Luminosity Calibration at ATLAS in 2011

Mark Tibbetts (LBNL) on behalf of the ATLAS Luminosity Measurement Task Force

Lumi Days Workshop  1st March 2012
Outline

- Introduction
- 2011 7TeV Proton-Proton Calibration
- Luminosity for Other Running Conditions in 2011
- Plans and Requests for 2012
- Summary
Introduction
Detectors and Algorithms in 2011

- **Bunch by bunch**
  - BCM
    - Evt OR, AND
    - Separate measurement from H/V modules
  - LUCID
    - Evt OR, AND, A, C
    - Hit OR
  - ZDC (HI only)
    - Evt Incl OR A & C
  - Vertex methods
    - Evt & vtx counting

- **BCID-blind**
  - FCal (forward LAr)
    - Gap currents
  - TILE cal
    - PMT currents

Redundancy essential for assessing long term stability & $\mu$ dependence
Physics Running in 2011

- 'Afterglow' in physics running
  - Collision induced radiation $>>$ instrumentation
  - Proportional to inst. lumi
  - $\sim$1% for LUCID_EvtOR
  - $\sim$0.5% for BCMH_EvtOR
- Level inferred from activity in adjacent empty BCIDs
  - Stable level across train
  - Subtracted with precision of 0.2%

LHC page 1 & ATLAS control room preferred algorithm
- Insensitive to bkgd & afterglow
- Linear for physics pileup conditions
- Sufficient statistical sensitivity
- 2011 7TeV pp was BCMH_EvtAND

Offline (physics analysis) preferred algorithm
- Best absolute luminosity determination
- 2011 7TeV pp was BCMH_EvtOR

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$\epsilon = \sigma_{\text{vis}} / \sigma_{\text{inel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LUCID_EvtOR</td>
<td>59.5 %</td>
</tr>
<tr>
<td>LUCID_EvtAND</td>
<td>18.8 %</td>
</tr>
<tr>
<td>BCMH_EvtOR</td>
<td>6.6 %</td>
</tr>
<tr>
<td>BCMH_EvtAND</td>
<td>0.2 %</td>
</tr>
</tbody>
</table>

Hit counting commissioned for LUCID in preparation for 2012 $<\mu>$
van der Meer Scan Overview

- **Scan Sequence**
  - Initial horizontal followed by vertical scan
    - Recentering after scan (usually unnecessary)
  - Second horizontal followed by vertical scan in same fill
    - Check reproducibility of key luminosity variables
    - Essential for evaluation of beam dependent systematic uncertainties

\[
L = \frac{f_r}{\sigma_{vis}} \sum_{i} \mu_{vis}^i
\]

Derive algorithm calibration constant \( \sigma_{vis} \)

\[
\sum_\delta = \frac{1}{\sqrt{2\pi}} \int \frac{\mu_{vis}^{sp}(\delta)}{\mu_{vis}^{sp}(0)} d\delta
\]

Calculated analytically or numerically depending on scan curve fit method

\[
\sigma_{vis} = 2\pi \sum_x \sum_y \left| \frac{\mu_{vis}}{n_1 n_2} \right|_{\delta=0}
\]

Algorithm dependent

\[
L_{sp} = 10^{22} \left( \text{p/bunch} \right)^2 \frac{f_r}{2\pi \sum_x \sum_y}
\]

