Some thoughts on the generalized Panofsky-Wenzel theorem and its Proof

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The generalized Panofsky-Wenzel theorem

(1) $$\Delta p_x\left(x,y,t_0\right) = -\frac{i\cdot e}{\omega}\cdot\frac{d}{dx}\int_{-\infty}^{+\infty} dz\cdot E_z\left(x,y,z,t=z/v+t_0\right) = -\frac{i\cdot e}{\omega}\cdot\frac{\partial V_z}{\partial x}$$

is valid for <u>any</u> ω-periodic field map with vanishing field at entry and exit of the particle path and holds for any particle speed v (provided v is constant, stiff beam).

Due to its general validity (1) is also valid for any 'infinitely' short cavity. For a newcomer this might then lead to the erroneous conclusion that (1) is trivially true since it is always locally true, i.e. the 'local' transverse momentum kick δp_x induced on any slice δz is simply –i·e/ω times the x-gradient of the 'local' acceleration $E_z(x)\cdot\delta z$ and (1) means simply adding up δz-slices with always the same proportionality factor.

However, in 'macroscopic' cavities such a local proportionality is not true at all and it is just the 'miracle' of PW that – whatever the details of the field map – the <u>integrated</u> momentum kick is identical to the gradient of the <u>integrated</u> longitudinal voltage. PW means that the cavity can be treated as a black-box: measuring the longitudinal voltage at two different (close) offset paths will exactly predict the deflection of another particle without knowing what is really in the black-box (with the above restrictions for (1)).

To demonstrate this non-locality, we use a simple to handle field-map but having both local electric and magnetic deflection: a coaxial λ/2 TEM-resonator with a particle path parallel but off axis, see Fig. 1.

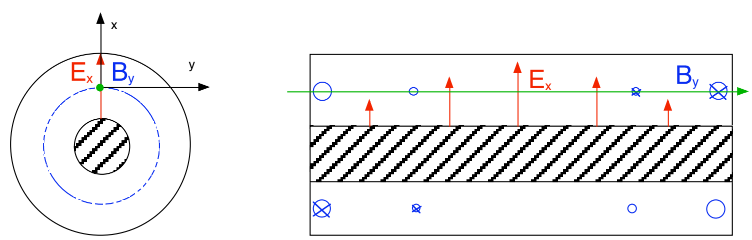

Fig. 1: The λ/2 resonator: At the particle path at $r=r_0$ (green) $B_\phi=B_y$ and $E_r=E_x$.

The electric and magnetic field on the particle path, starting at z=0, are

(2a) $$E_x = E_0 \sin(2\pi/\lambda\cdot z)\ \cos(\omega\cdot t)\quad \left\{\cdot r_0/r\right\}$$

(2b) $$B_y = B_0 \cos(2\pi/\lambda\cdot z)\ \sin(\omega\cdot t)\quad \left\{\cdot r_0/r\right\}$$

while all other field components are zero, especially E_z. From this and (1) one concludes immediately that the net deflection of any particle with any speed should be zero !

We will show now that locally there is no zero deflection at all but both electric and magnetic deflection enforce non-zero local deflection. However we will see that physics constraints are such that, whatever the conditions for speed and starting phase, the net deflection adds up to zero. Both fields scale radial as r_0/r. Hence to fulfill Maxwell's equation we have to constrain E_0 and B_0 by

(3) $$-\frac{\partial B_\phi}{\partial t} = curl_\phi(E) = \frac{\partial E_r}{\partial z} \quad\Rightarrow\quad \frac{r_0}{r}E_0\frac{2\pi}{\lambda} \equiv -\frac{r_0}{r}B_0\cdot\omega \quad\Rightarrow\quad B_0 = -\frac{E_0}{c}$$

Furthermore ω and λ are constrained so that (2) can be expressed by a wave-length λ and a common excitation constant E_0 as

(4a) $$E_x = E_0 \sin(2\pi/\lambda \cdot z)\ \cos(2\pi \cdot c/\lambda \cdot t)$$

(4b) $$B_y = -E_0/c \cdot \cos(2\pi/\lambda \cdot z)\ \sin(2\pi \cdot c/\lambda \cdot t)$$

The 'local' deflection caused by the Lorentz force (particle charge e) on a slice of width δz is in full generality

(5) $$\delta p \ = \ e(E + v \times B)\cdot \delta t \ = \ e(E + v \times B)\cdot \delta z/|v|$$

and specially for the present field map with the deflection direction '$\perp$' = 'x' and the (constant) speed v in z-direction

(6) $$\delta p_\perp \ = \ e\cdot(E_x/v - B_y)\cdot \delta z$$

On purpose we keep the deflection contribution from the electric and magnetic field separate, i.e.

