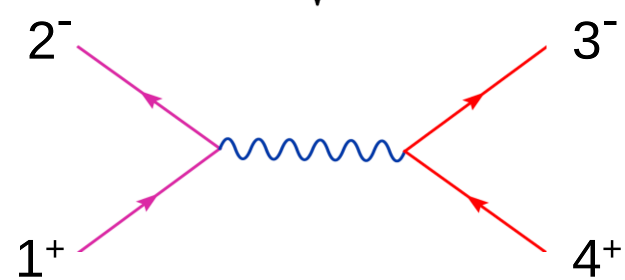
P $\Downarrow$ $\langle\,\rangle \leftrightarrow [\,]$

$\Downarrow$



$$A_4 = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

$\Rightarrow$



$$A_4 = -\frac{[1\,4]^2}{[1\,2][3\,4]}$$

Unpolarized, helicity-summed cross sections

(the norm in QCD)

$$\frac{d\sigma(e^+e^- \to q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2\left\{\left|\frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2 + \left|\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2\right\}$$

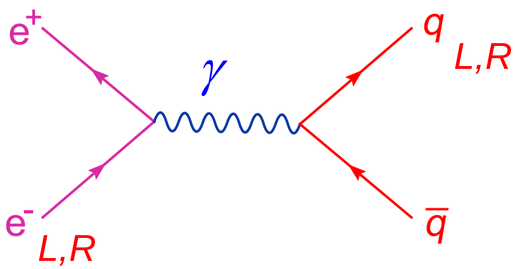
$$= 2\,\frac{s_{24}^2 + s_{14}^2}{s_{12}^2}$$

$$= \frac{1}{2}[(1-\cos\theta)^2 + (1+\cos\theta)^2]$$

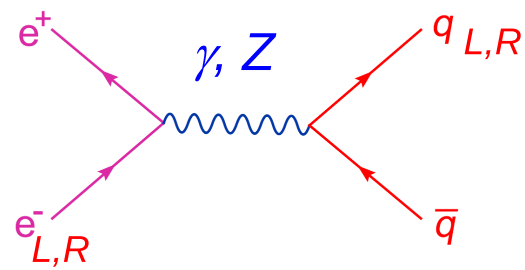
$$= 1 + \cos^2\theta$$

Reweight helicity amplitudes → electroweak/QCD processes

For example, Z exchange



$\Rightarrow$

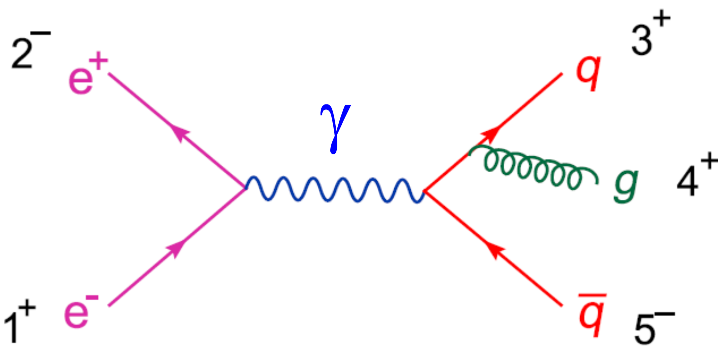


$$Q_e Q_q \quad \Rightarrow \quad Q_e Q_q + \frac{v^e_{L,R} v^q_{L,R} \; s}{s - M_Z^2 + i\Gamma_Z M_Z}$$

$$v_L^f = \frac{2I_3^f - 2Q_f \sin^2\theta_W}{\sin 2\theta_W} \qquad v_R^f = -\frac{2Q_f \sin^2\theta_W}{\sin 2\theta_W}$$

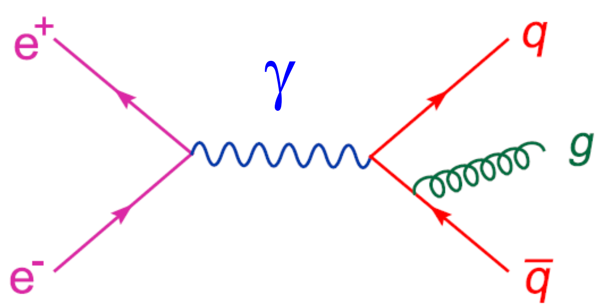
Next most famous pair of Feynman diagrams