Algorithm independent
## vdM Calibrations Analysed in 2011

<table>
<thead>
<tr>
<th></th>
<th>2010 PbPb</th>
<th>2.76TeV pp</th>
<th>7TeV May</th>
<th>(7TeV Oct)</th>
<th>2011 PbPb</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHC Fill(s)</td>
<td>1533</td>
<td>1653 &amp; 1658</td>
<td>1783</td>
<td>N/A</td>
<td>2337</td>
</tr>
<tr>
<td>Paired Bunches</td>
<td>113</td>
<td>64 (sans pilots)</td>
<td>14</td>
<td>N/A</td>
<td>344</td>
</tr>
<tr>
<td>Peak $\mu$</td>
<td>$1.4x10^{-4}$</td>
<td>0.36</td>
<td>2.3</td>
<td>N/A</td>
<td>$6x10^{-4}$</td>
</tr>
<tr>
<td>$\beta^*$</td>
<td>3.5m</td>
<td>11m</td>
<td>1.5m</td>
<td>N/A</td>
<td>1m</td>
</tr>
<tr>
<td>Length scale calibration</td>
<td>done</td>
<td>yes</td>
<td>yes</td>
<td>N/A</td>
<td>yes</td>
</tr>
<tr>
<td>Calibrated</td>
<td>LUCID, BCM, ZDC</td>
<td>LUCID, BCM, Vtx</td>
<td>LUCID, BCM, Vtx</td>
<td>N/A</td>
<td>LUCID, BCM, ZDC, Vtx</td>
</tr>
<tr>
<td>Status</td>
<td>Ongoing</td>
<td>Ongoing</td>
<td>Being finalised for publication</td>
<td>N/A</td>
<td>Ongoing</td>
</tr>
</tbody>
</table>

- Focus in ATLAS has been on the May 7TeV pp scan and long term stability of this calibration
- With reduction in bunch current uncertainties, systematics at sub-percent level become important and must be exhaustively evaluated
7 TeV Proton-Proton Calibration
May 2011 7TeV pp Scan

- Only precision 7TeV calibration in 2011 for ATLAS
- 14 colliding bunches
  - Beam size and bunch current product vary 10-20% bunch to bunch
  - Determine $\sigma_{\text{vis}}$ for each colliding bunch
- Moderate emittance growth between scans
- Fill had additional pair of x-y scans with offset of $3\Sigma$ in non-scanning plane
  - Constrain x-y coupling
  - Analysis on-going, not presented here

<table>
<thead>
<tr>
<th>BCIDs</th>
<th>Collide At</th>
</tr>
</thead>
<tbody>
<tr>
<td>2602, 2652, 2702, 2752</td>
<td>IP1, IP5 &amp; IP8</td>
</tr>
<tr>
<td>817, 867, 917, 967</td>
<td>IP1, IP5 &amp; IP2</td>
</tr>
<tr>
<td>81, 131, 181, 231, 281, 331</td>
<td>IP1 &amp; IP5</td>
</tr>
</tbody>
</table>
Scan Curve Interpretation

- 2010 vdM data well described by sum of 2 Gaussians & constant term
  - Constant then interpreted as background from beam gas
- 2011 data favours single Gaussian component only
- Rates can be corrected for background
  - Beam gas from unpaired bunches
  - Noise & afterglow from adjacent empty bunches
- Fit to background corrected data can still have statistically significant constant component
  - 2\textsuperscript{nd} Gauss with $\sigma \rightarrow \infty$?
  - Must be included in evaluation of luminosity observables

Gauss+p0 fit quality suffers in some BCIDs due to non-Gaussian beam profile component particularly in horizontal plane
Scan Curve Fit Model

- Refit data with alternative models
  - Cubic spline
  - Gauss+p0 fit to background subtracted data
    - p0 included in $\Sigma \& \mu_{\text{vis}}(0)$
  - Gauss+p0 fit to uncorrected data
    - p0 as background
- Largest deviation in means for any one algorithm/scan assigned as fit model systematic (0.3%)

REMINDER: $\sigma_{\text{vis}}$ should be identical for all BCIDs

Statistical uncertainties for spline method (0.3-0.4%) not shown here.
Scan Curve Model (Aside)

- Fitted peak $\mu_{\text{vis}}$ values violate expected chronology for some BCIDs
  - Correlated to fits with poor $\chi^2$ (significant non-Gaussian pulls)
  - Measured peak ok within statistics
    - Hence spline fit too
- Effect not observed in 2010 scan data
Bunch by Bunch Calibration Consistency

