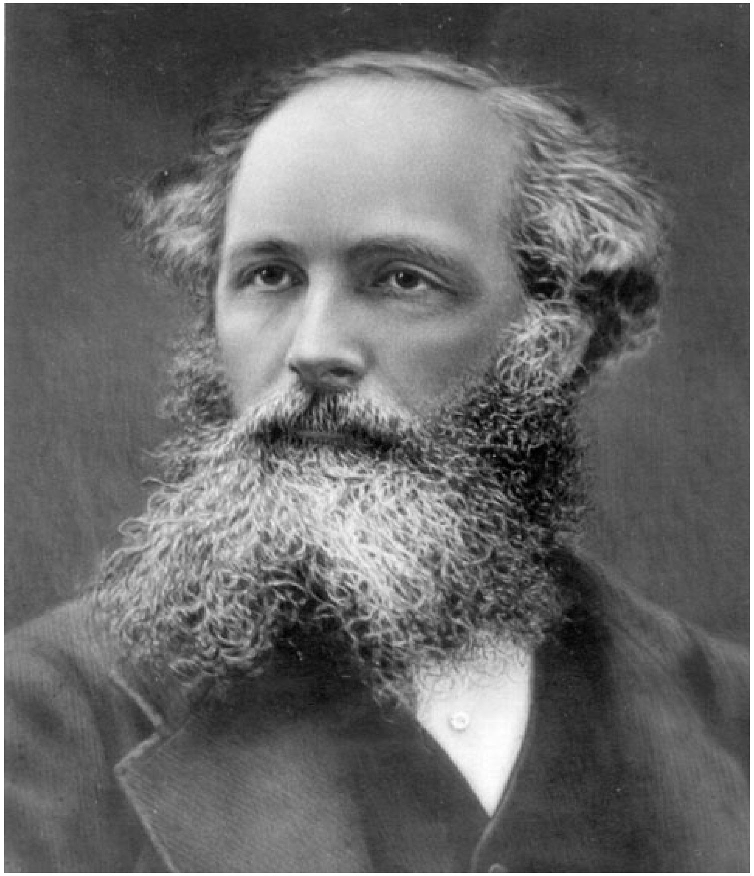
Review of Electromagnetism



This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Maxwell's equations

(in integral form)

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$

$$\oiint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\, dV$$

$$\oiint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$

$\vec{E}, \vec{H}$ electric and magnetic field

$\vec{D}, \vec{B}$ electric displacement and magnetic induction

$\vec{J}$ electric current density

ρ electric charge density

$\iint \vec{J}(\vec{r},t)\cdot d\vec{A}$ stands for all currents going through the area A. It may consist of 3 parts

$$\vec{J}(\vec{r},t)=\vec{J}_c(\vec{r},t)+\vec{J}_{cv}(\vec{r},t)+\vec{J}_i(\vec{r},t)$$

$\vec{J}_c(\vec{r},t)=\kappa\vec{E}(\vec{r},t)$ *conduction current*

$\vec{J}_{cv}(\vec{r},t)=\rho(\vec{r},t)\vec{v}(\vec{r},t)$ *convection current*

$\vec{J}_i(\vec{r},t)$ *impressed current*

$\iiint \rho(\vec{r},t)\,dV$ stands for all charges in the volume V

Time-harmonic fields can be written as complex quantity

$$\vec{e}(\vec{r},t)=\vec{e}(\vec{r})\cos(\omega t+\varphi)=$$
$$=\Re[\vec{E}(\vec{r})e^{i\varphi}e^{i\omega t}]=\Re[\tilde{\vec{E}}(\vec{r})e^{i\omega t}]$$

$\tilde{\vec{E}}(\vec{r})$ is called phasor.

Advantages are:

$\frac{\partial}{\partial t} \rightarrow i\omega$,

phasors are vectors in a coordinate system rotating with ωt,

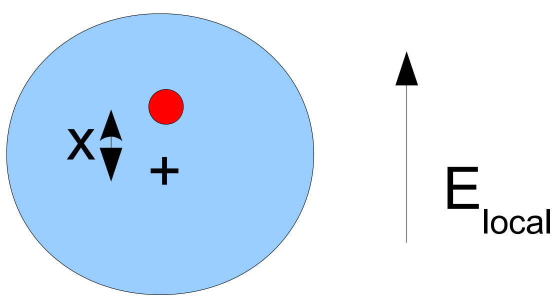
$e^{i\omega t}$ cancels out in the equations.

Fields in matter

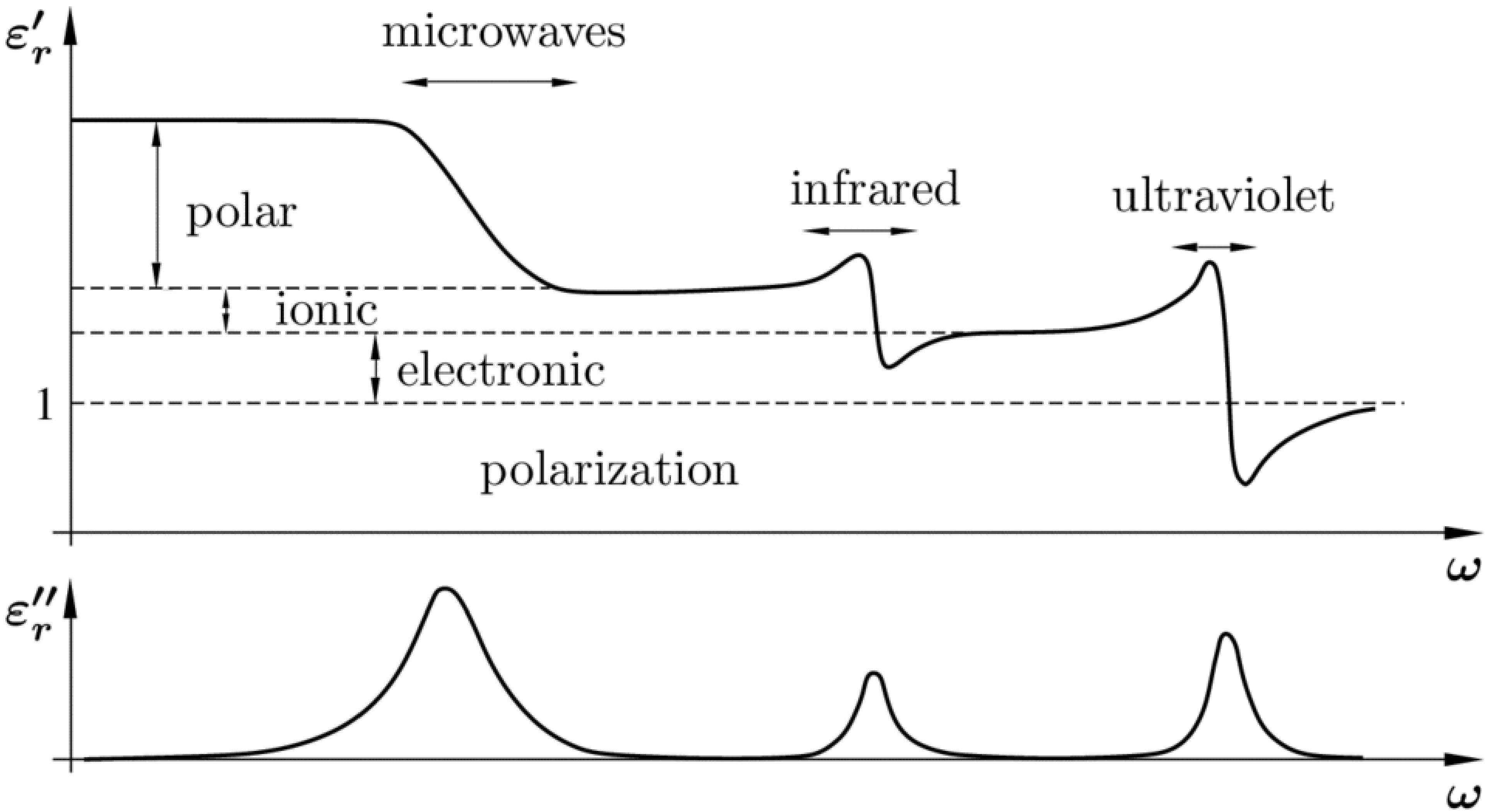
The effect of electric fields on matter can be described by a polarization P and the effect of magnetic fields by a magnetization M. Then, for linear materials

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}=\epsilon_0\vec{E}+\epsilon_0\chi_e\vec{E}=\epsilon\vec{E}$$
$$\vec{B}=\mu_0\vec{H}+\mu_0\vec{M}=\mu_0\vec{H}+\mu_0\chi_m\vec{H}=\mu\vec{H}$$

P and M result from averaging over atomic / molecular electric and magnetic dipoles induced by the fields, e.g.



$$p_e=qx \quad \rightarrow \quad P=np_e=\epsilon_0\chi_e E$$


ε_r'
microwaves
polar
infrared
ultraviolet
ionic
electronic
1
polarization
ω
ε_r''
ω

In dielectric material with losses the total current density is in the time-harmonic case

$$\vec{J}+i\omega\epsilon\vec{E}=\kappa\vec{E}+i\omega\epsilon\vec{E}=i\omega(\epsilon-i\frac{\kappa}{\omega})\vec{E}$$

$$\epsilon_c=\epsilon-i\frac{\kappa}{\omega}=\epsilon'-i\epsilon''=\epsilon'(1-i\tan(\delta_\epsilon))$$

$$\tan(\delta_\epsilon)=\frac{\epsilon''}{\epsilon'},$$ δ_ϵ electric loss angle

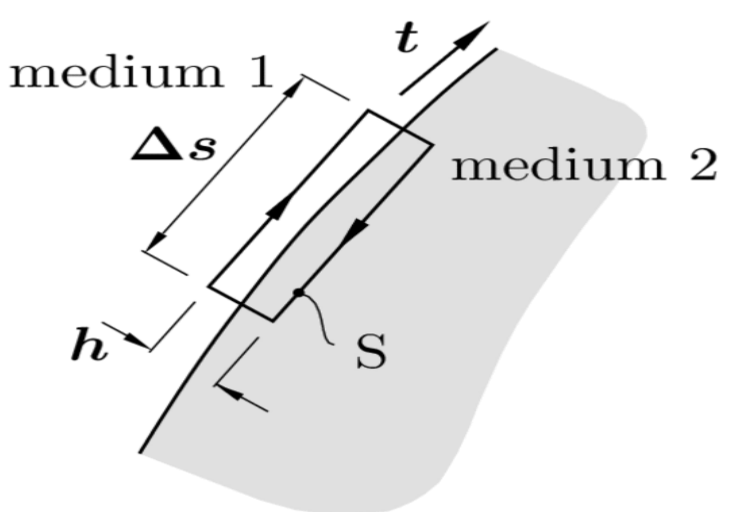
In most dielectrics is $\tan(\delta)\ll 1$.
In good conductors is $\tan(\delta)\gg 1$, $\varepsilon\approx\kappa/i\omega$.