(to a higher-order QCD person)



$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})_{i_3}^{\bar{\imath}_5} A_5$$

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$



Helicity formalism for massless vectors

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Xu, Zhang, Chang (1984); Kleiss, Stirling (1985); Gunion, Kunszt (1985)

$$(\varepsilon_i^+)_\mu = \varepsilon_\mu^+(k_i, q) = \frac{\langle i^+ | \gamma_\mu | q^+ \rangle}{\sqrt{2}\, \langle i\, q \rangle}$$

$$(\not{\varepsilon}_i^+)_{\alpha\dot\alpha} = \not{\varepsilon}_{\alpha\dot\alpha}^+(k_i, q) = \frac{\sqrt{2}\, \tilde\lambda_i^{\dot\alpha} \lambda_q^\alpha}{\langle i\, q \rangle}$$

reference vector q^μ is null, $q^2 = 0$

$\not{q}|q^\pm\rangle = 0$

obeys $\varepsilon_i^+ \cdot k_i = 0$ (required transversality)

$\varepsilon_i^+ \cdot q = 0$ (bonus)

under azimuthal rotation about k_i axis, helicity +1/2 $\tilde\lambda_i^{\dot\alpha} \to e^{i\phi/2} \tilde\lambda_i^{\dot\alpha}$

helicity -1/2 $\lambda_i^\alpha \to e^{-i\phi/2} \lambda_i^\alpha$

so $\not{\varepsilon}_i^+ \propto \frac{\tilde\lambda_i^{\dot\alpha}}{\lambda_i^\alpha} \to e^{i\phi}\, \not{\varepsilon}_i^+$ as required for helicity +1

$e^+e^- \to q g \bar{q}$ (cont.)

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q\,5\rangle}{s_{45}\langle 4\,5\rangle}$$

Choose $q = k_5$ to remove 2nd graph

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

$$= -\frac{\langle 2\,5\rangle[1\,2]\langle 2\,5\rangle[4\,3]}{\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle[4\,3]\langle 4\,5\rangle}$$

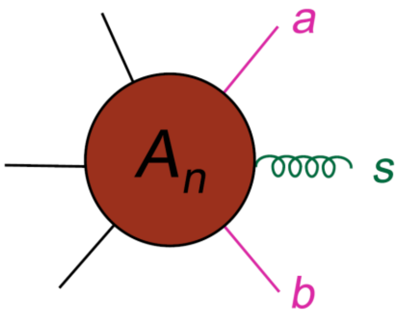
$$\boxed{A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}}$$

Properties of $\mathcal{A}_5(e^+e^- \to qg\bar{q})$

1. Soft gluon behavior

$$k_4 \longrightarrow 0$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$
$$\to \mathcal{S}(3,4^+,5) \times A_4(1^+,2^-,3^+,5^-)$$



Universal "eikonal" factors
for emission of soft gluon *s*
between two hard partons *a* and *b*

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$
$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current:
independent of parton type (*q* *vs.* *g*) and helicity
– only depends on momenta of *a*,*b*, and color charge:

$$\frac{\varepsilon_s^+(q)\cdot k_a}{k_a\cdot k_s} - \frac{\varepsilon_s^+(q)\cdot k_b}{k_b\cdot k_s} \propto \frac{\langle a\,q\rangle}{\langle s\,q\rangle\langle a\,s\rangle} - \frac{\langle b\,q\rangle}{\langle s\,q\rangle\langle b\,s\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$ (cont.)

2. Collinear behavior

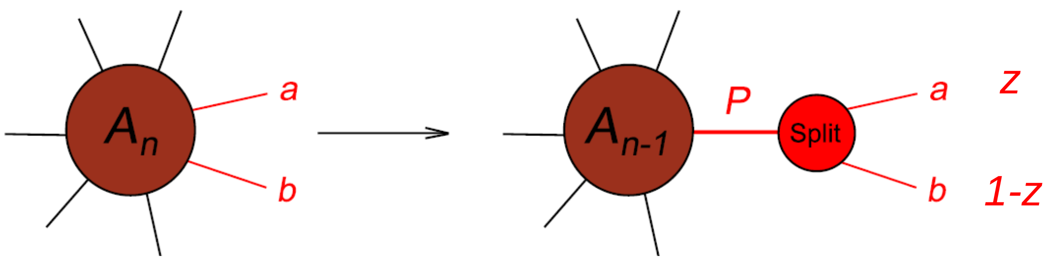
$k_3 \,||\, k_4$: $\quad k_3 = z\, k_P, \quad k_4 = (1-z)\, k_P$

$k_P \equiv k_3 + k_4, \quad k_P^2 \to 0$

$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P,$ etc.