- Independent algorithms (same fit) show correlated systematic spreads within a scan
  - Bunch by bunch spread of $\sigma_{\text{vis}}$ values taken as consistency systematic (0.55%)
- Difference between scan VII & scan VIII mean interpreted as emittance growth and other non-reproducibility
  - Largest difference for any one algorithm assigns overall systematic (0.8%)
Comparison of $L_{sp}$ between LUCID and BCM algorithms shows good consistency

- Difference in $L_{sp}$ bunch by bunch reflects different beam overlap profiles
- Largest difference in ratio to 1 taken as systematic uncertainty of 0.3\% on $\Sigma_x \Sigma_y$ product & hence $\sigma_{vis}$
Beam Recentering

- Observe up to 2 micron non-reproducibility in fitted peak position
  - NB. Beams not recentered between scans
  - Observed effect larger at larger $\beta^*$; should scale as $\sqrt{\beta^*}$
  - Affects estimate of $\mu_{\text{vis}}$ at peak, impact on $\sigma_{\text{vis}}$ is 0.1%
- Visible systematic difference depending on cross-talk with other IPs
**μ Dependence & BCMH vs BCMV**

- In vdM fill algorithms consistent across μ range to 0.5%
- **Under physics conditions BCMH & BCMV show ~0.7% discrepancy**
  - Instrumentation effect not currently understood
    - Diamond charge pumping?

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![Graph](image_url)

**May vdM fill**

**Oct physics fill**
Non-Linear Correlations

- Do beam overlap profiles transversely factorise?
  - NB. Not linear x-y coupling
- ATLAS use 'pessimistic' correlated sum of 2 Gaussian as alternative fit
  - Model arbitrary; effect if any fill dependent
- Potential for vertex imaging of luminous region to address this in future
  - Requires larger $\beta^*$ (11m) to allow sufficient resolution of beam overlap region

<table>
<thead>
<tr>
<th>vdM Scan</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>April 2010</td>
<td>3%</td>
</tr>
<tr>
<td>May 2010</td>
<td>2%</td>
</tr>
<tr>
<td>Oct 2010</td>
<td>0.9%</td>
</tr>
<tr>
<td>May 2011</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

Reminder: 2010 data exhibited 2nd Gaussian component not seen in 2011

![Graphs](image-url)
Length Scale Calibration ($\beta^*=1.5\text{m}$)

- $\sigma_{\text{vis}}$ relies on LHC nominal separation determined by 4 closed orbit bumps
- Scale checked with dedicated scan:
  - Move one beam by nominal amount
  - Follow with other beam & scan around 1st beam to find maximum
  - Beamspot position at peak calibrates nominal movement of 1st beam
  - Repeat to extract four calibration constants
- In vdM scan beams moved simultaneously in opposite directions
  - Overall separation scale is average of two beams in given plane
- Calibration uncertainty of 0.3%
  - Non-linear residuals observed at $\sim0.2$ micron level

<table>
<thead>
<tr>
<th>Scale</th>
<th>Horizontal</th>
<th>Vertical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam 1</td>
<td>$-1.0010\pm0.0004$</td>
<td>$+0.9955\pm0.0003$</td>
</tr>
<tr>
<td>Beam 2</td>
<td>$-0.9981\pm0.0003$</td>
<td>$+0.9983\pm0.0003$</td>
</tr>
<tr>
<td>Separation</td>
<td>$-0.9996\pm0.0003$</td>
<td>$+0.9969\pm0.0002$</td>
</tr>
</tbody>
</table>
# 7TeV vdM ATLAS Uncertainties