(7a) $$\delta p_{\perp,E} \ = \ e\cdot E_0/v\cdot \sin(2\pi/\lambda\cdot z)\ \cos(2\pi\cdot c/\lambda\cdot t)\cdot \delta z$$

(7b) $$\delta p_{\perp,B} \ = \ e\cdot E_0/c\cdot \cos(2\pi/\lambda\cdot z)\ \sin(2\pi\cdot c/\lambda\cdot t)\cdot \delta z$$

Here one sees already that the local deflection of both components at any z and any t is not zero individually, nor that both parts add up locally to zero!

A particle sees the field at its present location z at the present time t, but z and t are constrained here by $t=z/v+\psi/\omega$, where ψ expresses the common phase of the E- and B-fields when the particle enters the cavity, a completely free parameter. From this one gets

(8a) $$\Delta p_{\perp,E} \ = \ \frac{e\cdot E_0}{v}\cdot \int_0^{\lambda/2} \sin(2\pi/\lambda\cdot z)\ \cos(2\pi/\lambda\cdot(c/v)\cdot z + \psi)\cdot dz$$

(8b) $$\Delta p_{\perp,B} \ = \ \frac{e\cdot E_0}{c}\cdot \int_0^{\lambda/2} \cos(2\pi/\lambda\cdot z)\ \sin(2\pi/\lambda\cdot(c/v)\cdot z + \psi)\cdot dz$$

These integrals can be executed straightforward and we get with $\beta=v/c$

(9a) $$\Delta p_{\perp,E} \ = \ -\frac{e\cdot E_0\cdot\lambda\cdot\beta}{c}\cdot\frac{\cos(\psi)+\cos(\pi/\beta+\psi)}{2\pi\ (1-\beta^2)}$$

(9b) $$\Delta p_{\perp,B} \ = \ \frac{e\cdot E_0\cdot\lambda\cdot\beta}{c}\cdot\frac{\cos(\psi)+\cos(\pi/\beta+\psi)}{2\pi\ (1-\beta^2)}$$

(9c) $$\Delta p_\perp \ = \ \Delta p_{\perp,B} \ + \ \Delta p_{\perp,B} \ \equiv \ 0$$

We see that neither the integrated $\Delta p_{\perp,E}$ nor the integrated $\Delta p_{\perp,B}$ are individually zero but both add up under any condition for β or ψ to perfectly zero as required by PW.

Conclusion:

In arbitrary ω-proportional field-maps (single mode) the local deflection along the (stiff beam) particle path as well as the local transverse acceleration gradient are not proportional at all. But, provided that the field vanishes at entry and exit of the path, local deflection and longitudinal acceleration always add up such that PW is respected, irrespectively of the particle speed and any other details of the field map

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(9) remains also finite and valid for the limiting case $\beta\to 1$, i.e. $v\to c$.

$$\Delta p_{\perp,E} \ = \ -\frac{e\cdot E_0\cdot\lambda\cdot\beta}{4\ c}\sin(\psi); \qquad \Delta p_{\perp,B} \ = \ \frac{e\cdot E_0\cdot\lambda\cdot\beta}{4\ c}\sin(\psi); \qquad \Delta p_\perp \ \equiv \ 0$$

Panofsky Wenzel Theorem(s) for Pedestrians

The theorem

$$\Delta p_x(x,y,t_0) = -\frac{i\cdot e}{\omega}\cdot\frac{d}{dx}\int_{-\infty}^{+\infty} dz\cdot E_z(x,y,z,t=z/v+t_0) \tag{1}$$

is very useful e.g. to determine transversal impedances from longitudinal measurements[1]. It relates the transversal gradient of the accelerating voltage and the transversal momentum kick; generally is cited 'Panofsky-Wenzel' [1]. To verify the assumptions, the validity range and definition of variables of (1) the author of these lines has consulted the original paper and found to his surprise that the original theorem (PW) does not tell anything about longitudinal components and is restricted to pure TE or pure TM modes.

The 'enlarged' theorem (1) was e.g. derived by Browman [2], but [2] takes a detour in expressing fields by the vector and scalar potentials A and V. Chao [3] and Vaganian/Henke [4] have given other proofs following completely different approaches but delivering as byproduct other interesting relations.

On the remainder of this page we will derive (1) in a very straightforward way. Also we will at the same time <u>highlight the necessary conditions and assumptions</u>.

Very generally the Lorentz force is

$$\vec{f} = e\cdot\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

We assume that<u>fields are ω-periodic</u> (with the convention of [2] exp(-i·ω·t)), i.e. we consider one mode but <u>NO SYMMETRY whatsoever is assumed</u> concerning fields or cavity geometry. Then, using Maxwell's equation <u>curl(E)=-dB/dt</u> we have

$$curl\left(\vec{E}\right) = -\frac{d\vec{B}}{dt} \;\rightarrow\; curl\left(\vec{E}\right) = i\cdot\omega\cdot\vec{B} \quad \text{hence}$$

$$\vec{f} = e\cdot\left(\vec{E} - \frac{i}{\omega}\cdot\vec{v}\times curl\left(\vec{E}\right)\right)$$

We assume v in z-direction. Then the transversal force in x-direction (y similar) is

$$f_x = e\cdot\left(E_x + \frac{i}{\omega}\cdot v\cdot curl_y\left(\vec{E}\right)\right) = e\cdot\left(E_x + \frac{i}{\omega}\cdot v\cdot\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\right)$$

and hence for a <u>stiff beam</u> with <u>z=v·(t-t$_0$) and x and y constant</u> we get the transversal momentum kick

$$\Delta p_x = \int_{-\infty}^{+\infty} f_x\cdot dt = \frac{i\cdot e}{\omega}\cdot\int_{-\infty}^{+\infty} dt\cdot\left[\left.\left(-i\cdot\omega\cdot E_x + v\cdot\frac{\partial E_x}{\partial z} - v\cdot\frac{\partial E_z}{\partial x}\right)\right|_{z=v\cdot(t-t_0)}\right]$$

We have the mathematical identity

$$\frac{d}{dt}E_x\left(z=v\cdot(t-t_0),t\right) = \frac{\partial E_x}{\partial t} + v\cdot\frac{\partial E_x}{\partial z} = -i\cdot\omega\cdot E_x + v\cdot\frac{\partial E_x}{\partial z}$$

and therefore

[1] The sign depends on the convention for the fields, exp(–iωt) is used here following [2], not exp(+iωt)

$$\Delta p_x = \frac{i\cdot e}{\omega}\cdot E_x\left(z = v\cdot\left(t-t_0\right),t\right)\Big|_{t\to-\infty}^{t\to+\infty} - \frac{i\cdot e}{\omega}\int_{-\infty}^{+\infty} v\cdot dt\cdot\frac{\partial E_z}{\partial x}\Big|_{z=v\cdot\left(t-t_0\right)}$$

The <u>field is assumed to vanish at both ends of the considered volume</u>; hence the first term is zero. In the second term we replace the t-integration by a z-integration with v·dt=dz and infinite limits in z yielding in fact theorem (1).

References

[1] W.K.H. Panofsky, W.A. Wenzel, "Some Considerations Concerning the Transverse Deflection of Charged Particles in Radio Frequency Fields", Review of Scientific Instruments, Nov. 1956, p. 967
[2] M. J. Browman, "Using the Panofsky-Wenzel Theorem in the Analysis of Radio-Frequency Deflectors", Proc. PAC93, Washington, p. 800-802
[3] A. Chao, Lecture notes, to be found at http://www.slac.stanford.edu/pubs/slacpubs/9500/slac-pub-9574-ch01.pdf
[4] S. Vaganian and H. Henke, "The Panofsky-Wenzel Theorem and General Relations for the Wake Potential", Particle Accelerators 44, (1995) p. 239-242