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$



Time-like kinematics (fragmentation).
Space-like (parton evolution)
related by crossing

Universal collinear factors, or splitting amplitudes
$\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

Collinear limits (cont.)

We found, from $k_3 \,||\, k_4$:

$$\mathrm{Split}_-(a_q^+, b_g^+) = \frac{1}{\sqrt{1-z}\,\langle a\, b\rangle}$$

Similarly, from $k_4 \,||\, k_5$:

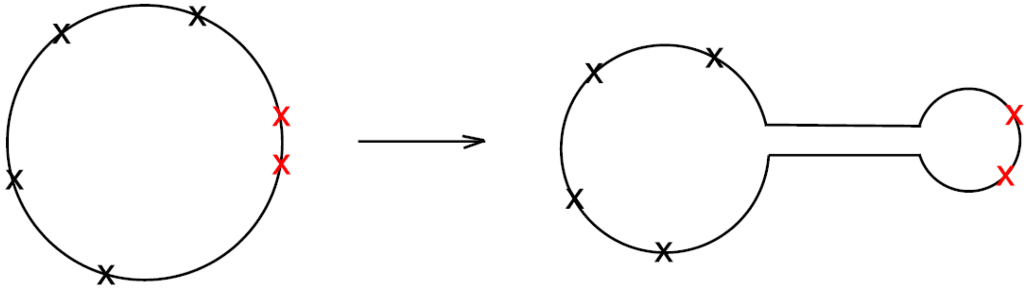
$$\mathrm{Split}_+(a_g^+, b_{\bar{q}}^-) = \frac{1-z}{\sqrt{z}\,\langle a\, b\rangle}$$

$$\Downarrow$$

Applying C and P:

$$\mathrm{Split}_-(a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z}\,[a\, b]}$$

Universality can be argued various ways, including from factorization + operator product expansion in string theory:



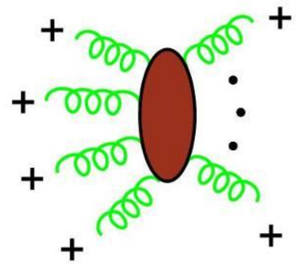
Mangano, Parke, Phys. Rept. 200, 301 (1991)

Simplest pure-gluonic amplitudes

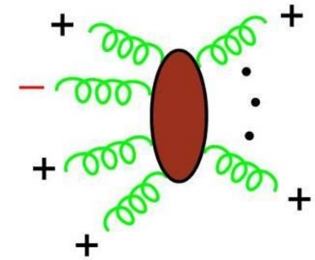
Note: helicity label assumes particle is outgoing; reverse if it's incoming

Strikingly, many vanish:

$$A_n^{\text{tree}}(1^\pm, 2^+, \ldots, n^+) =$$



$=$

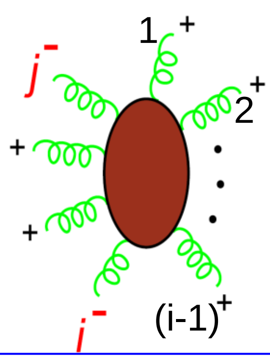


$= \quad 0$

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{ij,\ \text{MHV}} = A_n^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

$=$



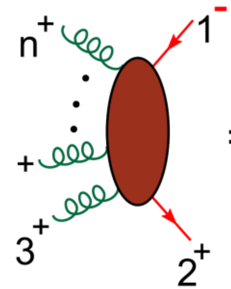
$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