Boundary / continuity conditions

At interfaces between different media we use the first two Maxwell equations

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

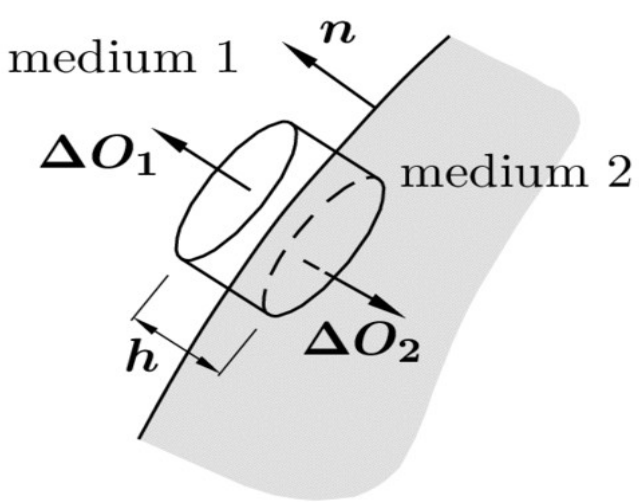
$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$



$$E_{t1} = E_{t2}, \quad H_{t1} - H_{t2} = J_A$$

$E_{t1} = 0, \quad H_{t1} = J_A$ if 2 is pec and the 3[d] and 4[th] equation

$$\oint\!\!\!\oint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\,dV$$
$$\oint\!\!\!\oint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$



$$B_{n1} = B_{n2}, \quad D_{n1} - D_{n2} = \rho_A$$
$$B_{n1} = 0, \quad D_{n1} = \rho_a$$ if 2 is pec

With Gauss' and Stokes' theorem we transform Maxwell's equations from integral in differential form

$$\oiint \vec{E}\cdot d\vec{A} = \iiint \vec{\nabla}\cdot\vec{E}\, dV\,, \quad \oint \vec{E}\cdot d\vec{s} = \iint (\vec{\nabla}\times\vec{E})\cdot d\vec{A}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (1)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad (3)$$

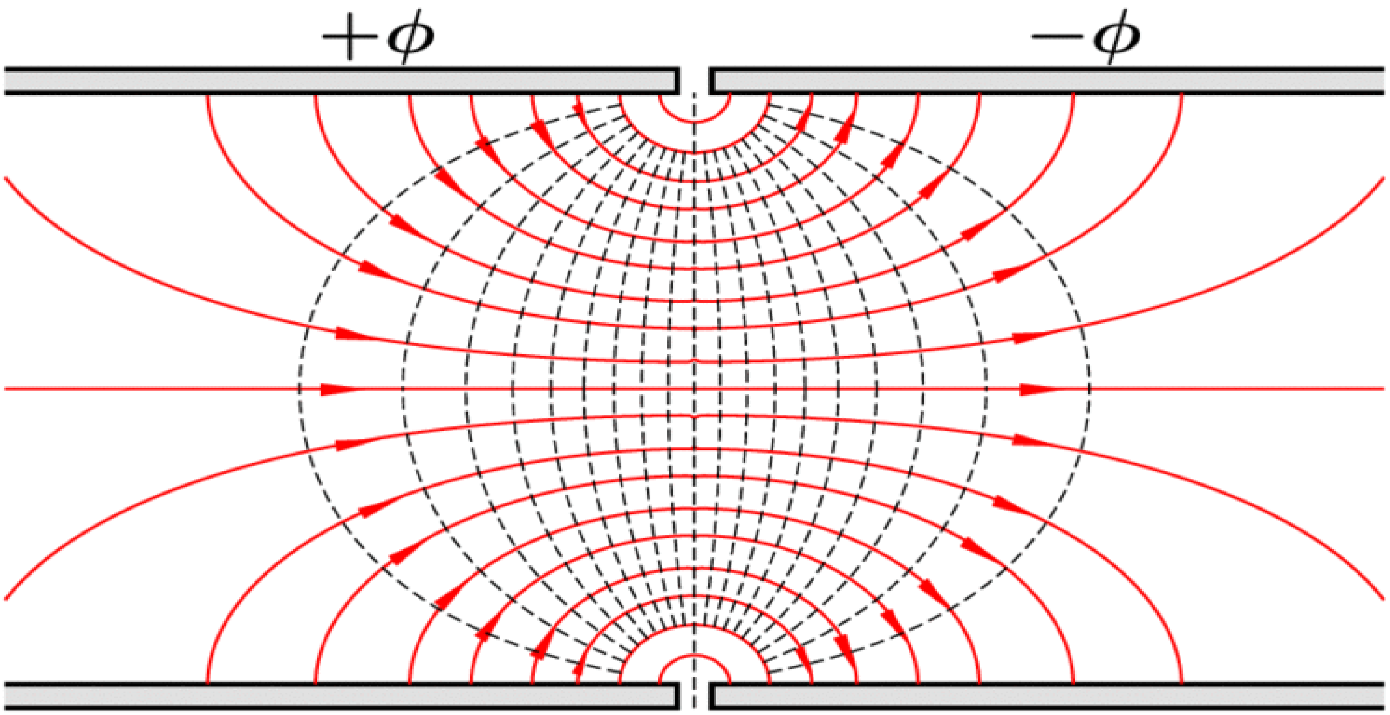
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (4)$$

Electrostatic fields

(ε=const.)

$$\vec{\nabla}\times\vec{E}=0 \quad \rightarrow \quad \vec{E}=-\vec{\nabla}\Phi$$

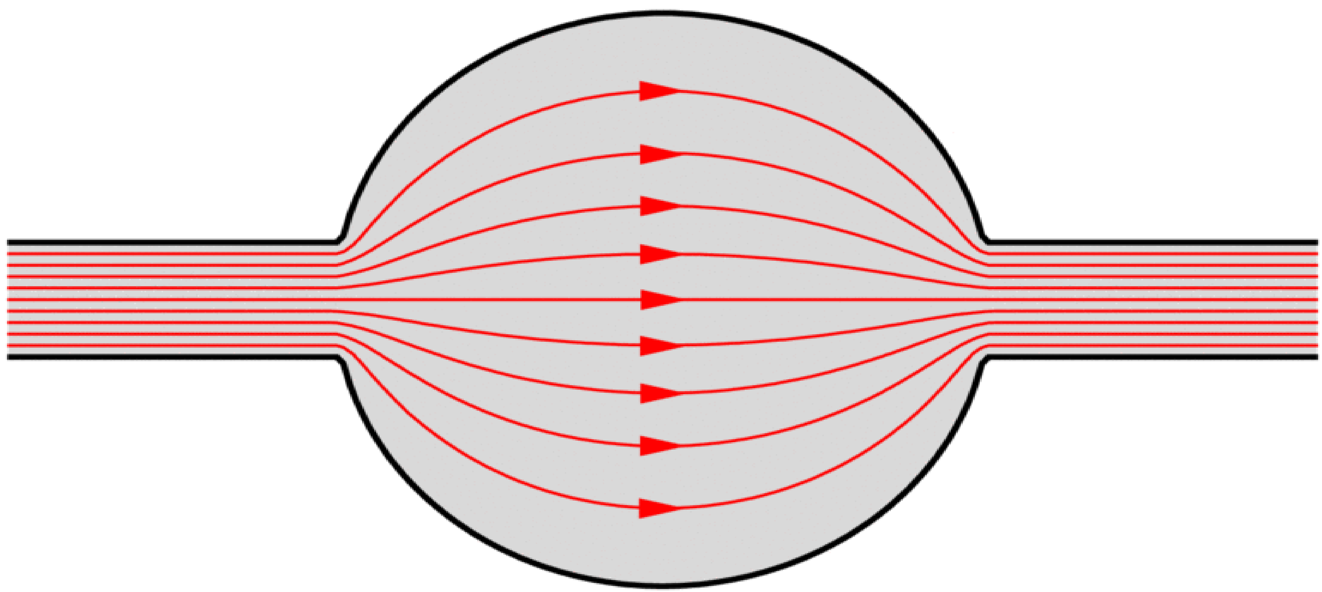
$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot(\epsilon\vec{E})=\rho \quad \rightarrow \quad \vec{\nabla}^2\Phi=-\frac{\rho}{\epsilon}$$



Stationary currents (κ=const.)

$$\vec{\nabla}\times\vec{E}=0 \quad \rightarrow \quad \vec{E}=-\vec{\nabla}\Phi$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=0=\vec{\nabla}\cdot\vec{J}=\vec{\nabla}\cdot(\kappa\vec{E}) \quad \rightarrow \quad \vec{\nabla}^2\Phi=0$$