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam centering</td>
<td>0.1</td>
<td>0.1</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Beam jitter</td>
<td>0.3</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Non-reproducibility</td>
<td>0.4</td>
<td>0.8</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Bunch consistency</td>
<td>0.4</td>
<td>0.5</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Fit model</td>
<td>0.8</td>
<td>0.3</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Bkgd subtraction</td>
<td>0.3</td>
<td>0.3</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Reference $L_{sp}$</td>
<td>0.3</td>
<td>0.3</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>Dynamic beta</td>
<td>0.8</td>
<td></td>
<td></td>
<td>Collisions at IP1 only</td>
</tr>
<tr>
<td>Non-linear correlations</td>
<td>0.5</td>
<td>0.5</td>
<td>Measured in scan</td>
<td>Fill dependent</td>
</tr>
<tr>
<td>$\mu$-dependence in scan</td>
<td>0.5</td>
<td>0.5</td>
<td>Measured in scan</td>
<td>~2011?</td>
</tr>
<tr>
<td>Length scale calibration</td>
<td>0.3</td>
<td>0.3</td>
<td></td>
<td>Depends on $\beta^*$</td>
</tr>
<tr>
<td>Inner det. Length scale</td>
<td>0.3</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BCM H/V consistency</td>
<td>0.7</td>
<td>0.7</td>
<td></td>
<td>?</td>
</tr>
<tr>
<td>Total vdM uncertainty</td>
<td>1.5</td>
<td>1.75</td>
<td>~2011?</td>
<td>Without beam current</td>
</tr>
</tbody>
</table>

*Fill dependent uncertainties will be uncorrelated between 7TeV (2011) and 8TeV (2012)*

*5/fb UNCERTAINTY ANALYSIS BEING FINALISED*

*THES NUMBERS PRELIMINARY & SUBJECT TO CHANGE*
Long Term Stability & $\mu$ Dependence

- vdM determines $\sigma_{vis}$ at single point in time with peak $\mu \sim 2.5$
- Must confirm algorithm ($\mu_{vis}/\mu$) stable over time and for $\mu$ range of physics running
- ATLAS strategy is to partially separate the 2 effects
  - Scan significant range of $\mu$ in single LHC fill
    - Algorithm stability in $\mu$ at single point in time
  - Investigate $\mu$ dependence for all 2011 physics running 7TeV pp data
    - Algorithm stability in $\mu$ for conditions across the year
Luminosity From Calorimeters

- FCal (gap current) and Tile (PMT current) calorimeters have response proportional to luminosity
  - Calibrating relative response to known luminosity algorithm at specific time & $\mu$ allows extrapolation to other times & $\mu$
- Tile is calibrated to LUCID_EvtOR at peak $\mu$ of May vdM fill
- FCal is calibrated to BCMH_EvtOR in dedicated physics fill scanning across $\mu$
- For 2011 cross-calibration of LUCID & BCM to calorimeter methods proved essential in addressing initial discrepancies observed after winter shutdown
LHC Fill 2086 beams separated at IP1 then recentered

- Effectively samples wide range of $\mu$
  - Peak $\mu \approx 12$
- FCal calibrated to BCM in this fill
  - Observe residual non-linearity

Comparison between BCM, LUCID & Tile suggests consistency across physics $\mu$ range (2-12) of 1%

- AND algorithms diverge at $\mu < 2$
  - Imperfect background subtraction in the region it becomes important
  - Subject of investigation, plot likely to change in this region
Long Term Stability

- Can check relative variation of $\langle \mu \rangle$ between algorithms for all 2011 7TeV pp data
  - See no deviation beyond 1%
    - Assign this as uncertainty in addition to previous $\mu$ dependence
- Work ongoing to add vertexing algorithms to these $\mu$ dependence studies
  - Pileup scan in particular
# Full 2011 7TeV Uncertainty Table

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>DCCT</td>
<td>2.73</td>
<td>0.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FBCT</td>
<td>1.30</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ghost charge</td>
<td>0.18</td>
<td>0.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total BCP</td>
<td>3.0</td>
<td>0.35</td>
<td>~2011?</td>
<td>Thanks to BCNWG!</td>
</tr>
<tr>
<td>Total vdM</td>
<td>1.5</td>
<td>1.75</td>
<td>~2011?</td>
<td></td>
</tr>
<tr>
<td>Total $\sigma_{\text{vis}}$ uncertainty</td>
<td>3.4</td>
<td>1.8</td>
<td>~2011?</td>
<td></td>
</tr>
<tr>
<td>Long term stability</td>
<td>1.0</td>
<td>1.0</td>
<td>~2011</td>
<td>Detector aging in 2012?</td>
</tr>
<tr>
<td>$\mu$-dependence in physics running</td>
<td>1.0</td>
<td>1.0</td>
<td>~2011</td>
<td>Larger $\mu$ range for 2012</td>
</tr>
<tr>
<td>Afterglow subtraction</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total monitoring</td>
<td>1.4</td>
<td>1.4</td>
<td>~2011</td>
<td>Unchanged from summer despite more data &amp; higher $\mu$ values</td>
</tr>
<tr>
<td>Total 2011 7TeV pp</td>
<td>3.7</td>
<td>2.3</td>
<td>~2011?</td>
<td></td>
</tr>
</tbody>
</table>