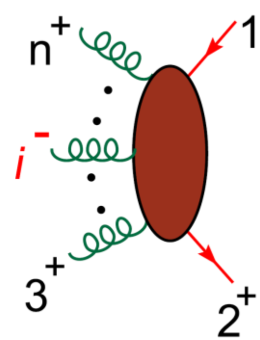
Remarkable simplicity – has inspired many formal developments

MHV amplitudes with massless quarks

Helicity conservation on fermion line → $A_n^{\text{tree}}(1_{\bar{q}}^{\pm}, 2_q^{\pm}, 3^{h_3}, \ldots, n^{h_n}) \equiv 0$

more vanishing ones: $A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, 3^{+}, \ldots, n^{+}) =$  $= 0$

the MHV amplitudes:

$A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, \ldots, i^{-}, \ldots, n^{+}) =$  $= \dfrac{\langle 1\, i\rangle^3 \langle 2\, i\rangle}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$

Related to pure-gluon MHV amplitudes by a secret supersymmetry: after stripping off color factors, massless quarks ~ gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985); Kunszt (1986); Nair (1988)

Properties of MHV amplitudes

1. Verify soft limit

$$k_s \to 0$$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a\,s\rangle\langle s\,b\rangle\cdots\langle n\,1\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a\,b\rangle\cdots\langle n\,1\rangle}$$

$$\to \text{Soft}(a,s^+,b) \times A_{n-1}^{ij,\,\text{MHV}}$$

2. Extract gluonic collinear limits:

$$k_a \,||\, k_b \quad (b = a+1)$$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a-1,a\rangle\langle a\,b\rangle\langle b,b+1\rangle\cdots\langle n\,1\rangle} = \frac{1}{\sqrt{z(1-z)}\langle a\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a-1,P\rangle\langle P,b+1\rangle\cdots\langle n\,1\rangle}$$

$$\to \text{Split}_-(a^+,b^+) \times A_{n-1}^{ij,\,\text{MHV}}$$

So

$$\text{Split}_-(a^+,b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

and

$$\text{Split}_+(a^-,b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

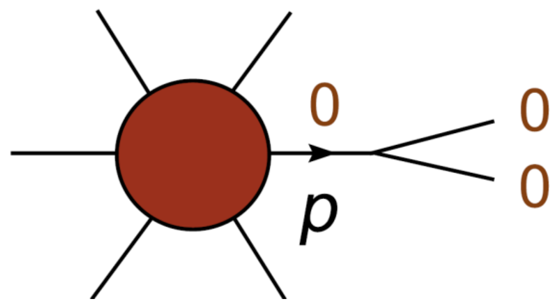
$$\text{Split}_+(a^+,b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

plus parity conjugates

Spinor Magic

Spinor products precisely capture
square-root + phase behavior in **collinear limit**.
Excellent variables for **helicity amplitudes**