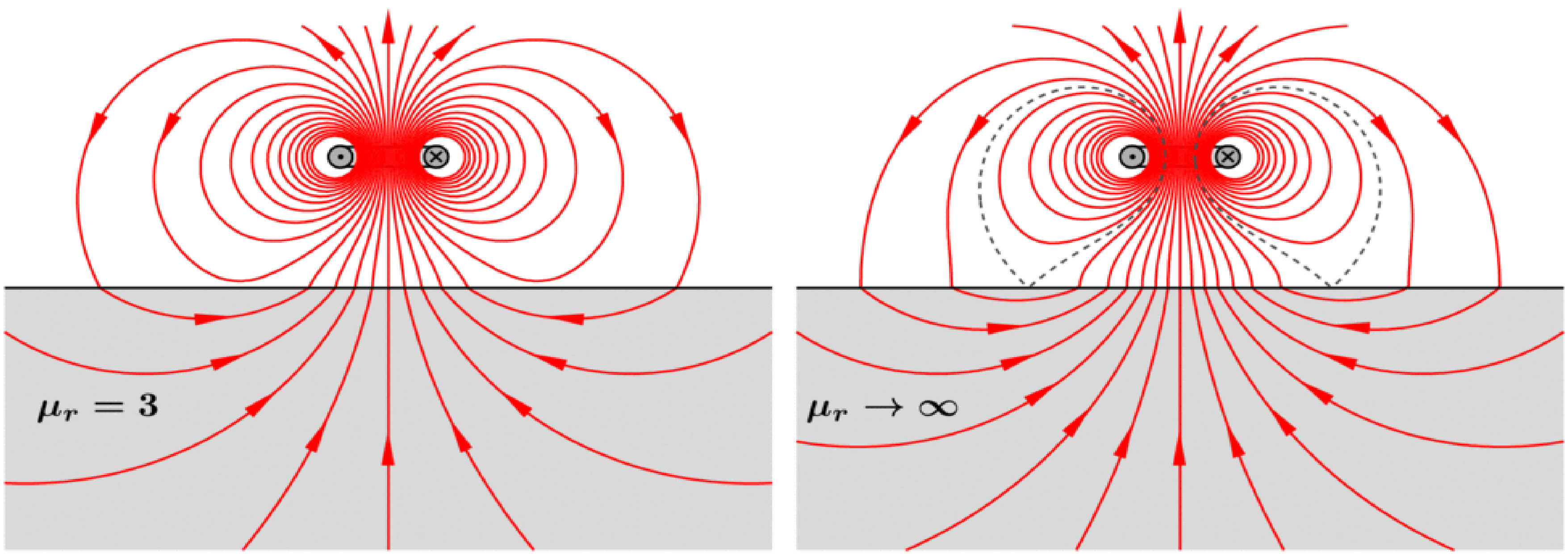


Magnetostatic fields

(μ=const.)

$$\vec{\nabla}\cdot\vec{B}=0 \quad \rightarrow \quad \vec{B}=\vec{\nabla}\times\vec{A}=\mu\,\vec{H}$$

$$\vec{\nabla}\times\vec{H}=\vec{J} \quad \rightarrow \quad \vec{\nabla}^2\vec{A}=\mu\,\vec{J}$$



Quasi-stationary fields

(ε, μ, κ=const.)

$$\vec{\nabla}\cdot\vec{B}=0 \quad \rightarrow \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \rightarrow \quad \vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$$

$$\textit{good conductors}: \qquad \rho=0$$

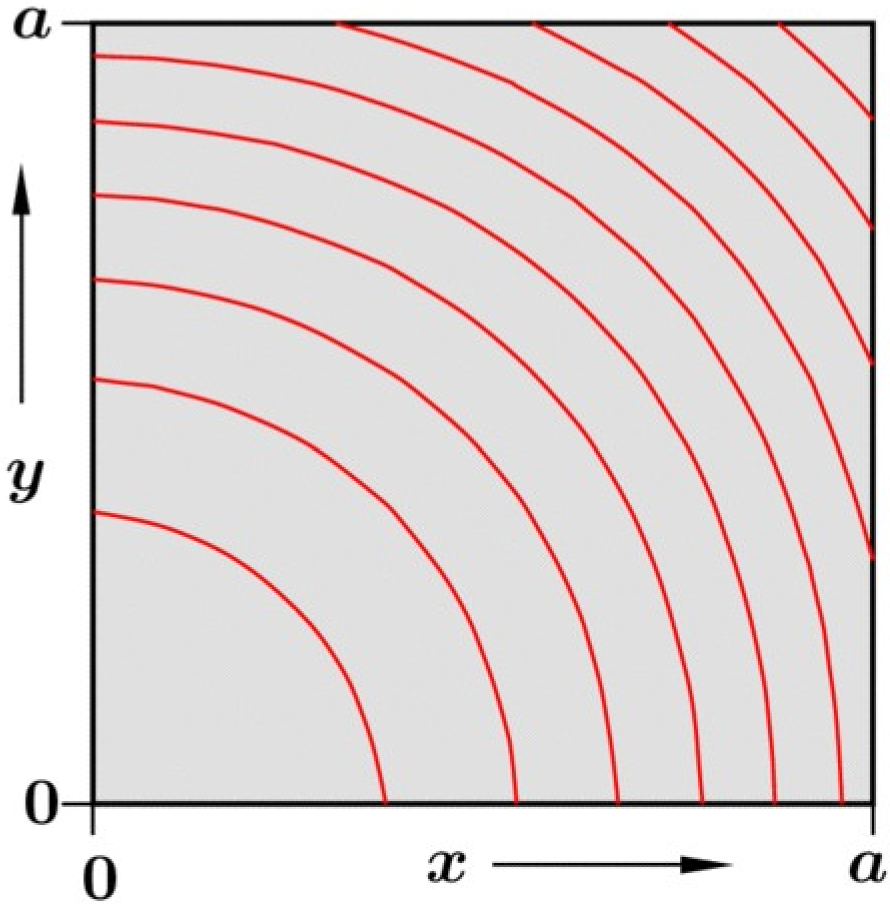
$$\textit{no impressed voltages}: \Phi=0$$

$$\vec{\nabla}\cdot\vec{D}=-\epsilon\vec{\nabla}^2\Phi-\epsilon\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=0 \quad \rightarrow \quad \vec{\nabla}\cdot\vec{A}=0$$

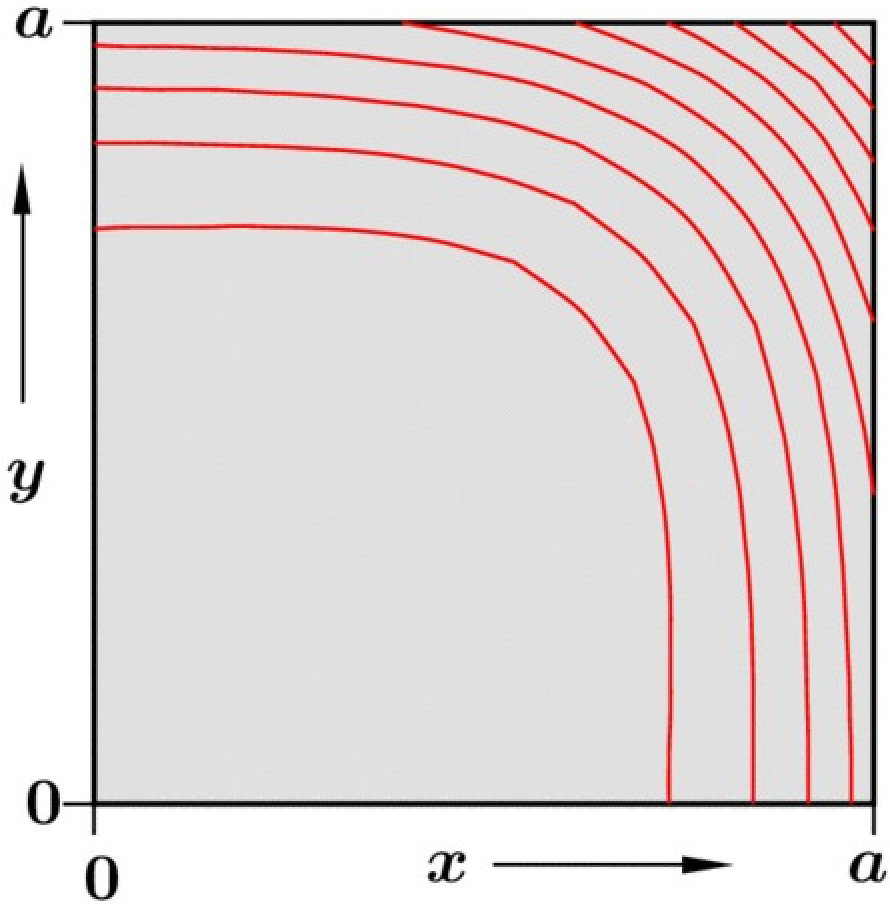
$$\vec{\nabla}\times\vec{H}=\vec{J}=\kappa\vec{E} \quad \rightarrow \quad \vec{\nabla}^2\vec{A}-\mu\kappa\frac{\partial\vec{A}}{\partial t}=0$$

Aluminum: $\kappa = 17\ 10^6\ \Omega^{-1}\text{m}^{-1}$, a=1cm

f=50 Hz



f=5 kHz



Poynting's theorem

(ε, μ, κ=const., ρ=0, J=κE, full set of Maxwell's equations)

Work done by the fields on charges ρ

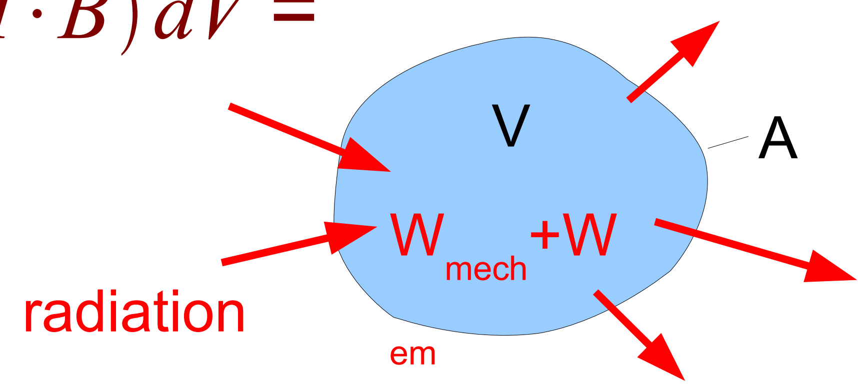
$$d\frac{\delta W}{\delta t} = d\vec{f}\cdot\frac{\delta\vec{s}}{\delta t} = \rho\frac{\delta\vec{s}}{\delta t}\cdot(\vec{E}+\vec{v}\times\vec{B})\,dV = \vec{J}\cdot\vec{E}\,dV$$

using Maxwell's equations

$$\vec{E}\cdot\vec{J} = -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \frac{\partial}{\partial t}\left[\frac{1}{2}\vec{H}\cdot\vec{B}+\frac{1}{2}\vec{E}\cdot\vec{D}\right]$$

$$\iiint \vec{E}\cdot\vec{J}\,dV + \frac{\partial}{\partial t}\iiint\left(\frac{1}{2}\vec{E}\cdot\vec{D}+\frac{1}{2}\vec{H}\cdot\vec{B}\right)dV =$$

$$= -\oiint(\vec{E}\times\vec{H})\cdot d\vec{A}$$



Poynting vector (radiation flux) $\vec{S}=\vec{E}\times\vec{H}$

Dissipated power density $p_d=\vec{E}\cdot\vec{J}$

Electric energy density $w_e=\frac{1}{2}\vec{E}\cdot\vec{D}$

Magnetic energy density $w_m=\frac{1}{2}\vec{H}\cdot\vec{B}$

Poynting's theorem for time-harmonic fields

Example:

decompose $\vec{E}=\Re[\tilde{\vec{E}}\,e^{i\omega t}]=\frac{1}{2}[\tilde{\vec{E}}\,e^{i\omega t}+\tilde{\vec{E}}^{*}\,e^{-i\omega t}]$

$$w_E = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,e^{i2\omega t} + \tilde{\vec{E}}^*\cdot\tilde{\vec{D}}^*\,e^{-i2\omega t}] + \frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}^* + \tilde{\vec{E}}^*\cdot\tilde{\vec{D}}] =$$

$$= \frac{1}{4}\Re[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,e^{i2\omega t}] + \frac{1}{4}\Re[\tilde{\vec{E}}\cdot\tilde{\vec{D}}^*]$$