No reason why 2012 would be worse than 2011 but much depends on what the data looks like!

For the most part (vdM & monitoring) 2011 and 2012 uncertainties will be uncorrelated.

5/fb UNCERTAINTY ANALYSIS BEING FINALISED
THESE NUMBERS PRELIMINARY & SUBJECT TO CHANGE
Luminosity for Other Running Conditions in 2011
Pb-Pb Scans

- Both 2010 and 2011 vdM scan analyses maturing
  - Uncertainties to be evaluated
    - DCCT, FBCT
    - Satellites, ghost charge
    - Background
    - 2-4% inconsistencies between algorithms under investigation

Nov 2010 PbPb preliminary analysis shows reasonable agreement between ZDC A & C side algorithms for beam profile determination.
March 2.76TeV pp Scan

- Beam dump between scans leads to complications in reproducibility
  - Different $\Sigma$
  - Vertical beam profile changes significantly between fills
- Preliminary investigation of scan I with LUCID Evt algorithms only

Bunch by bunch consistency in scan I is $\sim$1-2%; Partially explained by BCP offset
Limits in Understanding at Low $\mu$

- Relevant to ALFA run luminosity determination
  - $\mu \sim 0.03-0.04$, $\beta^*=90m$
- Preliminary investigation suggests large difference in algorithms relative to BCMH_EvtOR over the range of $\mu$
  - Apparent residual spread of BCMV_EvtOR, LUCID_EvtOR, LUCID_EvtAND relative to BCMH_EvtOR under investigation
  - Significant average deviation for some algorithms
    - LUCID_EvtOR: -8%
    - LUCID_EvtAND: -15%
    - BCMV_EvtOR: -2%
- Currently investigating vertex based algorithms
  - Initial results suggest consistency with BCM at $\sim 2\%$ level
Plans and Requests for 2012
Luminosity Calibration in 2012

- Initial calibrations extrapolating from 2011 likely to be crude
  - Change in $\sigma_{\text{inel}}$ & algorithm efficiencies with $\sqrt{s}$ increase of 7TeV → 8TeV difficult to predict
    - Preliminary study suggests $dL/L \sim 5-17\%$?
  - Experience from 2011 suggests uncertainties in detector consistency can become significant after extended winter shutdown
    - For 2011 startup effect was as large as $\sim 8\%$
- To address these points & for $L$ uncertainty comparable to 2011 calibration scans are essential
ATLAS Preferred Calibration Strategy

**Option 1 (Chamonix)**
- Early precision vdM scan with $\beta^* = 11m$
- Distance scale calibration

**Option 2**
- Early scan at normal optics with sparse pattern
- Precision scan after closure of ICHEP dataset
  - $\beta^* = 11m$ & Distance scale calib.

