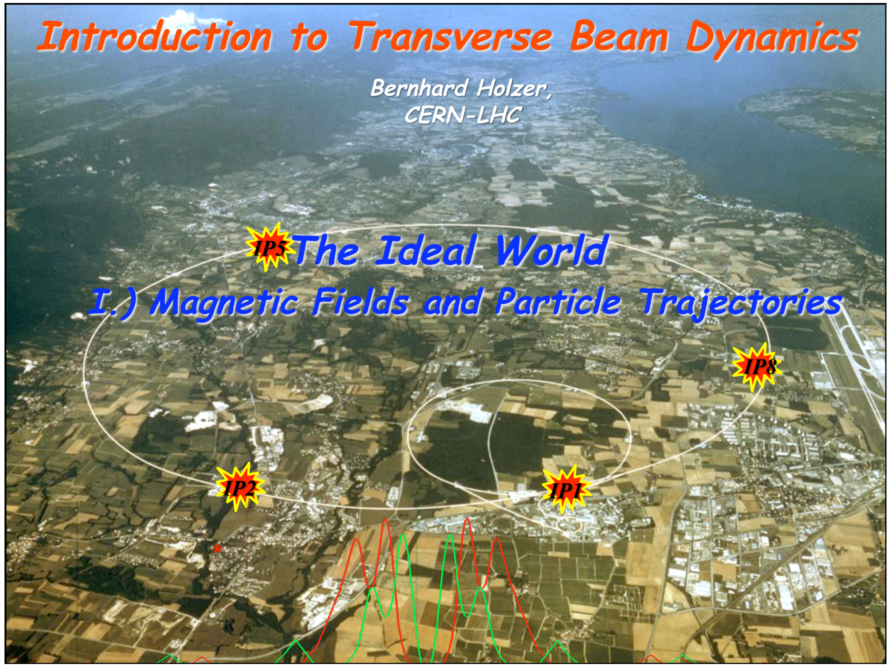
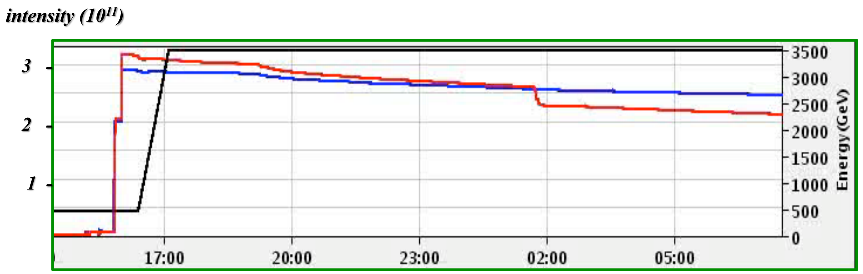



## Luminosity Run of a typical storage ring:

LHC Storage Ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about $v \approx c$
$L = 10^{10}\text{-}10^{11}$ km
... several times Sun - Pluto and back




→ guide the particles on a well defined orbit („design orbit")
→ focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.

## *I.) Introduction and Basic Ideas*

*„ ... in the end and after all it should be a kind of circular machine"*

*→ need transverse deflecting force*

*Lorentz force* $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

*typical velocity in high energy machines:* $v \approx c \approx 3*10^8 \, {}^{m}/_{s}$

*Example:*

$$B = 1\,T \quad \rightarrow \quad F = q*3*10^8 \frac{m}{s} * 1\frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

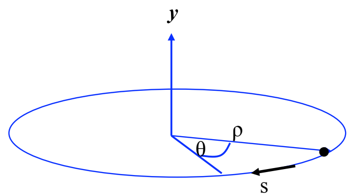
*equivalent el. field ...* $E$

*technical limit for el. field*

$$E \leq 1 \frac{MV}{m}$$

*old greek dictum of wisdom:*
*if you are clever, you use magnetic fields in an accelerator wherever it is possible.*

*The ideal circular orbit*




*circular coordinate system*

*condition for circular orbit:*

*Lorentz force* $\boldsymbol{F_L = e\,v\,B}$

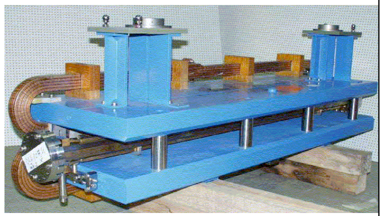
*centrifugal force* $\boldsymbol{F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}}$

$$\boldsymbol{\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\, \cancel{v}\, B}$$

$$\boxed{\boldsymbol{\frac{p}{e} = B\,\rho}}$$

*$B\rho$ = "beam rigidity"*

## 1.) The Magnetic Guide Field



***Dipole Magnets:***

***define the ideal orbit***
***homogeneous field created***
***by two flat pole shoes***

$$B = \frac{\mu_0\, n\, I}{h}$$

***Normalise magnetic field to momentum:***

$$\frac{p}{e} = B\rho \longrightarrow \frac{1}{\rho} = \frac{e\,B}{p}$$

***convenient units:***

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

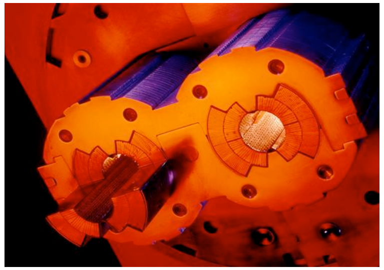
***Example LHC:***

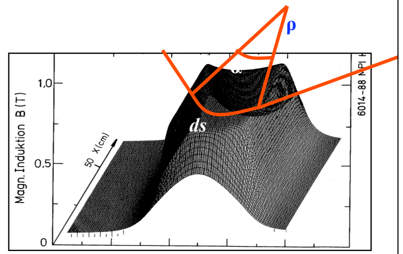
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\, Vs/m^2}{7000*10^9\, eV/c} = \frac{8.3\, s*3*10^8\, m/s}{7000*10^9\, m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,{}^{1}\!/_{m}$$

## The Magnetic Guide Field





*field map of a storage ring dipole magnet*

$$\rho = 2.53\,km \longrightarrow 2\pi\rho = 17.6\,km$$
$$\approx 66\%$$

$$B \approx 1 \ldots 8\ T$$

***rule of thumb:*** $\frac{1}{\rho} \approx 0.3\,\frac{B[T]}{p[GeV/c]}$ ***„normalised bending strength"***

## 2.) Quadrupole Magnets:

*required:* *focusing forces* *to keep trajectories in vicinity of the ideal orbit*

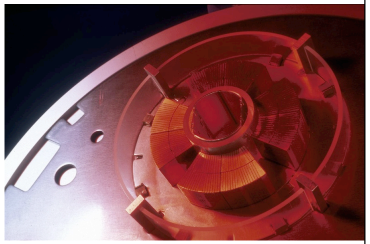
*linear increasing Lorentz force*

*linear increasing magnetic field* $B_y = g\,x \qquad B_x = g\,y$

*normalised quadrupole field:*

*gradient of a quadrupole magnet:* $g = \dfrac{2\mu_0 nI}{r^2}$

$$\longrightarrow \quad k = \frac{g}{p/e}$$



*LHC main quadrupole magnet*

*simple rule:* $\boxed{k = 0.3\,\dfrac{g(T/m)}{p(GeV/c)}}$

$$g \approx 25 \ldots 220 \;\; T/m$$

*what about the vertical plane:*
*... Maxwell*

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \quad \Rightarrow \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

## 3.) The equation of motion:

### Linear approximation:

* ideal particle → design orbit

* any other particle → coordinates x, y small quantities
$x,y \ll \rho$

→ magnetic guide field: only linear terms in x & y of B have to be taken into account

*Taylor Expansion of the B field:*

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2 B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3 B_y}{dx^3}x^3 + \ldots$$

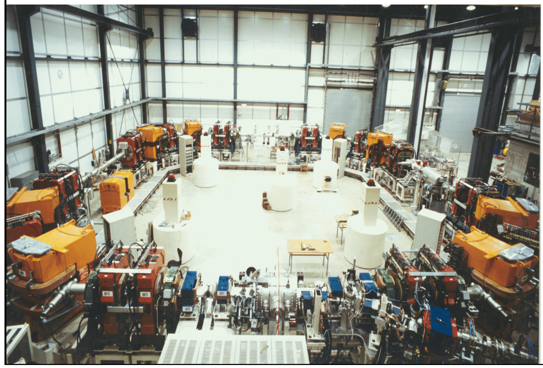
normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

## The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}\,m\,x^2} + \cancel{\frac{1}{3!}\,n\,x^3} + \dots$$

*only terms linear in x, y taken into account* *dipole fields*
*quadrupole fields*



*Separate Function Machines:*

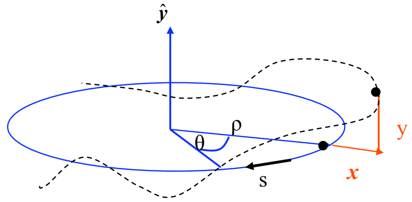
*Split the magnets and optimise them according to their job:*

*bending, focusing etc*

*Example:*
*heavy ion storage ring TSR*

* *man sieht nur dipole und quads → linear*

## *Equation of Motion:*




*Consider local segment of a particle trajectory*
*... and remember the old days:*
*(Goldstein page 27)*

*radial acceleration:*

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

*Ideal orbit:* $\rho = const, \quad \frac{d\rho}{dt} = 0$

*Force:* $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

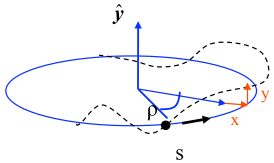
$$F = mv^2/\rho$$

*general trajectory:* $\rho \rightarrow \rho + x$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2




1 $\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$ *... as $\rho = const$*

2 *remember: $x \approx mm$ , $\rho \approx m$ ... → develop for small x*

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

*Taylor Expansion*

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + $$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

*guide field in linear approx.*

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

*independent variable: $t \rightarrow s$*

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\, g}{mv}$$

$$m\,v = p$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\, g}{p/e}$$

*normalize to momentum of particle*

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$

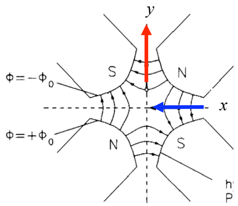
$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* ***Equation for the vertical motion:***

$\frac{1}{\rho^2} = 0$ ***no dipoles … in general …***

$k \leftrightarrow -k$ ***quadrupole field changes sign***

$$\boxed{y'' + k\, y = 0}$$




## Remarks:

* $x'' + \left(\frac{1}{\rho^2} - k\right)\cdot x = 0$

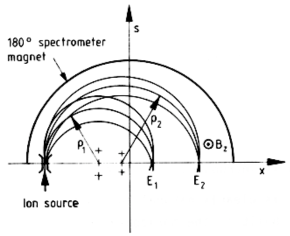
***… there seems to be a focusing even without a quadrupole gradient***

***„weak focusing of dipole magnets"***

$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2}\,x$

***even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets***

***… in large machines it is weak.*** *(!)*




*Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole*

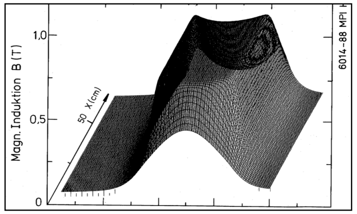
## Hard Edge Model:

$$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

***this equation is not really correct !!!***

$$x''(s) + \left\{\frac{1}{\rho^2(s)} - k(s)\right\} * x(s) = 0$$

***bending and focusing forces – even inside a magnet – depend on the position "s"***




*magnetic field of a storage ring dipole*

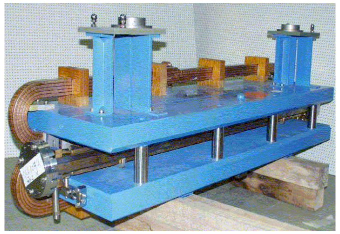
*k = const within a magnet*

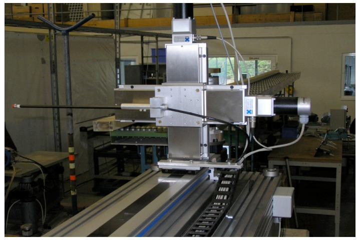
***But still: Inside a magnet the focusing properties are constant !***

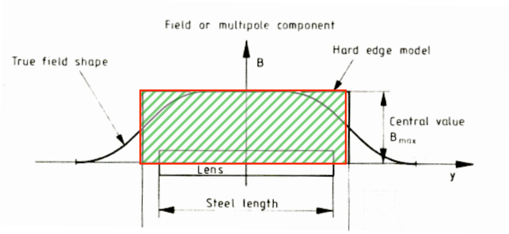
$$\frac{1}{\rho} = const$$

$$k = const$$

## „effective magnetic length"








$$B * l_{eff} = \int_0^{l_{mag}} B\,ds$$

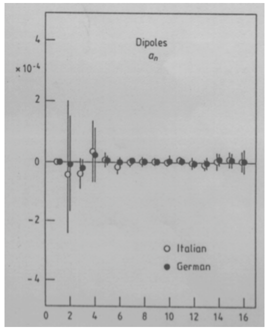
*** **Multipoles**

***Taylor Expansion of the B field:***

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \dots \quad : B_{y0}$$

***divide by the main field***
***to get the relative error contribution***

***→ definition of multipole coefficients.***




*Multipole contributions to the HERA s.c. dipole field*

## 4.) Solution of Trajectory Equations

***Define … hor. plane:*** $K = 1/\rho^2 - k$

***… vert. Plane:*** $K = k$

$$x'' + K\,x = 0$$

***Differential Equation of harmonic oscillator … with spring constant K***

***Ansatz:*** $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

***general solution: linear combination of two independent solutions***

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

***general solution:***

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

*determine $a_1$, $a_2$ by boundary conditions:*

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 , & a_1 = x_0 \\ x'(0) = x'_0 , & a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

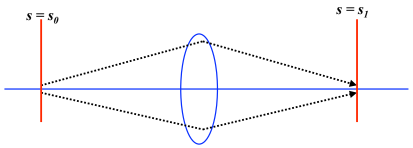
*Hor. Focusing Quadrupole $K > 0$:*

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

***For convenience expressed in matrix formalism:***

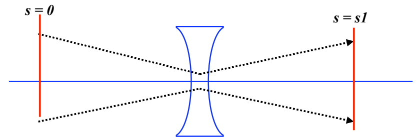
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$




$$M_{foc} = \begin{pmatrix} \cos\left(\sqrt{|K|}l\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}l\right) \\ -\sqrt{|K|} \sin\left(\sqrt{|K|}l\right) & \cos\left(\sqrt{|K|}l\right) \end{pmatrix}$$

*hor. defocusing quadrupole:*

$$x'' - K\,x = 0$$




***Remember from school:***

$$f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s)$$

***Ansatz:*** $x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}} \sinh\sqrt{|K|}l \\ \sqrt{|K|} \sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

*drift space:* $K = 0$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

**! *with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“***

## Thin Lens Approximation:

*matrix of a quadrupole lens*

$$M = \begin{pmatrix} \cos\sqrt{|k|}\,l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}\,l \\ -\sqrt{|k|}\sin\sqrt{|k|}\,l & \cos\sqrt{|k|}\,l \end{pmatrix}$$

*in many practical cases we have the situation:*

$$f = \frac{1}{k l_q} >> l_q$$ *... focal length of the lens is much bigger than the length of the magnet*

*limes:* $l_q \rightarrow 0$ *while keeping* $k\, l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

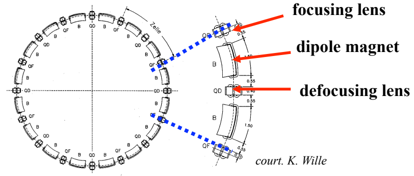
*... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !*

## Transformation through a system of lattice elements

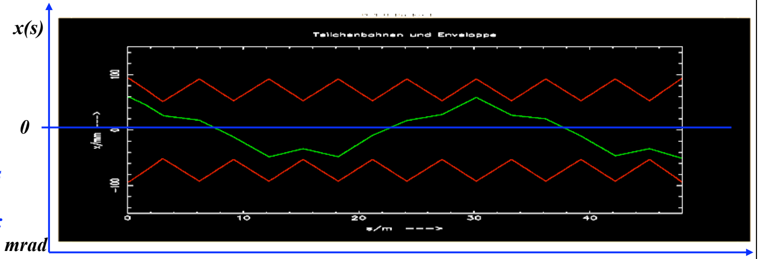
*combine the single element solutions by multiplication of the matrices*

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*} .....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

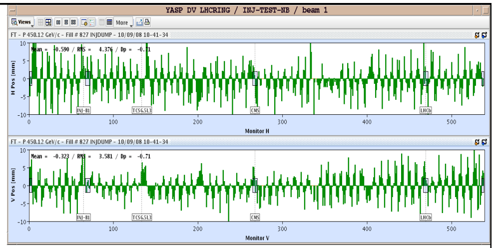



*in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „*




*typical values in a strong foc. machine:*
$x \approx mm, x' \le mrad$

## 5.) Orbit & Tune:

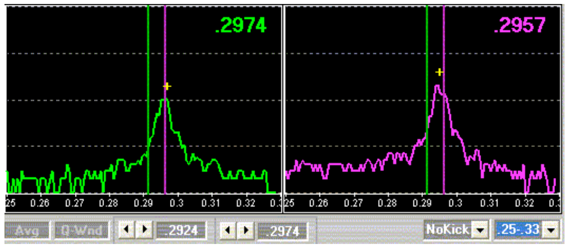


*Tune: number of oscillations per turn*

*64.31*
*59.32*

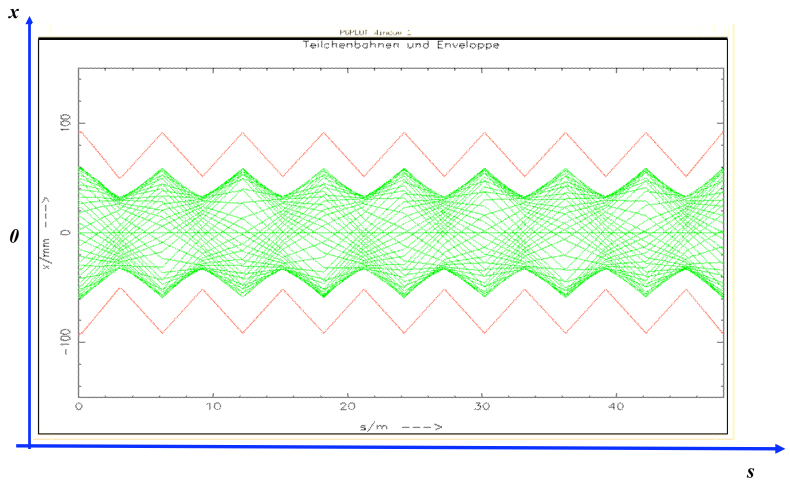
*Relevant for beam stability:*
*non integer part*

*LHC revolution frequency: 11.3 kHz* $0.31 * 11.3 = 3.5 kHz$




***Question:*** ***what will happen, if the particle performs a second turn ?***

*... or a third one or ... $10^{10}$ turns*



## *Résumé:*

*beam rigidity:* $B \cdot \rho = p/q$

*bending strength of a dipole:* $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

*focusing strength of a quadrupole:* $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

*focal length of a quadrupole:* $f = \frac{1}{k \cdot l_q}$

*equation of motion:* $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

*matrix of a foc. quadrupole:* $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

## 6.) Bibliography: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4.) Bernhard Holzer: *Lattice Design*, CERN Acc. School: Interm.Acc.phys course, http://cas.web.cern.ch/cas/ZEUTHEN/lectures-zeuthen.htm

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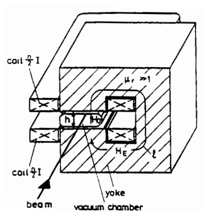
7.) Frank Hinterberger: *Physik der Teilchenbeschleuniger*, Springer Verlag 1997

8.) Mathew Sands: *The Physics of e+ e- Storage Rings*, SLAC report 121, 1970

9.) D. Edwards, M. Syphers : *An Introduction to the Physics of Particle Accelerators*, SSC Lab 1990

## Dipole Magnets:

homogeneous field created by two flat pole shoes




***Field Calculation:***

*3rd Maxwell equation for a static field:* $\vec{\nabla} \times \vec{H} = \vec{j}$

*according to Stokes theorem:* $\int_S (\vec{\nabla} \times \vec{H})\vec{n}\, da = \oint \vec{H}\, d\vec{l} = \int_S \vec{j} \cdot \vec{n}\, da = N \cdot I$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + H_{Fe} * l_{Fe}$$

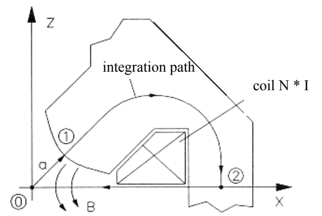
*in matter we get with $\mu_r \approx 1000$* $\oint \vec{H}\, d\vec{l} = H_0 * h + \cancel{\frac{H_0}{\mu_r} * l_{Fe}} \approx H_0 * h$

*Magnetic field of a dipole magnet:* $H_0 = B_0 / \mu_0 \longrightarrow B_0 = \frac{\mu_0 N I}{h}$

*h = gap height*

## Quadrupole Magnets:

***Calculation of the Quadrupole Field:***




*court. K. Wille*

$$\oint \boldsymbol{Hds} = \boldsymbol{N * I}$$

$$\oint \boldsymbol{Hds} = \int_0^1 \boldsymbol{H_0 ds} + \underbrace{\cancel{\int_1^2 \boldsymbol{H_{Fe} ds}}}_{H_{Fe}=H_0/\mu_{Fe},\ \mu_{Fe} \approx 1000} + \underbrace{\cancel{\int_2^0 \boldsymbol{H ds}}}_{H \perp ds} = \boldsymbol{N * I}$$

*now we know that* $\boldsymbol{H} = \boldsymbol{B} / \mu_0$

*and we require* $\boldsymbol{B(r)} = -\boldsymbol{g * r}$

$$\longrightarrow \int_0^1 \boldsymbol{H_0 ds} = \int_0^a \frac{\boldsymbol{B_0}}{\mu_0} \boldsymbol{dr} = \int_0^a \frac{\boldsymbol{g * r}}{\mu_0} \boldsymbol{dr} = \boldsymbol{N * I}$$

*gradient of a quadrupole field:* $\boldsymbol{g} = \frac{2\mu_0 * \boldsymbol{N * I}}{\boldsymbol{r^2}}$