Time-averaged quantities

$$\vec{S}_c = \frac{1}{2}\vec{E}\times\vec{H}^* \quad \rightarrow \quad \vec{S} = \frac{1}{2}\Re[\vec{E}\times\vec{H}^*]$$

$$\bar{p}_d = \frac{1}{2}\vec{E}\cdot\vec{J}^*$$

$$\bar{w}_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* \qquad \bar{w}_m = \frac{1}{4}\vec{H}\cdot\vec{B}^*$$

Conservation of energy flow

$$-\oint\!\!\!\oint \vec{S}_c \cdot d\vec{A} = \iiint \bar{p}_d \, dV + i2\omega \iiint (\bar{w}_m - \bar{w}_e) \, dV$$

active power (time-averaged Joulean heat)

$$\bar{P}_{act} = \bar{P}_d = \oint\!\!\!\oint \vec{S} \cdot d\vec{A} = -\oint\!\!\!\oint \Re[\vec{S}_c] \cdot d\vec{A}$$

reactive power

$$\bar{P}_{react} = -\oint\!\!\!\oint \Im[\vec{S}_c] \cdot d\vec{A} = 2\omega \iiint (\bar{w}_m - \bar{w}_e) \, dV$$

In good conductors is $W_m \gg W_e$ ($|E| \ll |H|$)

$$\oiint \vec{S}_c \cdot d\vec{A} = -\bar{P}_d - i2\omega \bar{W}_m = -\bar{P}_c$$

$$We\ define: \quad \bar{I}^* = \oint \vec{\bar{H}}^* \cdot d\vec{s}$$

$$\bar{U} = \int_1^2 \vec{\bar{E}} \cdot d\vec{s} = \bar{I}(R + i\omega L_i)$$

and obtain the resistance and internal inductance

$$\bar{P}_c = \frac{1}{2}\bar{U}\bar{I}^* = \frac{1}{2}|\bar{I}^2|(R + i\omega L_i) = \bar{P}_d + i2\omega \bar{W}_m$$

Electromagnetic waves

(ε, μ=const., ρ, J, κ=0)

The simplest electromagnetic wave is a plane wave.
It depends only on one space variable (propagation direction) and on the time variable.

$$\vec{E}=\vec{E}(z,t), \qquad \vec{H}=\vec{H}(z,t):$$

$$-\frac{\partial H_y}{\partial z}=\epsilon\frac{\partial E_x}{\partial t} \qquad \frac{\partial E_x}{\partial z}=-\mu\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_x}{\partial z}=\epsilon\frac{\partial E_y}{\partial t} \qquad -\frac{\partial E_y}{\partial z}=-\mu\frac{\partial H_x}{\partial t}$$

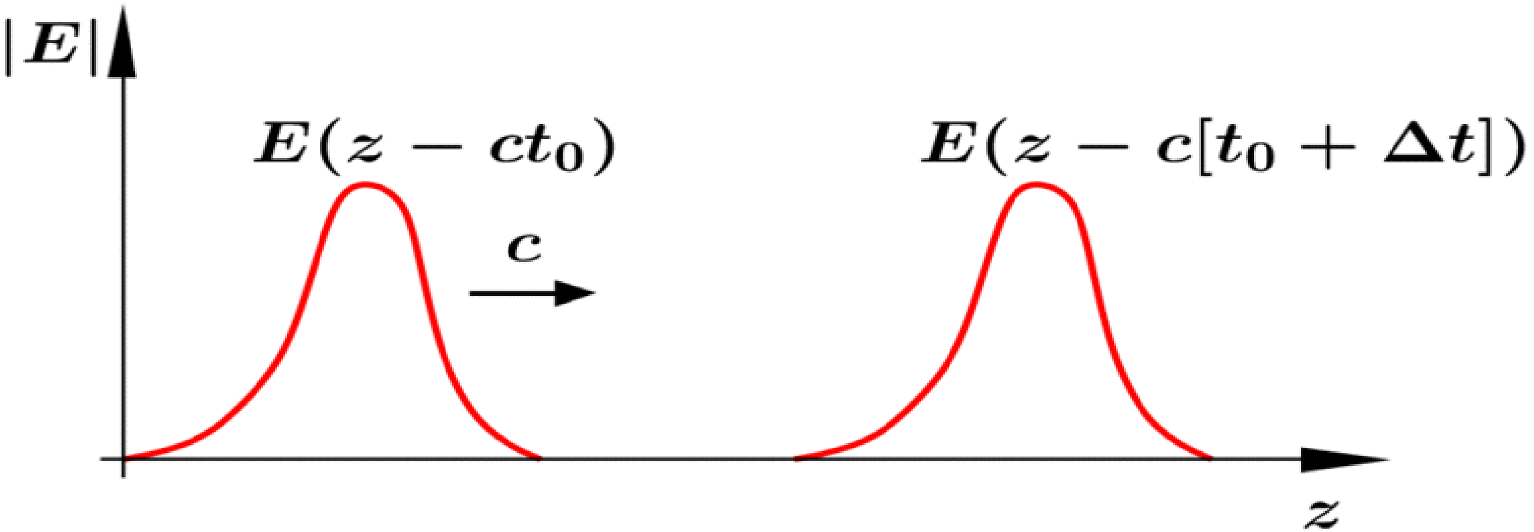
Wave equation

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu\epsilon}}$$

Alembert's solution

$$E_x = f(z-ct) + g(z+ct) \quad \rightarrow$$

$$H_y = \frac{1}{Z}[f(z-ct) - g(z+ct)], \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$



$$\textit{velocity of light}: \qquad c=\frac{1}{\sqrt{\mu\,\epsilon}}$$

$$\textit{wave impedance}: \qquad Z=\sqrt{\frac{\mu}{\epsilon}}$$

$$\approx 377\,\Omega \quad \textit{in free space}$$

$$\vec{E}\perp\vec{H}\,, \quad \vec{E}\times\vec{H}\rightarrow \textit{direction of propagation}$$

$$\vec{E}\,,\vec{H} \;\perp\; \textit{direction of propagation}$$

$$E^{+}/H^{+}=-E^{-}/H^{-}=Z$$

Time-harmonic plane wave

$$\frac{\partial^2 E_x}{\partial z^2}+k^2 E_x=0, \qquad k=\frac{\omega}{c}=\omega\sqrt{\mu\epsilon}=\frac{2\pi}{\lambda}$$

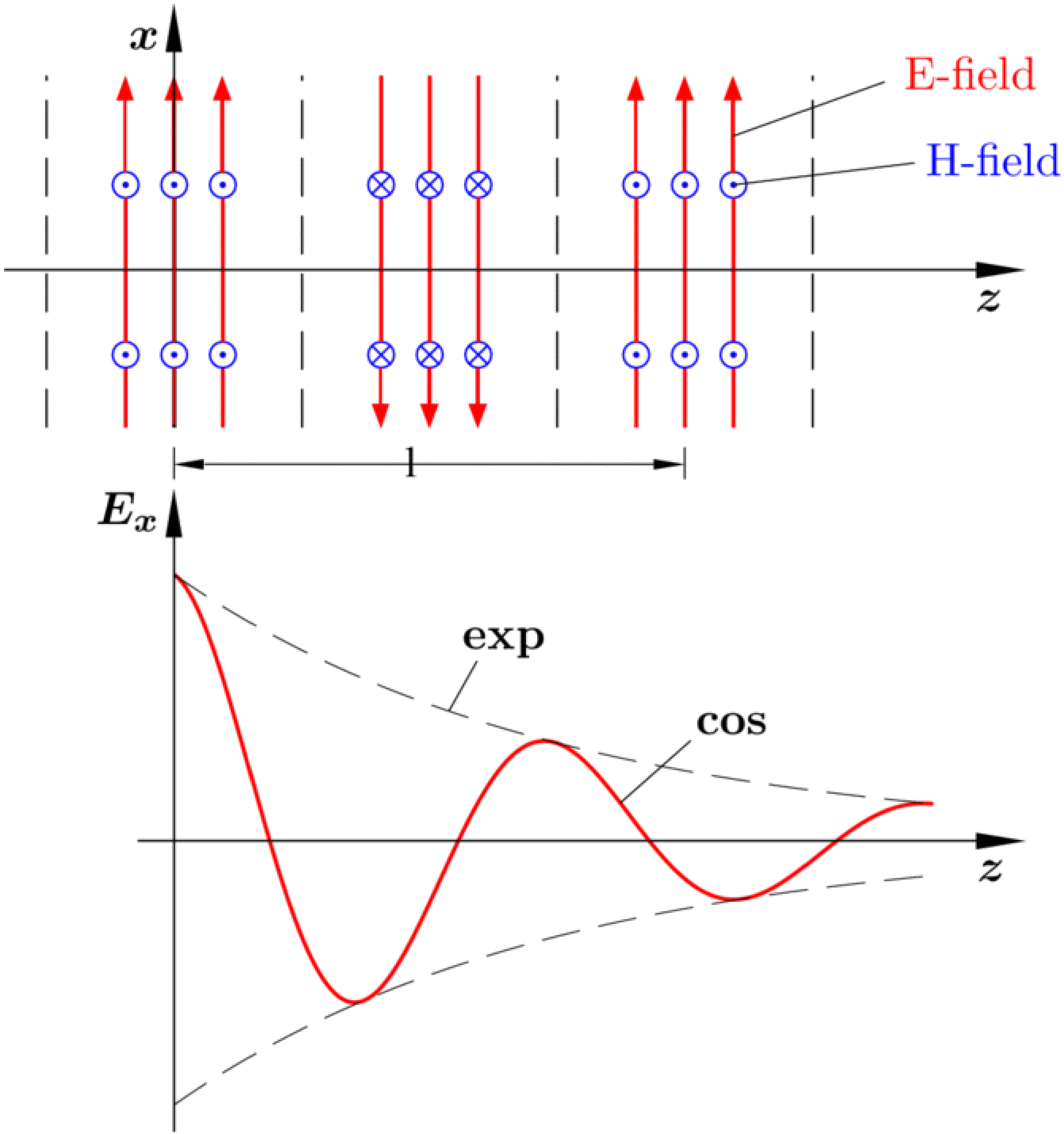
$$E_x=A\,e^{i(\omega t-kz)}+B\,e^{i(\omega t+kz)}$$

$$H_y=\frac{1}{Z}\left(A\,e^{i(\omega t-kz)}-B\,e^{i(\omega t+kz)}\right)$$

$$lossy\ material: \quad \epsilon_c=\epsilon_r\epsilon_0\left(1-i\frac{\kappa}{\omega\epsilon_r\epsilon_0}\right)=\epsilon_0(\epsilon_r{}'-i\epsilon_r{}'')$$

$$k=\omega\sqrt{\mu\epsilon_c}=\beta-i\alpha$$

$$\frac{\beta}{k_0}=\pm\sqrt{\frac{\epsilon_{r'}}{2}+\frac{\epsilon_{r'}}{2}\sqrt{1+\left(\frac{\epsilon_{r''}}{\epsilon_{r'}}\right)^2}}, \quad \frac{\alpha}{k_0}=\pm\sqrt{-\frac{\epsilon_{r'}}{2}+\frac{\epsilon_{r'}}{2}\sqrt{1+\left(\frac{\epsilon_{r''}}{\epsilon_{r'}}\right)^2}}$$


x
E-field
H-field
z
l
E_x
exp
cos
z

Phase velocity

$$\phi = \omega t - \beta z = const. \quad \rightarrow \quad \omega - \beta \frac{dz}{dt} = \omega - \beta v_{ph} = 0$$

$$v_{ph} = \frac{d\omega}{d\beta}$$

Group velocity

(wave packet with $\Delta\omega$)

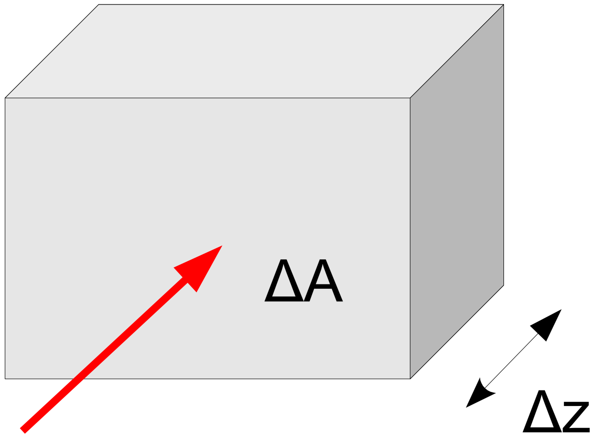
$$\omega_1 = \omega_0 + \delta\omega, \qquad \omega_2 = \omega_0 - \delta\omega$$

$$\beta_1 = \beta_0 + \delta\beta, \qquad \beta_2 = \beta_0 - \delta\beta$$

$$\Re\left[e^{i(\omega_1 t - \beta_1 z)} + e^{i(\omega_2 t - \beta_2 z)}\right] = 2\cos(\omega_0 t - \beta_0 z)\cos(\delta\omega t - \delta\beta z)$$

$$v_g = \frac{\delta\omega}{\delta\beta} \quad \rightarrow \quad v_g = \frac{d\omega}{d\beta}$$

Energy velocity



$$\frac{\bar{w}\,\Delta A\,\Delta z}{\Delta t}=\bar{S}_z\,\Delta A \quad \rightarrow \quad v_e=\frac{\Delta z}{\Delta t}=\frac{\bar{S}_z}{\bar{w}}$$

for plane waves

$$\bar{S}_z=\frac{1}{2}(\vec{E}\times\vec{H}^*)_z=\frac{|E_0{}^2|}{2Z}, \qquad \bar{w}=\frac{1}{4}\vec{E}\cdot\vec{D}^*+\frac{1}{4}\vec{H}\cdot\vec{B}^*=\frac{1}{2}\epsilon|E_0{}^2|$$

$$v_e=\frac{1}{Z\epsilon}=\frac{1}{\sqrt{\mu\epsilon}}=c$$

Low-loss dielectrics: ε''«ε'

$$\beta \approx \sqrt{\epsilon_r{}'}\,k_0, \qquad \alpha \approx \frac{1}{2}\frac{\epsilon_r{}''}{\sqrt{\epsilon_r{}'}}k_0, \qquad Z \approx \frac{Z_0}{\sqrt{\epsilon_r{}'}}\left(1+\frac{i}{2}\frac{\epsilon_r{}''}{\epsilon_r{}'}\right)$$

Example: Polyamide (nylon), $\kappa=10^{-8}\,\Omega^{-1}\text{m}^{-1}$, $\varepsilon_r=3$, f=10MHz

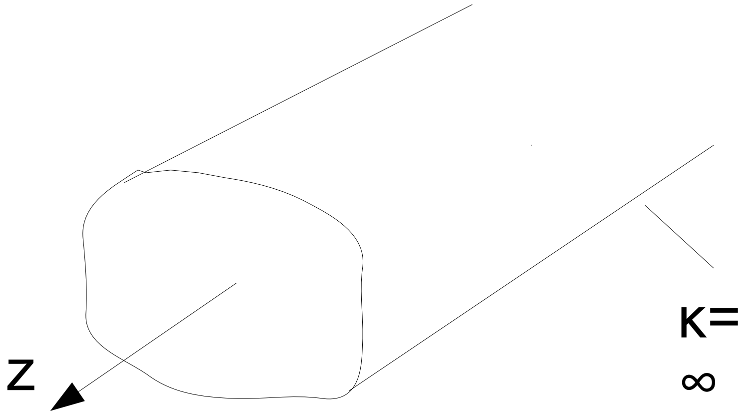
11% attenuation in 100km, arc $Z\approx10^{-4}\,°$

Very good conductors (metallic): ε''≈-iκ/ω » ε'

$$\beta \approx \alpha \approx \sqrt{\frac{\omega\mu_0\kappa}{2}}, \qquad Z \approx (1+i)\frac{\alpha}{\kappa}, \qquad arc\,Z = 45\,°$$

Skin depth: $$\delta = \sqrt{\frac{2}{\omega\mu\kappa}}$$

Cylindrical, ideal conducting waveguides



$$\vec{\nabla}\cdot\vec{E}=0 \quad \rightarrow \quad \vec{E}^{TE}=\vec{\nabla}\times\vec{A}^{TE}$$

$$\vec{\nabla}\cdot\vec{H}=0 \quad \rightarrow \quad \vec{H}^{TM}=\vec{\nabla}\times\vec{A}^{TM}$$

$$substituting\; e.g.\quad \vec{E}=\vec{\nabla}\times\vec{A}\quad into\quad \vec{\nabla}\times\vec{H}=\epsilon\frac{\partial\vec{E}}{\partial t}$$

$$yields\qquad \vec{H}=\vec{\nabla}\Phi+\epsilon\frac{\partial\vec{A}}{\partial t}$$

Next using $\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$ *gives*

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=-\mu\vec{\nabla}\Phi-\mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

Because $\vec{A},\Phi$ *are not fully determined, use Lorenz' gauge*

$$\vec{\nabla}\cdot\vec{A}=-\mu\Phi$$

yielding a vectorial wave equation

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$$

Similarily, we proceed for the TM−case and obtain the same equation.

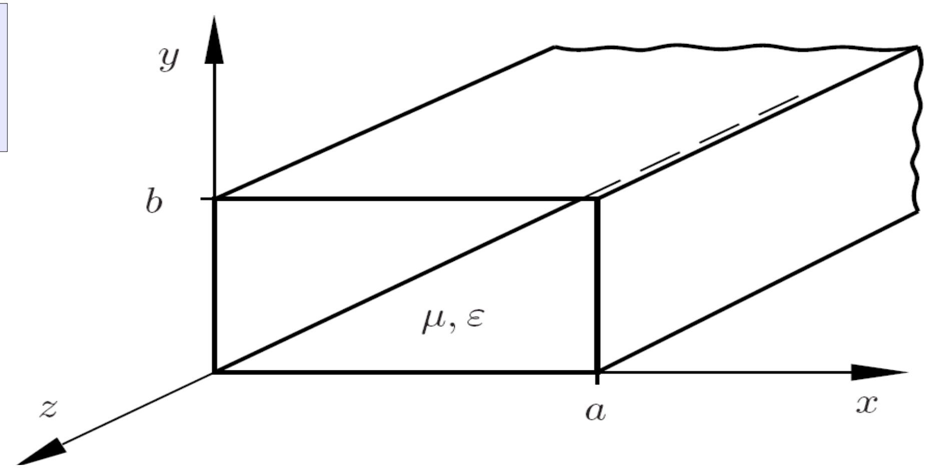
Since only two independent functions are needed, we choose

$$\vec{A}^{TE} = A^{TE}\,\vec{e}_z, \qquad \vec{A}^{TM} = A^{TM}\,\vec{e}_z$$

which for time-harmonic fields results in a scalar Helmholtz equation

$$\vec{\nabla}^2 A^p + k^2 A^p = 0, \quad k = \frac{\omega}{c} = \omega\sqrt{\mu\,\epsilon}, \qquad p = \begin{Bmatrix} TE \\ TM \end{Bmatrix}$$

Rectangular waveguide



$$\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2}+k^2 A=0$$

Bernoulli ansatz: $A(x,y,z)=X(x)Y(y)Z(z)$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{-k_x^2}+\underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{-k_y^2}+\underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{-k_z^2}+k^2=0$$

Dispersion relation: $k^2=k_x^2+k_y^2+k_z^2$

e.g. for X(x)

$$\frac{d^2X}{dx^2}+k_x^2X=0 \quad \rightarrow \quad X=A\cos(k_x x)+B\sin(k_x x)$$

$$=\begin{Bmatrix}\cos(k_x x)\\ \sin(k_x x)\end{Bmatrix} \; or \; \begin{Bmatrix}e^{ik_x x}\\ e^{-ik_x x}\end{Bmatrix}$$

for $Y(y), Z(z)$ *correspondingly.*

General solution

$$A(x,y,z)=\begin{Bmatrix}\cos(k_x x)\\ \sin(k_x x)\end{Bmatrix}\begin{Bmatrix}\cos(k_y y)\\ \sin(k_y y)\end{Bmatrix}\begin{Bmatrix}e^{ik_z z}\\ e^{-ik_z z}\end{Bmatrix}e^{i\omega t}$$

$$TE-waves: \quad \vec{E}=\vec{\nabla}\times A\,\vec{e}_z$$

$$E_x = \frac{\partial A}{\partial y} = k_y \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} -\sin(k_y y) \\ +\cos(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$E_y = -\frac{\partial A}{\partial x} = k_x \begin{Bmatrix} +\sin(k_x x) \\ -\cos(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$H_x = -\frac{1}{i\omega\mu}\frac{\partial^2 A}{\partial x \partial z} = \frac{\mp k_x k_z}{\omega\mu} \begin{Bmatrix} -\sin(k_x x) \\ +\cos(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$H_y = -\frac{1}{i\omega\mu}\frac{\partial^2 A}{\partial y \partial z} = \frac{\mp k_y k_z}{\omega\mu} \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} -\sin(k_y y) \\ +\cos(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$H_z = -\frac{k_x^2+k_y^2}{i\omega\mu} A = -\frac{k_x^2+k_y^2}{i\omega\mu} \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

Boundary conditions:

$$E_x(y=0,b)=0, \qquad E_y(x=0,a)=0$$

$$\sin(k_x a)=0 \quad \rightarrow \quad k_{xm}=m\pi/a, \quad m=(0),1,2,3,....$$

$$\sin(k_y b)=0 \quad \rightarrow \quad k_{yn}=n\pi/b, \quad n=(0),1,2,3,....$$

Fields

$$E_x=-k_{yn} C_{mn} \cos(k_{xm} x) \sin(k_{yn} y) \mathrm{e}^{\pm i k_z z}$$

$$E_y=+k_{xm} C_{mn} \sin(k_{xm} x) \cos(k_x x) \mathrm{e}^{\pm i k_z z}, \qquad E_z=0$$

$$H_x=\pm\frac{k_{xm} k_z}{\omega\mu} C_{mn} \sin(k_{xm} x) \cos(k_{yn} y) \mathrm{e}^{\pm i k_z z}$$

$$H_y=\pm\frac{k_y k_z}{\omega\mu} C_{mn} \cos(k_{xm} x) \sin(k_{yn} y) \mathrm{e}^{\pm i k_z z}$$

$$H_z=-\frac{k_{xm}^2+k_{yn}^2}{i\omega\mu} C_{mn} \cos(k_x x) \cos(k_y y) \mathrm{e}^{\pm i k_z z}$$

$$TM-waves: \quad \vec{H}=\vec{\nabla}\times A\vec{e}_z$$

$$H_x = \frac{\partial A}{\partial y} = k_y \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} -\sin(k_y y) \\ +\cos(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$H_y = -\frac{\partial A}{\partial x} = k_x \begin{Bmatrix} +\sin(k_x x) \\ -\cos(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$E_x = \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial x \partial z} = \frac{\pm k_x k_z}{\omega\epsilon} \begin{Bmatrix} -\sin(k_x x) \\ +\cos(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$E_y = \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial y \partial z} = \frac{\pm k_y k_z}{\omega\epsilon} \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} -\sin(k_y y) \\ +\cos(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

$$E_z = \frac{k_x^2+k_y^2}{i\omega\epsilon} A = \frac{k_x^2+k_y^2}{i\omega\epsilon} \begin{Bmatrix} \cos(k_x x) \\ \sin(k_x x) \end{Bmatrix} \begin{Bmatrix} \cos(k_y y) \\ \sin(k_y y) \end{Bmatrix} e^{\pm i k_z z}$$

Boundary conditions:

$$E_z(x=0,a\,;\,y=0,b)=0$$

$$\sin(k_x a)=0 \quad \rightarrow \quad k_{xm}=m\pi/a\,, \quad m=1,2,3\,,....$$

$$\sin(k_y b)=0 \quad \rightarrow \quad k_{yn}=n\pi/b\,, \quad n=1,2,3\,,....$$

Fields

$$H_x=k_{yn}D_{mn}\sin(k_{xm}x)\cos(k_{yn}y)\mathrm{e}^{\pm ik_z z}$$

$$H_y=-k_{xm}D_{mn}\cos(k_{xm}x)\sin(k_x x)\mathrm{e}^{\pm ik_z z}\,, \qquad H_z=0$$

$$E_x=\pm\frac{k_{xm}k_z}{\omega\epsilon}D_{mn}\cos(k_{xm}x)\sin(k_{yn}y)\mathrm{e}^{\pm ik_z z}$$

$$E_y=\pm\frac{k_{yn}k_z}{\omega\epsilon}D_{mn}\sin(k_{xm}x)\cos(k_{yn}y)\mathrm{e}^{\pm ik_z z}$$

$$E_z=\frac{k_{xm}^2+k_{yn}^2}{i\omega\epsilon}D_{mn}\sin(k_x x)\sin(k_y y)\mathrm{e}^{\pm ik_z z}$$

Wave impedance

$$Z_F = \begin{cases} Z_F^{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{k_z} \\ Z_F^{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{k_z}{\omega\epsilon} \end{cases}$$

Dispersion relation

$$k^2 = k_{xm}^2 + k_{yn}^2 + k_{zmn}^2$$

$$k_{zmn} = \sqrt{k^2 - (k_{xm}^2 + k_{yn}^2)} = \sqrt{k^2 - k_{cmn}^2}$$

$$k_{zmn} = \begin{cases} \beta_{mn}, \quad real & for\ k > k_{cmn} \\ 0 & for\ k = k_{cmn} \\ -i\alpha_{mn}, \quad imaginary & for\ k < k_{cmn} \end{cases}$$

critical wavenumber: $k_{cmn} = \sqrt{k_{xm}^2 + k_{yn}^2}$

cutoff frequency: $f_{cmn} = c\,k_{cmn}/2\pi$

cutoff wavelength: $\lambda_{cmn} = 2\pi/k_{cmn}$

guide wavelength: $\lambda_{zmn} = 2\pi/k_{zmn} = \dfrac{\lambda}{\sqrt{1-(\lambda/\lambda_{cmn})^2}}$

energy flux density:

$$\bar{S}_{cz} = \frac{1}{2}(\vec{E}\times\vec{H}^*)_z = \frac{1}{2} Z_{Fmn}\left[|H_{xmn}|^2 + |H_{ymn}|^2\right]$$

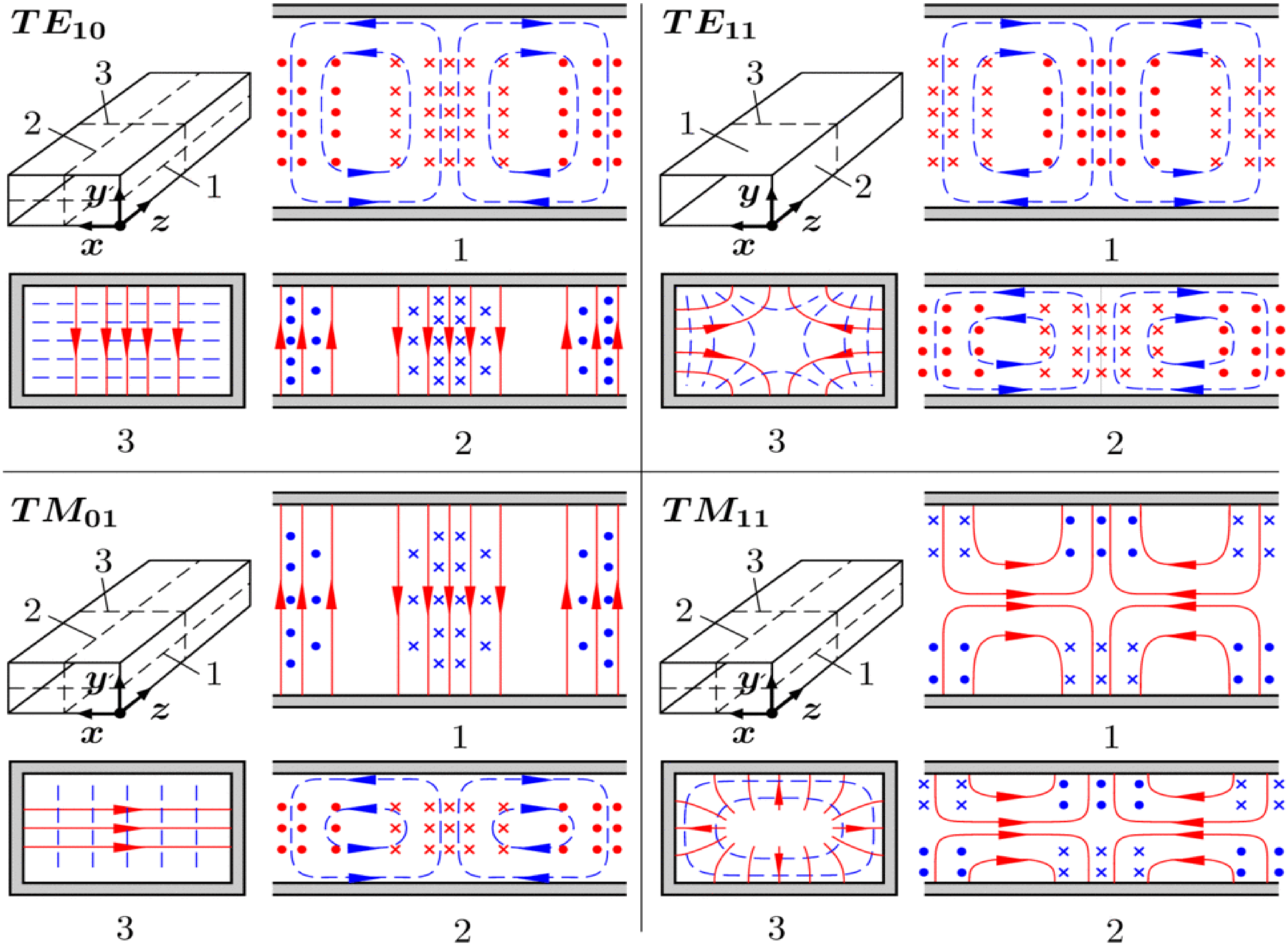
$$= \begin{cases} imaginary & \\ 0 & for \\ real & \end{cases} \begin{matrix} k < k_c \\ k = k_c \\ k > k_c \end{matrix}$$

Each mn defines a certain (eigen-) mode. The general solution is the linear combination of all modes

$$\vec{E} = \sum \left(\vec{E}_{mn}^{TE} + \vec{E}_{mn}^{TM} \right), \qquad \vec{H} = \sum \left(\vec{H}_{mn}^{TE} + \vec{H}_{mn}^{TM} \right)$$

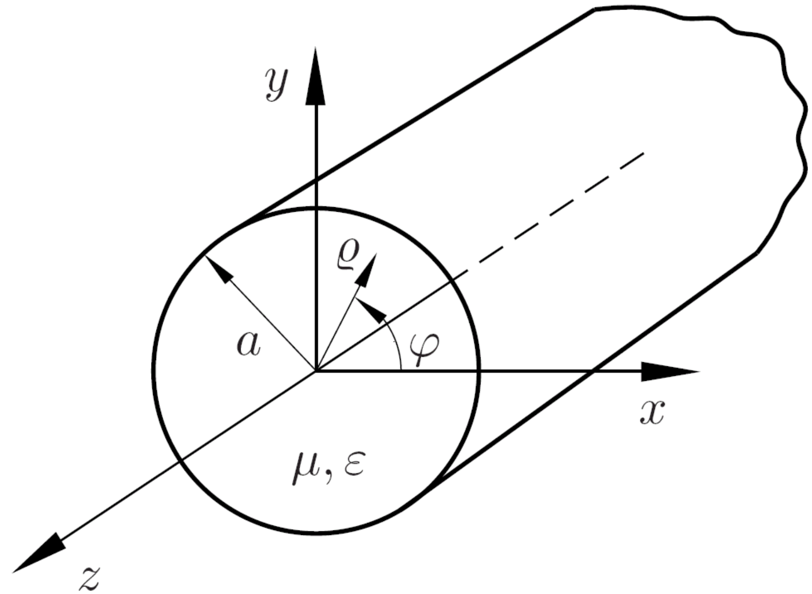
Modes are normally sorted referring to their cutoff frequency.
Example: S-band 2.68 – 3.95 GHz, a=7.214cm, b=3.404cm

type	m	n	f_c / GHz
TE	1	0	2.08
TE	2	0	4.16
TE	0	1	4.41
TE, TM	1	1	4.87
TE, TM	2	1	6.06



TE10
TE11
TM01
TM11
1
2
3
x
y
z

Circular waveguide



Helmholtz equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 A}{\partial\varphi^2}+\frac{\partial^2 A}{\partial z^2}+k^2 A=0$$

Bernoulli ansatz:

$$A=R(\rho)\Phi(\varphi)Z(z)$$

$$A=\begin{Bmatrix}\cos(\mu\varphi)\\ \sin(\mu\varphi)\end{Bmatrix}\begin{Bmatrix}J_\mu(K\rho)\\ N_\mu(K\rho)\end{Bmatrix}\mathrm{e}^{-ik_z z}, \qquad K=\sqrt{k^2-k_z^2}$$

1. Because of rotational symmetry we choose the coordinate system such that only cos-function is needed.

2. Because of the 2π-periodicity it is μ=m.

3. Neumann fuction is infinite at ρ=0, so only Bessel function is needed.

$$A = C\cos(m\varphi) J_m(K\rho) e^{-ik_z z}$$

$$TE-waves: \quad \vec{E} = \vec{\nabla} \times A \vec{e}_z$$

$$E_\varphi = -\frac{\partial A}{\partial r} \sim J_m'(K\rho)$$

$$E_\varphi(\rho = a) = 0 \quad \rightarrow \quad K_{mn} a = j'_{mn}$$

$$E_\rho = -\frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$E_\varphi = -\frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}, \qquad E_z = 0$$

$$H_\rho = \frac{k_z}{\omega\mu} \frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$H_\varphi = -\frac{k_z}{\omega\mu} \frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$H_z = -\frac{j'^2_{mn}/a^2}{i\omega\mu} C_{mn} \cos(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$TM-waves: \quad \vec{H}=\vec{\nabla}\times A\vec{e}_z$$

$$E_z=\frac{K^2}{i\omega\epsilon}A\sim J_m(K\rho), \qquad E_z(\rho=a)=0 \;\rightarrow\; K_{mn}a=j_{mn}$$

$$H_\rho=-\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m(j_{mn}\frac{\rho}{a})\mathrm{e}^{-ik_z z}$$

$$H_\varphi=-\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m(j_{mn}\frac{\rho}{a})\mathrm{e}^{-ik_z z}, \qquad H_z=0$$

$$E_\rho=-\frac{k_z}{\omega\epsilon}\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m(j_{mn}\frac{\rho}{a})\mathrm{e}^{-ik_z z}$$

$$E_\varphi=\frac{k_z}{\omega\epsilon}\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m(j_{mn}\frac{\rho}{a})\mathrm{e}^{-ik_z z}$$

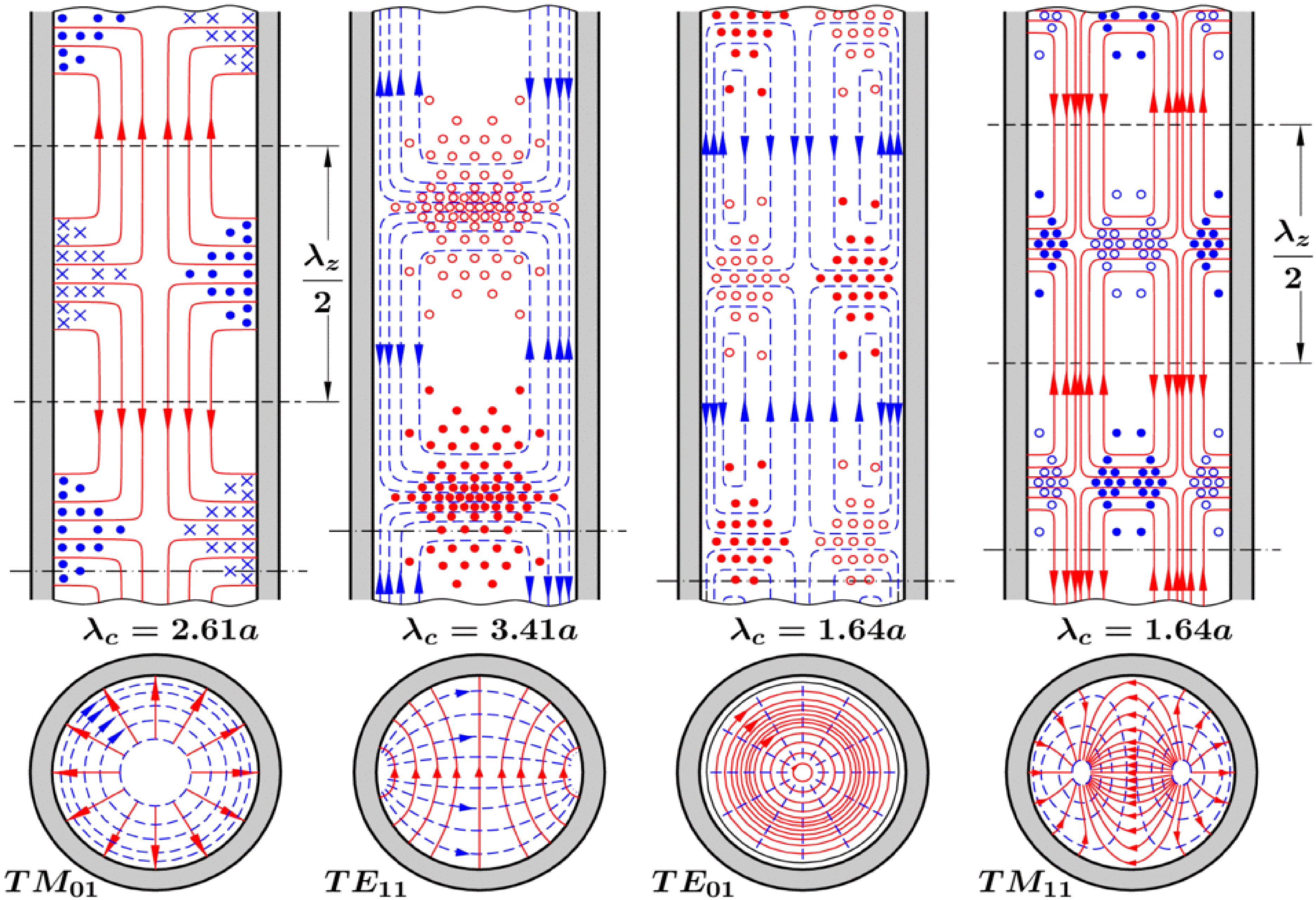
$$E_z=\frac{j_{mn}^2/a^2}{i\omega\epsilon}D_{mn}\cos(m\varphi)J_m(j_{mn}\frac{\rho}{a})\mathrm{e}^{-ik_z z}$$