- Both options under discussion within ATLAS
  - Early scan from option 2 expected to achieve <5% precision
  - Precision scan aims at matching 2011 uncertainty or better

- Additional needs
  - **Pileup scan (cf. Fill 2086 in 2011)**
    - Essential for controlling systematics given potential detector aging & higher $\mu$ in 2012
    - Requires $L > 2 \times 10^{33}$ for calorimeters & $\mu$ as high as stable running permits
  - **Afterglow scan**
    - Calibrate afterglow subtraction for BCM & LUCID at highest possible $\mu$
    - <5 colliding bunches >700 BCIDs apart, no unpaired bunches in between
Further Prospects for 2012

• Calibration of still more luminosity detectors & algorithms for consistency checks
  • RPC, Medipix, LUCID tube hit counting
  • Dedicated Inner Detector DAQ stream with prompt track & vertex reco for online lumi monitoring from these objects
    - Well advanced, cut definitions only limited by CPU constraints
  • Z counting luminosity comparisons
Summary

- Close to a precision calibration for 7TeV pp data in 2011 approaching 2% uncertainty
  - Order of magnitude improvement in bunch current uncertainties requires exhaustive evaluation of sub-percent uncertainties
    - Detailed analysis of vdM scan calibration uncertainties
    - Examination of $\mu$ dependence and algorithm consistency for
      - Single high luminosity fill
      - Across the entire 2011 data taking period
- In 2012 we hope to achieve comparable results
  - Prepared for challenges of higher pileup
  - 2012 uncertainty for the most part uncorrelated with 2011
- Thanks to the LHC for superb performance in 2011!
Backup
Event Counting Pileup Corrections

Inclusive counting (EventOR):

\[ P_{\text{EventOR}}(\mu_{\text{vis}}^{\text{OR}}) = 1 - e^{-\mu_{\text{vis}}^{\text{OR}}} = \frac{N_{\text{OR}}}{N_{\text{BC}}} \Rightarrow \mu_{\text{vis}}^{\text{OR}} = -\ln \left(1 - \frac{N_{\text{OR}}}{N_{\text{BC}}} \right) \]

Coincidence counting (EventAND):

\[ P_{\text{EventAND}}(\mu_{\text{vis}}^{\text{AND}}) = 1 - 2e^{-(1+\frac{\sigma_{\text{vis}}^{\text{OR}}}{\sigma_{\text{vis}}^{\text{AND}}})\mu_{\text{vis}}^{\text{AND}}/2} + e^{-(\frac{\sigma_{\text{vis}}^{\text{OR}}}{\sigma_{\text{vis}}^{\text{AND}}})\mu_{\text{vis}}^{\text{AND}}} = \frac{N_{\text{AND}}}{N_{\text{BC}}} \]

- Cannot analytically invert (use e.g. lookup table)
- Dependence on \( \frac{\sigma_{\text{vis}}^{\text{OR}}}{\sigma_{\text{vis}}^{\text{AND}}} \) requires analysis iterated
Less systematic variation in each scan, all fits with good quality for double Gaussian+p0
Beam Jitter Uncertainty

- Random deviations from nominal position at given scan point
- Length scale calibration gives 0.8 $\mu$m RMS on relative beam centering
- Randomly vary toy MC scan points by RMS to estimate $\sigma_{vis}$ uncertainty
Inner Detector Length Scale Uncertainty

- Length scale calibration relies on absolute ID length scale
- Uncertainty from study of 9 ID misalignment MC samples
  - True vs. reconstructed vertex position
- Correlated across all scans
More on Double Gaussian Non-Linear Modelling

- Fit strategy assumes Gaussian sums factor transversely

\[
[f_{1x} G_{1x}(h) + (1 - f_{1x}) G_{2x}(h)][f_{1y} G_{1y}(k) + (1 - f_{1y}) G_{2y}(k)] \Rightarrow \frac{1}{\Sigma_{x(y)}} = \frac{f_{1x(y)}}{\sigma_{1x(y)}} + \frac{(1 - f_{1x(y)})}{\sigma_{2x(y)}}
\]

- In case where overlap region has distinct sub-populations in tail/core sum will be of two 2D Gaussians

\[
[f_1 G_{1x}(h) G_{1y}(k) + (1 - f_1) G_{2x}(h) G_{2y}(k)] \Rightarrow \frac{1}{\Sigma_x \Sigma_y} = \frac{f}{\sigma_{1x} \sigma_{1y}} + \frac{(1 - f)}{\sigma_{2x} \sigma_{2y}}
\]