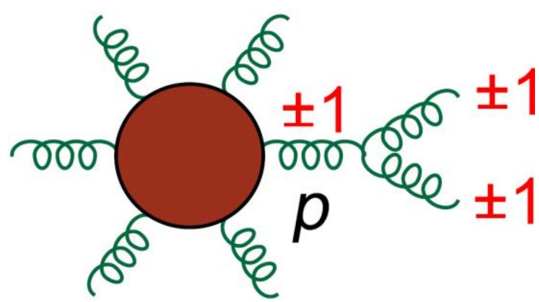
scalars



$$\sim \frac{1}{p^2} \sim \frac{1}{s_{ij}}$$

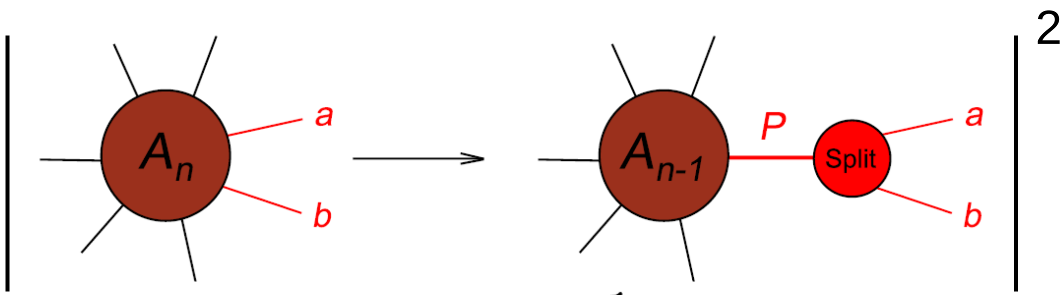
gauge theory

angular momentum mismatch



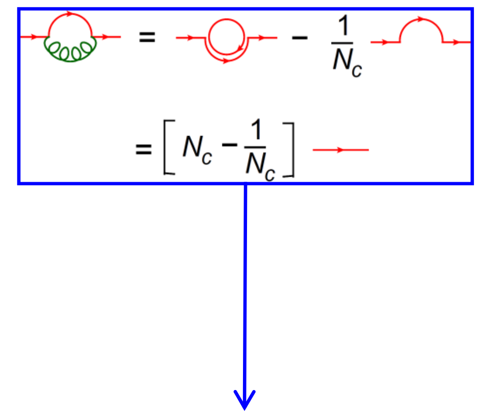
$$\sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i\, j\rangle} \text{ or } \frac{1}{[i\, j]}$$

From splitting amplitudes to probabilities



$$d\sigma_n \sim d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z)$$

$$P(z) \propto \sum_{h_P,h_a,h_b} |\mathrm{Split}_{-h_P}(a^{h_a}, b^{h_b})|^2 \, s_{ab}$$



$q \to qg$:

$$P_{qq}(z) \propto C_F \left\{ \left| \frac{1}{\sqrt{1-z}} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \right|^2 \right\}$$

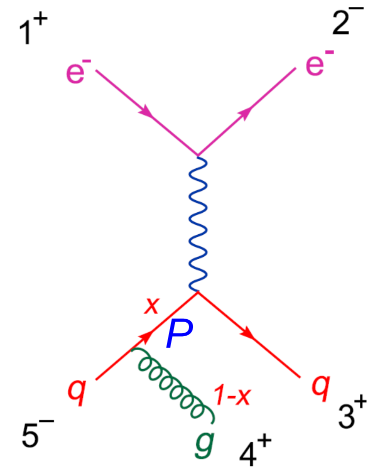
$$= C_F \frac{1+z^2}{1-z} \qquad z < 1$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

Note soft-gluon singularity as $z_g = 1 - z \to 0$

Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however



$$q \to qg: \quad k_P = x\, k_5, \quad k_4 = (1-x)\, k_5$$

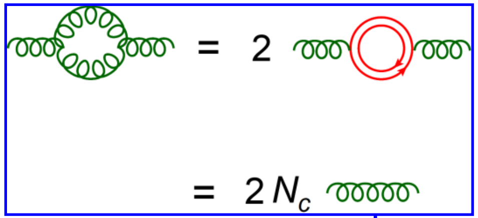
$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{\frac{1}{x}}{\sqrt{\frac{1-x}{x}}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

$$= \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1-x}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

absorb into flux factor:
$$d\sigma_5 \propto \frac{1}{s_{15}}$$
$$d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x\, s_{15}}$$

When dust settles, get exactly the same splitting kernels (at LO)

Similarly for gluons



$g \to gg$:

$$P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\}$$

$$= C_A \frac{1 + z^4 + (1-z)^4}{z(1-z)}$$

$$= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \qquad z < 1$$

$C_A = N_c$

Again a soft-gluon singularity. Gluon number not conserved.
But momentum is. Momentum conservation mixes $g \to gg$ with

$g \to q\bar{q}$:

$$\boxed{P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]}$$

$T_R = \frac{1}{2}$

(can deduce, up to color factors, by taking
$e^+ \,||\, e^-$ in $\mathcal{A}_5(e^+e^- \to qg\bar{q})$)

Exercise

Gluon splitting (cont.)

$g \to gg$:

Applying momentum conservation,

$$\int_0^1 dz\, z\, \left[P_{gg}(z) + 2n_f P_{qg}(z)\right] \;=\; 0$$

Exercise:
Work out b_0

gives

$$\boxed{\begin{aligned} P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + b_0\,\delta(1-z) \\ b_0 &= \frac{11C_A - 4n_f T_R}{6} \end{aligned}}$$

Amusing that first β-function coefficient enters,
since no loops were done, except implicitly via unitarity: