On the changing form of Maxwell's equations during the last 150 years — spotlights on the history of classical electrodynamics —

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I am grateful to Dima Vassiliev for the invitation.

See Itin, Obukhov et al., arXiv:0911.5175   *file MaxGestalt/MaxwellUCL2.tex*

The Maxwell equations 1862-1868

Five decisive papers of Maxwell (1831-1879) + his Treatise (see C.W.F. Everitt, Maxwell, 1975)

1. "On Faraday's Lines of Force" (1855-1856): Analogies between lines of force and streamlines in an incompressible fluid, electrotonic function $\mathbf{A}$, with $\mathbf{B} = \operatorname{curl} \mathbf{A}$ (the latter formula was used earlier also by Gauss)

2. "On Physical Lines of Force" (1861-1862): Molecular vortices and electric particles, induced electromotive force $\mathbf{E} = (-)\partial\mathbf{A}/\partial t$

3. "On the Elementary Relations of Electrical Quantities" (1863, missing in his scientific papers): electromagnetic quantities and their physical dimensions, forces and fluxes

4. "A Dynamical Theory of the Electromagnetic Field" (1865): Provides a new theoretical framework for the subject; systematic overview given of all equations, first clear formulation of his system of eqs.

5. "Note on the Electromagnetic Theory of Light" (1868): integral form *without* $\mathbf{A}$, four basic theorems provided: *MaxwellEqsP2.pdf,* later Murnaghan 1921, Kottler 1922, Cartan 1924, de Rham 1931...

We will provide some *spotlights* on the subsequent development of these eqs.

In Maxwell: " A Treatise on Electricity and Magnetism" (2nd edition, 1881) he gave his electromagnetic field equations their most compact form. Maxwell Monument in Edinburgh: *SchweigertMax1.jpg, SchweigertMax2.jpg*

On the history of Maxwell's equations of classical electrodynamics

1. In components: Maxwell 1862-1865

2. In quaternions (Hamilton 1843): Maxwell 1873

3. In symbolic vector calculus: Heaviside 1885-1888, Gibbs 1901, Föppl

4. In components (compact): Hertz 1890, ansatz for moving bodies

5. In components à la Maxwell-Hertz + Lorentz transf.: Einstein 1905

6. In symbolic 4d calculus: Minkowski 1907-1908

7. In 4d generally covariant tensor calculus: Einstein 1916

8. In premetric/integral formulation up to ~ 1960: (Maxwell), Murnaghan, Kottler, Cartan (formulated in differential forms), van Dantzig, Schrödinger, Schouten, Truesdell-Toupin, Post

9. In spinor calculus: After Pauli 1927 and Dirac 1928: Weyl, Fock, Infeld & van der Waerden,..., $\Rightarrow$ Penrose & Rindler [skip possibly]

10. 4d Clifford algebra formalism (vacuum) Riesz 1958 $\Rightarrow$ Baylis [skip]

11. In algebraic/discrete formulation in terms of (co)chains $\Rightarrow$ Bossavit, Tonti, Zirnbauer

12. 3d and 4d exterior calculus, premetric topological form of Maxwell's eqs. $\Rightarrow$ Kiehn, Post; Kovetz, Russer, Lindell, H. & Obukhov; signature of metric & Lenz rule & sign of energy $\Rightarrow$ Itin; metamat. Itin & Friedman

1. In components: Maxwell 1862-1865

See the *original* of 1865 where for the first time the "Maxwell equations" appeared systematically ordered: file *Maxwell1865_73.pdf,* and file *GerhardMaxwell1865004.pdf*

2. In quaternions (Hamilton 1843): Maxwell 1873

• Quaternion
The quaternions are a set of symbols of the form

$$\underbrace{a}_{\text{scalar p.}} + \underbrace{bi + cj + dk}_{\text{vector part}}, \tag{1}$$

where a, b, c, d are real numbers. They multiply using the rules

$$i^2 = j^2 = k^2 = -1 \qquad \text{and} \qquad ij = k. \tag{2}$$

They form a *non-commutative* division algebra.

• Hamilton 1843: The quotient of two vectors is generally a quaternion.
The name *vector* originates from Hamilton ($\Rightarrow$ Struik), also *nabla* ∇ (Assyrian harp)

• Quaternions: the most simple associative number system with more than 2 units (complex number has 2 units)

• Supporters of Hamilton against those of Grassmann (theory of extensions, exterior product, Grassmann algebra with anticommuting numbers)

• Clifford: Biquaternions: Quaternions the coefficients of which are a system of complex numbers $a + be$, with $e^2 = \pm 1$ or 0. Clifford algebra.

• Maxwell's equations in quaterionic form (Treatise, 2nd edition, 1881, Vol. II, p. 239–240; S = scalar and V vector part of quaternion) $\mathfrak{G}$ = velocity, ψ, Ω = scalar el./mg. pot., eq. numbers 1st column from 1865, 2nd one from 1881

(B_1) (A) $\qquad \mathfrak{B} = V\nabla\mathfrak{A} \qquad (S\nabla\mathfrak{A} = 0 \;\Rightarrow\; \mathfrak{B} = \nabla\mathfrak{A})$ eq. of mg. induction

(D) (B) $\qquad \mathfrak{E} = V\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\psi$ eq. of el.*motive* force

$\quad$ (C) $\qquad \mathfrak{F} = V\mathfrak{C}\mathfrak{B} - e\nabla\psi - m\nabla\Omega$ eq. of el.magn. force

$\quad$ (D) $\qquad \mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{I}$ eq. of magnetization

(C) (E) $\qquad 4\pi\mathfrak{C} = V\nabla\mathfrak{H}$ eq. of el. currents

(F) [G] $\qquad \mathfrak{K} = C\mathfrak{E}$ eq. of conductiv. (Ohm)

(E) [F] $\qquad \mathfrak{D} = \dfrac{1}{4\pi}K\mathfrak{E}$ eq. of el. displacement

(A) [H] $\qquad \mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$ eq. of true currents

(B_2) [L] $\qquad \mathfrak{B} = \mu\mathfrak{H}$ eq. of ind. magnetiz.

(G) [J] $\qquad e = S\nabla\mathfrak{D}$ [Coulomb-Gauss law]

$\qquad\qquad m = S\nabla\mathfrak{I}$

$\qquad\qquad \mathfrak{H} = -\nabla\Omega$

(H) $\qquad$ Number of eqs.(A) to (H) $= 20$ cont. eq. missing here

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3. In symbolic vector calculus: Heaviside 1885/88, Föppl 1894, Gibbs 1901

Heaviside's 'duplex system' of 1888 (see the *original Heaviside1888.pdf* in Phil. Mag. Ser. 5, 25: 153, pp. 130–156 (1888)) $\mathbf{e}, \mathbf{h} =$ impressed fields

$$\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{C} = k\mathbf{E}, \quad \mathbf{D} = (c/4\pi)\mathbf{E}$$

$$\mathrm{curl}\,(\mathbf{H} - \mathbf{h}) = 4\pi\mathbf{\Gamma}$$
$$\mathrm{curl}\,(\mathbf{e} - \mathbf{E}) = 4\pi\mathbf{G}$$
$$\mathbf{\Gamma} = \mathbf{C} + \dot{\mathbf{D}}, \quad \mathbf{G} = \dot{\mathbf{B}}/4\pi$$
$$\mathrm{div}\,\mathbf{B} = 0$$

$$\left[\text{Energy:} \qquad U = \frac{1}{2}\mathbf{E}\mathbf{D}\,, \quad T = \frac{1}{2}\mathbf{H}\mathbf{B}/4\pi\,, \quad Q = \mathbf{E}\mathbf{C}\,, \right.$$
$$\mathbf{W} = V(\mathbf{E} - \mathbf{e})(\mathbf{H} - \mathbf{h})/4\pi \quad \Leftarrow \text{ Poynting}$$
$$\left. \mathbf{e}\mathbf{\Gamma} + \mathbf{h}\mathbf{G} = Q + \dot{U} + \dot{T} + \mathrm{div}\,\mathbf{W}\right]$$

The electromagnetic field (H & O):
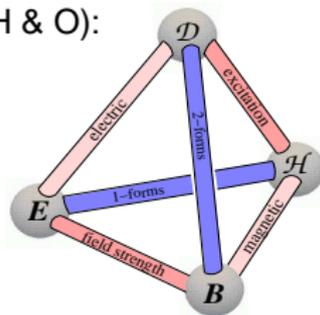
• Heaviside + Grassmann + Gibbs $\Rightarrow$ vector analysis: Hamilton's vectors, Grassmann's exterior product, Gibbs' dyadics; see also J. Crowe, A History of Vector Analysis, Dover

• 1872 Erlangen Program of Klein $\Rightarrow$ group theory + geometry: "Let be given a manifold and a transformation group in it. Develop the theory of invariants with respect to this group." 3d Euclidean group $T^3 \rtimes SO(3) \Rightarrow$ Poincaré group (4d translations $\rtimes$ Lorentz) $T^4 \rtimes SO(1,3)$ [also $SL(2,C)$ and $SO(3,C)$] $\Rightarrow$ diffeomorphism group

• Around 1900: Ricci + Levi-Civita $\Rightarrow$ absolute differential calculus, tensor analysis (tensor Voigt 1900) $\Rightarrow$ Einstein 1916, see history of Karin Reich

• In textbooks, Abraham-Föppl is a leading example (Einstein learned from Föppl), see *original AbrahamFoeppl1904.pdf,* 2nd edition

• Recommended textbook: Sommerfeld, Electrodynamics. Lectures on theoretical physics, Vol. III (1952)

• Bamberg + Sternberg: "...the most suitable framework for geometrical analysis is the exterior differential calculus of Grassmann and Cartan." (topological in constrast to metrical concepts are stressed)

4. In components (compact): Hertz 1890, ansatz for moving bodies
Hertz's system in vacuum, see the *original* Ann. Phys. 1890, $[A^{-1}]$ = velocity;
file *Hertz1890a.pdf*

$$A\frac{dL}{dt} = \frac{dZ}{dy} - \frac{dY}{dz} \qquad\qquad A\frac{dX}{dt} = \frac{dM}{dz} - \frac{dN}{dy}$$

$$A\frac{dM}{dt} = \frac{dX}{dz} - \frac{dZ}{dx} \qquad\qquad A\frac{dY}{dt} = \frac{dN}{dx} - \frac{dL}{dz}$$

$$A\frac{dN}{dt} = \frac{dY}{dx} - \frac{dX}{dy} \qquad\qquad A\frac{dZ}{dt} = \frac{dL}{dy} - \frac{dM}{dx}$$

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dx} = 0\,, \qquad \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0$$

$[\mathbf{H} = (L, M, N), \mathbf{E} = (X, Y, Z)$. For the first time we see all 4 Maxwell
vacuum equations together, cf. Darrigol, p.254 et seq.:

$$A\frac{d\mathbf{H}}{dt} = -\mathsf{curl}\,\mathbf{E}\,, \qquad\qquad A\frac{d\mathbf{E}}{dt} = \ \mathsf{curl}\,\mathbf{H}$$

$$\mathsf{div}\,\mathbf{H} = 0\,, \qquad\qquad \mathsf{div}\,\mathbf{E} = 0$$

Note: By differentiating with respect to the time t, we find the wave equation

$$A\frac{d^2\mathbf{H}}{dt^2} = -\mathsf{curl}\,\frac{d\mathbf{E}}{dt} = -\frac{1}{A}\mathsf{curl}\,\mathsf{curl}\,\mathbf{H} = \frac{1}{A}(\Delta\,\mathbf{H} - \mathsf{grad}\,\underbrace{\mathsf{div}\,\mathbf{H}}_{=0})\,.]$$

Hertz (continued): Ansatz for moving bodies by substituting the convective derivative (of Helmholtz). Let a (electric of magnetic) flux $\mathbf{F}$ be given; then, with the velocity $\mathbf{v}$ of the medium (at the time of Hertz d meant ∂),

$$\frac{d\mathbf{F}}{dt} \quad \Longrightarrow \quad \frac{D\mathbf{F}}{Dt} = \frac{d\mathbf{F}}{dt} - [\nabla \times (v \times \mathbf{F}) - v(\nabla \cdot \mathbf{F})] \,.$$

Substitute this in the l.h.s. of the 2 Maxwell equations containing a time derivative. Turned out to be unsuccessful, but it brought the electrodynamics of moving bodies under way $\Rightarrow$ Einstein 1905.

5. In components à la Maxwell-Hertz + Lorentz transf.: original
Einstein1905.pdf
In the kinematical part of his "On the Electrodynamics of Moving Bodies" he proves for a standard Lorentz transformation (boost in x-direction)

$$\begin{aligned}
\tau &= \beta(t - vx/c^2) \,,\\
\xi &= \beta(x - vt) \,,\\
\eta &= y \,,\\
\zeta &= z \,,
\end{aligned}$$

where $\qquad\qquad \beta = 1/\sqrt{1 - v^2/c^2}.$

Einstein 1905 (continued): He just took the Maxwell-Hertz equations for vacuum (with switched sign), electric field (X, Y, Z), magnetic field (L, M, N),

$$\frac{1}{V}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \qquad \frac{1}{V}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{V}\frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \qquad \frac{1}{V}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}$$

$$\frac{1}{V}\frac{\partial Z}{\partial t} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \qquad \frac{1}{V}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$$

Then he referred the electromagnetic process to the coordinate system above (τ, ξ, η, ζ) and uses the corresponding transformation formulas:

$$\frac{1}{V}\frac{\partial X}{\partial \tau} = \frac{\partial \beta\left(N - \frac{v}{V}Y\right)}{\partial \eta} - \frac{\partial \beta\left(M + \frac{v}{V}Z\right)}{\partial \zeta} \qquad \text{etc.}$$

Because of the relativity principle, we have

$$\frac{1}{V}\frac{\partial X'}{\partial \tau} = \frac{\partial N'}{\partial \eta} - \frac{\partial M'}{\partial \zeta} \qquad \frac{1}{V}\frac{\partial L'}{\partial \tau} = \frac{\partial Y'}{\partial \zeta} - \frac{\partial Z'}{\partial \eta}$$

$$\frac{1}{V}\frac{\partial Y'}{\partial \tau} = \frac{\partial L'}{\partial \zeta} - \frac{\partial N'}{\partial \xi} \qquad \frac{1}{V}\frac{\partial M'}{\partial \tau} = \frac{\partial Z'}{\partial \xi} - \frac{\partial X'}{\partial \zeta}$$

$$\frac{1}{V}\frac{\partial Z'}{\partial \tau} = \frac{\partial M'}{\partial \xi} - \frac{\partial L'}{\partial \eta} \qquad \frac{1}{V}\frac{\partial N'}{\partial \tau} = \frac{\partial X'}{\partial \eta} - \frac{\partial Y'}{\partial \xi}$$

Consequently,

$$X' = X \,, \qquad\qquad L' = L \,,$$
$$Y' = \beta \left(Y - \frac{v}{V} N \right) \,, \qquad M' = \beta \left(M + \frac{v}{V} Z \right) \,,$$
$$Z' = \beta \left(Z + \frac{v}{V} M \right) \,, \qquad N' = \beta \left(N - \frac{v}{V} Y \right) \,,$$

are the transformation formulas for the components of the electromagnetic field.

Similar derivations were given (partly earlier) by Poincaré and by Lorentz.

See also the books of von Laue (1911), Silberstein (quaternions! 1914), Pauli (1921), Einstein (1922),..., Møller (1952),...

6. In symbolic 4d calculus: Minkowski's way to: $\text{lor } \mathbf{f} = -\mathbf{s}$, $\text{lor } \mathbf{F}^* = \mathbf{0}$

Minkowski introduced fields f and F in Cartesian coordinates x, y, z and with imaginary time coo. it ($c = 1$); moreover, $x_1 := x$, $x_2 := y$, $x_3 := z$, $x_4 := it$. Euclidean metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = g_{hk} dx_h dx_k$, with $g_{hk} = \text{diag}(1, 1, 1, 1)$; there is no need to distinguish contravariant (upper) from covariant (lower) indices. The Maxwell equations in component form:

$$\frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4} = s_1 \,,$$

$$\frac{\partial f_{21}}{\partial x_1} + \frac{\partial f_{23}}{\partial x_3} + \frac{\partial f_{24}}{\partial x_4} = s_2 \,,$$

$$\frac{\partial f_{31}}{\partial x_1} + \frac{\partial f_{32}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_4} = s_3 \,,$$

$$\frac{\partial f_{41}}{\partial x_1} + \frac{\partial f_{42}}{\partial x_2} + \frac{\partial f_{43}}{\partial x_3} = s_4 \,,$$

$$\frac{\partial F_{34}}{\partial x_2} + \frac{\partial F_{42}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_4} = 0 \,,$$

$$\frac{\partial F_{43}}{\partial x_1} + \frac{\partial F_{14}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_4} = 0 \,,$$

$$\frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_4} = 0 \,,$$

$$\frac{\partial F_{32}}{\partial x_1} + \frac{\partial F_{13}}{\partial x_2} + \frac{\partial F_{21}}{\partial x_3} = 0 \,.$$

Minkowski (cont.): Modern notation and summ. conv. $h, k, \ldots = 1, 2, 3, 4$,

$$\frac{\partial f_{hk}}{\partial x_k} = s_h \qquad \text{and} \qquad \frac{\partial F_{hk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_h} + \frac{\partial F_{lh}}{\partial x_k} = 0 \qquad (\text{or } \partial_{[l} F_{hk]} = 0) \,.$$

Excitation f and the *field strength* F (in Maxwell's nomenclature[1])

$$(f_{hk}) = -(f_{kh}) = \begin{pmatrix} 0 & H_z & -H_y & -iD_x \\ -H_z & 0 & H_x & -iD_y \\ H_y & -H_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix}$$

$$(F_{hk}) = -(F_{kh}) = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix},$$

The 4d electric current denoted by s_h.

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[1] Minkowski took $\mathbf{D} = \mathbf{e}$, $\mathbf{H} = \mathbf{m}$; $\mathbf{E} = \mathbf{E}$, $\mathbf{B} = \mathbf{M}$, that is, for the excitation $f \sim (\mathbf{e}, \mathbf{m})$ and for the field strength $F \sim (\mathbf{E}, \mathbf{M})$.

Minkowski (cont.): He introduced the *dual* of F_{hk}, namely $F_{hk}^* := \frac{1}{2}\hat{\epsilon}_{hklm}F_{lm}$, with the Levi-Civita symbol $\hat{\epsilon}_{hklm} = \pm 1, 0$ and $\hat{\epsilon}_{1234} = +1$. Thus,

$$F^* = (F_{hk}^*) = \begin{pmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{pmatrix}.$$

Then both Maxwell equations read

$$\frac{\partial f_{hk}}{\partial x_k} = s_h, \qquad\qquad \frac{\partial F_{hk}^*}{\partial x_k} = 0.$$

Subsequently Minkowski developed a 4-dimensional type of Cartesian tensor calculus with a 4d differential operator called 'lor' (abbreviation of Lorentz). He introduces ordinary (co)vectors *(space-time vectors of the 1st kind)*, like x_h and $\text{lor}_h := \frac{\partial}{\partial x_h}$, and antisymmetric 2nd rank tensors *(space-time vectors of the 2nd kind)*, like f_{hk} and F_{hk}. Then, symbolically he wrote

$$\boxed{\text{lor}\, f = -s, \qquad \text{lor}\, F^* = 0}$$

(lor and * metric dependent).

Using his Cartesian tensor calculus, Minkowski has shown that these eqs. are *covariant under Poincaré transformations.* In vacuum, $f \sim F$. Compare with exterior calculus version with $dH = J$, $dF = 0$ and, in vacuum, $H \sim {}^\star F$.
• Minkowski also discovered 1907 the energy-momentum tensor for the electromagnetic field: densities of energy/momentum and their fluxes.

The next step occurred immediately after Einstein's fundamental 1915 paper on general relativity. Now Einstein was in command of tensor calculus in arbitrary coordinate systems. By picking suitable variables, he found ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with signature $(1,-1,-1,-1)$, here $\mu, \nu, ... = 0, 1, 2, 3$)

$$\frac{\partial F_{\rho\sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma\tau}}{\partial x^\rho} + \frac{\partial F_{\tau\rho}}{\partial x^\sigma} = 0\,, \qquad \mathfrak{F}^{\mu\nu} = \sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\,, \qquad \frac{\partial \mathfrak{F}^{\mu\nu}}{\partial x^\nu} = \mathcal{J}^\mu\,.$$

The field strength $F_{\rho\sigma}$ is a tensor, the excitation $\mathfrak{F}^{\mu\nu}$ a tensor density. Einstein's identifications, which were only worked out by him for vacuum, read

$$\mathfrak{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix}.$$

• These Maxwellian eqs. are generally covariant *and* metric independent. The gravitational potential only enters the "spacetime relation"—it is the 'constitutive law' of the vacuum.
Here *no* comma goes to semicolon rule " , $\rightarrow$; " (MTW) is necessary for eldyn.

For a mathematically precise presentation see Schouten "Tensor Analysis for Physicists" (Oxford 1951, Dover 1989).

8. In premetric/integral formulation, in tensor and exterior diff. calculus up to ~ 1960

Already initiated by Maxwell in his paper 5 (similar in Sommerfeld). Élie Cartan (1924) as an example. In special relativity:

$$\begin{aligned}
\Omega &= B_x[dy\,dz] + B_y[dz\,dx] + B_z[dx\,dy] \\
&\quad + E_x[dx\,dt] + E_y[dy\,dt] + E_z[dz\,dt] \\
\overline{\Omega} &= D_x[dy\,dz] + D_y[dz\,dx] + D_z[dx\,dy] \\
&\quad + H_x[dx\,dt] + H_y[dy\,dt] + H_z[dz\,dt] \\
S &= \rho[dx\,dy\,dz] - I_x[dy\,dz\,dt] - I_y[dz\,dt\,dx] - I_z[dx\,dy\,dt]
\end{aligned}$$

$$\Omega' = 0\,, \qquad \overline{\Omega}' = -4\pi S \qquad \Rightarrow S' = 0$$

Generalization:

$$\iint \Omega = 0\,, \qquad \iint \overline{\Omega} = -4\pi \iiint S\,,$$

where the integral on the right-hand-side extends over any 3-dimensional volume of spacetime and those on the left-hand-sides over the 2-dimensional boundary of this volume. Isn't that a beautiful representation?

Similar versions by Murnaghan, Kottler, van Dantzig, Schrödinger, Schouten, Truesdell-Toupin, Post

9. In spinor calculus: Spinors as *semi*vectors, tensor with rank $\frac{1}{2}$. Weyl 1928–29, Fock 1929, van der Waerden 1929, Schrödinger 1930, systematically: Infeld & van der Waerden, Sitzungsber. Preuss. Akad. Wiss. Physik.-Math. Klasse, p. 380 (1933). *E.M. Corson,* Tensors, Spinors and Rel. Wave Eqs., 1953. We take as example, Laporte & Uhlenbeck Phys. Rev. **37** (1931) 1380–1397: Group $SL(2, C)$ with transformations

$$\xi'_1 = \alpha_{11}\xi_1 + \alpha_{12}\xi_2\,, \qquad \overline{\xi'_1} = \overline{\alpha}_{11}\overline{\xi_1} + \overline{\alpha}_{12}\overline{\xi_2}\,,$$
$$\xi'_2 = \alpha_{21}\xi_1 + \alpha_{22}\xi_2\,, \qquad \overline{\xi'_2} = \overline{\alpha}_{21}\overline{\xi_1} + \overline{\alpha}_{22}\overline{\xi_2}\,,$$

and $\det \alpha = 1$, simple covering group of the proper orthochronous Lorentz group $SO_0(1, 3)$. Fundamental objects are the spinors a_k and $b_{\dot{r}}$; higher order objects $a_{kl}, b_{\dot{r}\dot{s}}, c_{\dot{r}k}, \ldots$. Spinor 'metric' $\epsilon^{kl} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \epsilon_{kl} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Relation between spinors and world vectors:

$$\frac{1}{2}(a_{\dot{2}1} + a_{\dot{1}2}) = A^1 = A_1\,, \qquad a_{\dot{2}1} = A_1 + iA_2\,,$$
$$\frac{1}{2i}(a_{\dot{2}1} - a_{\dot{1}2}) = A^2 = A_2\,, \qquad a_{\dot{1}2} = A_1 - iA_2\,,$$
$$\frac{1}{2}(a_{\dot{1}1} - a_{\dot{2}2}) = A^3 = A_3\,, \qquad a_{\dot{1}1} = A_3 - A_4\,,$$
$$\frac{1}{2}(a_{\dot{1}1} + a_{\dot{2}2}) = A^4 = -A_4\,, \qquad -a_{\dot{2}2} = A_3 + A_4\,.$$

In spinors (cont.) Definition of self-dual tensor $F_{kl}^* := \frac{i}{2}\epsilon_{kl\alpha\beta}F^{\alpha\beta}$. Introduce complex electromagnetic field strength (here for vacuum) and find the Maxwell equation

$$G^{kl} := F^{kl} + F^{*kl}, \qquad \frac{\partial G^{k\lambda}}{\partial x^\lambda} = S^k.$$

G^{kl} is an antisymmetric self-dual 2nd rank tensor with 6 independent components. We can assign to G^{kl} a *symmetric 2nd rank* spinor $g_{i\dot{m}}$ with 3 complex components, that is, with 6 independent components. Then we find Maxwell's vacuum eqs. in spinor form (field strength g, current s, potential ϕ):

$$\boxed{\partial^{\dot{\rho}}{}_l\, g_{\dot{\rho}\dot{m}} = 2s_{\dot{m}l}}, \qquad g_{\dot{r}\dot{s}} = \partial_{\dot{r}\lambda}\phi_{\dot{s}}{}^\lambda + \partial_{\dot{s}\lambda}\phi_{\dot{r}}{}^\lambda,$$

$$\partial_{\dot{\rho}\alpha}\partial^{\dot{\rho}\alpha}\phi_{\dot{m}l} = 2s_{\dot{m}l} \qquad \text{with Lorenz condition} \qquad \partial^{\dot{\mu}\lambda}\phi_{\dot{\mu}\lambda} = 0.$$

The covariant version of this 'Maxwell equation' is being used for the analysis of propagating electromagnetic waves in GR. Einstein's equation can also be put in spinor form: The curvature tensor becomes a totally symmetric 4th rank curvature spinor $\psi_{mnrs} = \psi_{(mnrs)}$; this uniform formalism facilitates sometimes the investigations on electromagnetic and gravitational waves.

10. 4d Clifford algebra formalism: consult for vacuum eldyn. the text of *Baylis,* e.g.; see also Puska, PIER 32 (2001) 413 for isotropic constitutive relations.

11. In algebraic/discrete formulation, Tonti's vers. of 2009, ACE'09 in Rome

$$\Psi := \int_S \mathbf{D} \cdot \mathbf{n} \, dS \text{ (electric flux)} \qquad \Phi := \int_S \mathbf{B} \cdot \mathbf{n} \, dS \text{ (magnetic flux)}$$

$$E := \int_L \mathbf{E} \cdot \mathbf{t} \, dL \qquad F := \int_L \mathbf{H} \cdot \mathbf{t} \, dL$$

$$I := \int_S \mathbf{J} \cdot \mathbf{n} \, dS \text{ (current)} \qquad Q^c := \int_V \rho \, dV \text{ (charge)}$$

Forget these defs., take the *global* quantities Ψ, Φ, E, F, I, Q^c as fundamental. Consider instant (of time) I, volume V and its boundary ∂V; furthermore, time interval T, surface S and its boundary ∂S. Inner — and outer orientation $\tilde{\ }$:

$$\Psi\left[\overline{I}, \partial\widetilde{V}\right] = Q^c\left[\overline{I}, \widetilde{V}\right], \quad \mathcal{F}\left[\overline{T}, \partial\widetilde{S}\right] = +\left\{\Psi\left[\overline{I}^+, \widetilde{S}\right] - \Psi\left[\overline{I}^-, \widetilde{S}\right]\right\} + Q^f\left[\overline{T}, \widetilde{S}\right],$$

$$\Phi\left[\widetilde{I}, \partial\overline{V}\right] = 0, \qquad \mathcal{E}\left[\widetilde{T}, \partial\overline{S}\right] = -\left\{\Phi\left[\widetilde{I}^+, \overline{S}\right] - \Phi\left[\widetilde{I}^-, \overline{S}\right]\right\}.$$

$\mathcal{F}$ = impulse of magnetomotive force, $\mathcal{E}$ = impulse of electromotive force. All laws refer to the *boundaries* of the space elements V and S. Compare with

$$\underline{d}\,\mathcal{D} = \rho, \qquad\qquad \underline{d}\,\mathcal{H} = +\dot{\mathcal{D}} + j,$$

$$\underline{d}\,B = 0, \qquad\qquad \underline{d}\,E = -\dot{B}.$$

Can be used for computer calculations. Start with global/discrete structures; don't discretize the differential Maxwell equations!

Tonti (2009), see `http://www.dica.units.it/perspage/tonti/`
The *differential* formulation requires the *field* vectors $\mathbf{B}$, $\mathbf{E}$, ρ, $\mathbf{J}$, $\mathbf{D}$, $\mathbf{H}$ as 16 *point* functions, the algebraic formulation requires the global 6 scalar variables Φ, $\mathcal{E}$, Q^c, Q^f, Ψ, $\mathcal{F}$ as domain variables. Avoiding the field vectors, we don't need to perform integrations, better for computational electromagnetism,

12. 3d and 4d exterior calculus, premetric topological form of Maxwell's eqs.

Post (Quantum Reprogramming... [1995], p. 105), in our version (Hehl & Obukhov, Foundations of Classical Eldyn., Boston 2003): Notions of de Rham 1931; for any cycle Z_3 with $\partial Z_3 = 0$ and any cycle Z_2 with $\partial Z_2 = 0$, we have

$$\oint_{Z_3} J = 0, \qquad f_\alpha = (e_\alpha \rfloor F) \wedge J, \qquad \oint_{Z_2} F = 0.$$

1st axiom $\Rightarrow$ matter and its conserved el. *charge.* 2nd axiom $\Rightarrow$ charge + *mechanical force* define elmg. *field strength.* 3rd axiom $\Rightarrow$ flux of field strength is sourcefree. *Twisted* diff. forms J, H; untwisted F, A [for twisted forms and generalized Clifford algebras, see Diane Demers `http://felicity.freeshell.org/math/index.htm`]

Differential version of electrodynamics:

$$dJ = 0, \qquad f_\alpha = (e_\alpha \rfloor F) \wedge J, \qquad dF = 0,$$
$$J = dH, \qquad\qquad\qquad\qquad\qquad F = dA.$$

Premetric/topological formulation (continued)

Because of the existence of (super)conductors, *we can measure the excitation H*. Thus, even if H emerges as for the electric current, it is more than that: It is measurable. This is in clear contrast to the potential A that is *not* measurable in class. eldyn. (see, however, the AB-effect).

The physical interpretation of the Maxwell equations can be found via the $(1 + 3)$-decomposition (signs embody the Lenz rule)

$$J = - j \wedge dt + \rho, \qquad\qquad H = - \mathcal{H} \wedge dt + \mathcal{D},$$
$$F = E \wedge dt + B, \qquad\qquad A = - \varphi \, dt + \mathcal{A},$$

Then, by substitutions, the (1 + 3)-decomposition of the Maxwell eqs. read

$$dH = J \begin{cases} \underline{d}\,\mathcal{D} = \rho & (1 \text{ constraint eq.}), \\ \dot{\mathcal{D}} = \underline{d}\,\mathcal{H} - j & (3 \text{ time evol. eqs.}), \end{cases}$$
$$dF = 0 \begin{cases} \underline{d}\,B = 0 & (1 \text{ constraint eq.}), \\ \dot{B} = -\underline{d}\,E & (3 \text{ time evol. eqs.}). \end{cases}$$

Accordingly, we have $2 \times 3 = 6$ time evolution equations for the $2 \times 6 = 12$ variables $(\mathcal{D}, B, \mathcal{H}, E)$ of the electromagnetic field. Thus the Maxwellian structure is underdetermined. We need, in addition, a constitutive (or electromagn. spacetime) relation that expresses the excitation $H = (\mathcal{H}, \mathcal{D})$ in terms of the field strength $F = (E, B)$, i.e., $H = H[F]$. In vacuo, $H = \lambda_0 {}^\star F$.

Topological quantum superstructure for Maxwell eqs. (Post, priv. comm. Feb. 2010) (Z_1, Z_2, Z_3 are 1d, 2d, 3d integration cycles, n, s are integers)

$$\sum_{Z_1} \oint_{Z_1} A = \underbrace{n\frac{h}{e}}_{\text{encl. mg. flux}} \quad , \quad \sum_{Z_2} \oint_{Z_2} H = \underbrace{se}_{\text{encl. el. charge}} \quad , \quad \sum_{Z_3} \oint_{Z_3} A \wedge H = \underbrace{ns\frac{h}{2}}_{\text{encl. action}} \quad .$$

Aharonov-Bohm, Gauss-Ampère, Kiehn-Post integrals with absolute dimensions of *action/charge, charge,* and *action,* respectively: macroscopic, phenomenological, and *topological* information stored in these integrals.

Let us add our knowledge from *quantum* theory. We know that all 'action' is quantized with $h/2$ and all charge is quantized with e. Consequently GA and KP should be valid universally. In *superconductors* mg. flux can be quantized.

So far, no information from the constitutive law was fed into AB, GA, KP.

Apply AB, GA, KP to the QHE/QSH effects. Since the charges are 2-dimensionally spread, we have to go to (1+2)-dimensional electrodynamics with $dH = J$, $dF = 0$, and H as twisted 1-form, :

$$\sum_{Z_1} \oint_{Z_1} A = n\frac{h}{e}, \quad \sum_{Z_1} \oint_{Z_1} H = se, \quad \sum_{Z_2} \oint_{Z_2} A \wedge H = ns\frac{h}{2}.$$

Now find the *topological* (metricfree) constitutive relation for the QHE:

$$J = -\sigma_H F \qquad \text{(quantized Hall resistance } 1/\sigma_H\text{)}.$$

We integrate $J = -\sigma_H F$ with the help of $J = dH$ and $F = dA$:

$$H = -\sigma_H A \qquad \text{or} \qquad A = -H/\sigma_H \,.$$

Now, in this physical case of the QHE in (1+2) dimensions, also the AB-integral is necessarily quantized.—

• Which system of Maxwell's equations should be taught to physics and engineering students? I opt for Sec.11 and Sec.12 in contrast to Jackson, Landau-Lifshitz,...

I used the *original* articles mentioned above, for secondary literature, see

- ▶ E. Whittaker, *A History of the Theories of Aether and Electricity,* 2 volumes, Humanities Press, NY, 1973 [1951].
- ▶ C.W.F. Everitt, *James Clerk Maxwell,* Physicis and Natural Philosopher, Charles Scibner's Sons, NY, 1975.
- ▶ J.L. Heilbron, *Electricity in the 17th and 18th Centuries,* A Study of Early Modern Physics, UC Press, Berkeley, 1979.
- ▶ O. Darrigol, *Electrodynamics from Ampère to Einstein,* Oxford, 2000.
- ▶ F. Steinle, *Explorative Experimente,* Ampère, Faraday und die Ursprünge der Elektrodynamik, Steiner, Stuttgart, 2005.

Soli Deo Gloria.