$$TE-waves: \quad K_{mn}a = j'_{mn}$$

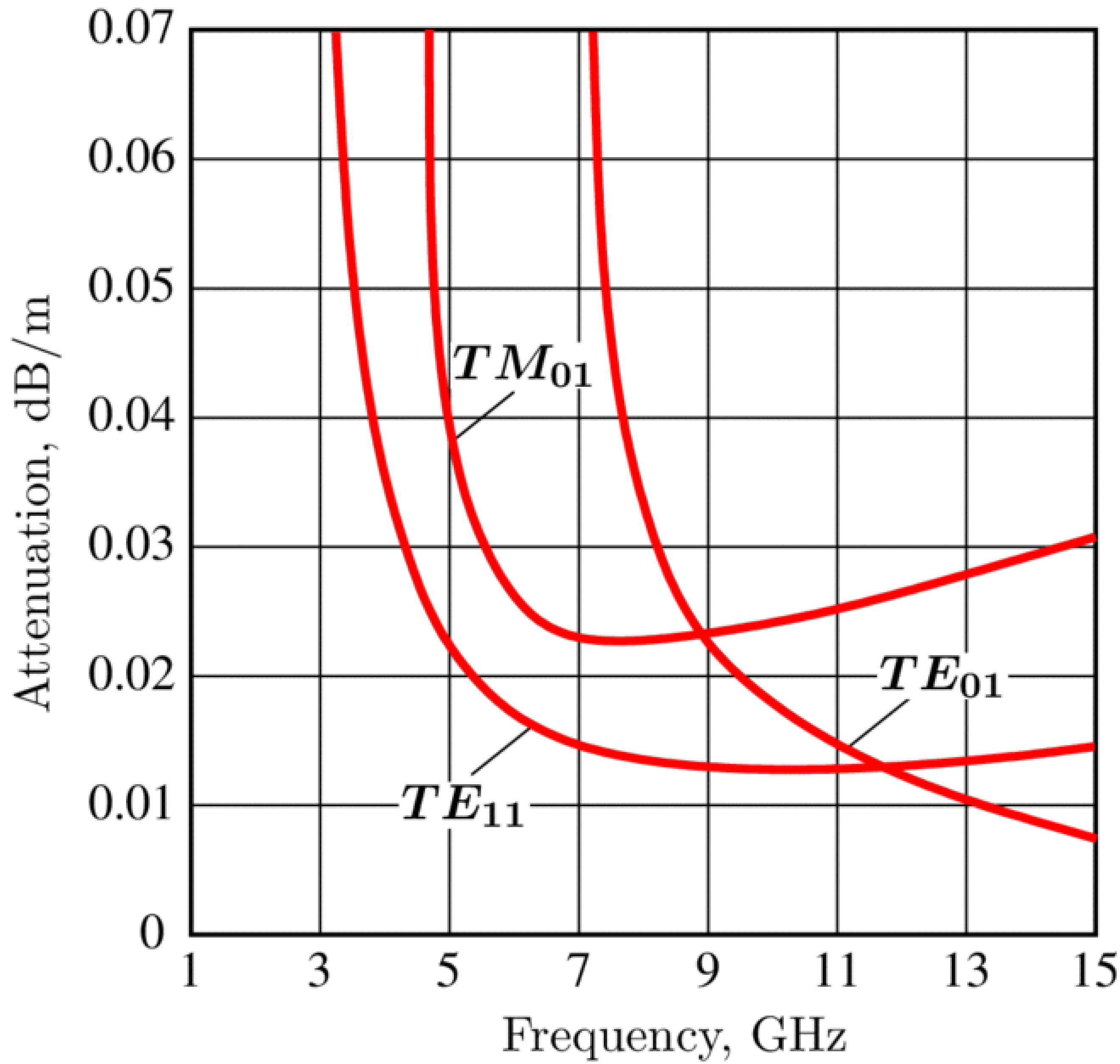
$$TM-waves: \quad K_{mn}a = j_{mn}$$

$$K=\sqrt{k^2-k_z^2} \quad \rightarrow \quad k_{cmn}=\frac{2\pi}{c}f_{cmn}=K_{mn}=\begin{Bmatrix} j'_{mn}/a \\ j_{mn}/a \end{Bmatrix}$$

type	m	n	f_c / GHz
TE	1	1	1.76
TM	0	1	2.30
TE	2	1	2.92
TE/TM	0/1	2/1	3.66
TM	3	1	4.01



λ_z
2
$\lambda_c = 2.61a$
$\lambda_c = 3.41a$
$\lambda_c = 1.64a$
$\lambda_c = 1.64a$
TM_{01}
TE_{11}
TE_{01}
TM_{11}


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
1
3
5
7
9
11
13
15
Attenuation, dB/m
Frequency, GHz
TM_{01}
TE_{01}
TE_{11}

Losses in very good conductors

Fields on metallic surfaces: E≈perp, H≈parallel

$$\vec{E}=\vec{E}_t+E_z\vec{e}_z, \quad \vec{H}=\vec{H}_t+H_z\vec{e}_z, \quad \vec{\nabla}=\vec{\nabla}_t+\vec{e}_z\frac{\partial}{\partial z}$$

$$\vec{\nabla}\times\vec{H}=\kappa\vec{E}: \qquad \vec{E}_t=-\frac{1}{\kappa}\vec{e}_z\times\vec{\nabla}_t H_z+\frac{1}{\kappa}\vec{e}_z\times\frac{\partial\vec{H}_t}{\partial z}$$

$$E_z\vec{e}_z=\frac{1}{\kappa}\vec{\nabla}_t\times\vec{H}_t$$

$$\vec{\nabla}\times\vec{E}=-i\omega\mu_0\vec{H}: \quad \vec{H}_t=-\frac{i}{\omega\mu_0}\vec{e}_z\times\vec{\nabla}_t E_z+\frac{i}{\omega\mu_0}\vec{e}_z\times\frac{\partial\vec{E}_t}{\partial z}$$

$$H_z\vec{e}_z=\frac{i}{\omega\mu_0}\vec{\nabla}_t\times\vec{E}_t$$

order of magnitude approximation : $\left|\vec{\nabla}_t\right| \sim \frac{1}{\lambda_0}$

$$\left|E_z\right| \sim \frac{1}{\kappa\lambda_0}\left|\vec{H}_t\right| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0\left|\vec{H}_t\right|$$

$$Z_0\left|H_z\right| \sim \frac{1}{\omega\mu_0}\frac{Z_0}{\lambda_0}\left|\vec{E}_t\right| = \frac{1}{2\pi}\left|\vec{E}_t\right|$$

$$\left|\frac{1}{\kappa}\vec{e}_z \times \vec{\nabla}_t H_z\right| \sim \frac{1}{\kappa\lambda_0}\left|H_z\right| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0\left|H_z\right| \sim \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2\left|\vec{E}_t\right|$$

$$\left|\frac{i}{\omega\mu_0}\vec{e}_z \times \vec{\nabla}_t E_z\right| \sim \frac{1}{\omega\mu_0\lambda_0}\left|E_z\right| = \frac{1}{2\pi Z_0}\left|E_z\right| \sim \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2\left|\vec{H}_t\right|$$

$$\kappa \vec{E}_t \approx \frac{\partial}{\partial z}(\vec{e}_z \times \vec{H}_t), \qquad i\omega\mu_0 \vec{H}_t \approx -\vec{e}_z \times \frac{\partial \vec{E}_t}{\partial z}$$

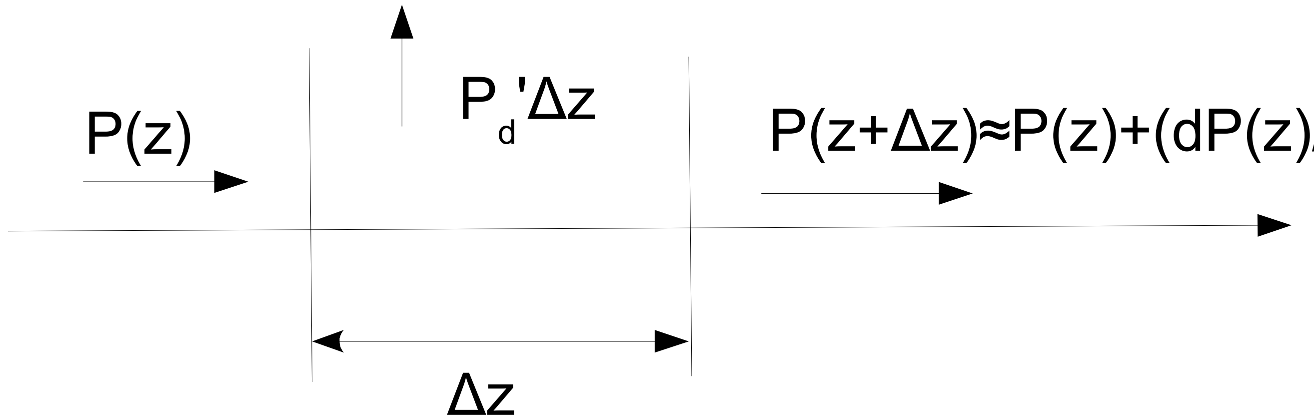
$$\kappa \frac{\partial \vec{E}_t}{\partial z} \approx \frac{\partial^2}{\partial z^2}(\vec{e}_z \times \vec{H}_t) \approx i\omega\mu_0\kappa(\vec{e}_z \times \vec{H}_t)$$

$$\frac{\partial^2 \vec{H}_t}{\partial z^2} - i\omega\mu_0\kappa \vec{H}_t = 0 \quad \rightarrow \quad \vec{H}_t = \vec{H}_{t0} \mathrm{e}^{-(1+i)z/\delta_s}$$

impedance boundary condition:

$$\vec{E}_{t0} \approx Z_W(\vec{n} \times \vec{H}_{t0}), \qquad Z_W = \frac{1+i}{\kappa\delta_s}$$

Attenuation in waveguides (power-loss method)



$$conservation\ of\ power: \quad \frac{dP(z)}{dz} = -2\alpha P(z) = -P_d'$$

$$dissipation\ per\ surface\ area:$$

$$\frac{\Delta P_d}{\Delta F} = -\vec{n}\cdot\Re(\vec{S}_c) = -\frac{1}{2}\Re(\vec{n}\cdot(\vec{E}\times\vec{H}^*)) = \frac{1}{2}\Re(Z_W)|\vec{H}_{tan0}|$$

$$= \frac{1}{2\kappa\delta_s}|\vec{H}_{tan0}|^2$$

dissipation per length:

$$P_d' = \frac{1}{2\kappa\delta} \oint \left|\vec{H}_{tan0}\right|^2 ds$$

transported power:

$$P(z) = \iint \Re(\vec{S}_c) \cdot d\vec{F} = \frac{1}{2} \Re(Z_F) \iint \left|\vec{H}_t\right|^2 dF$$

$$= \frac{1}{2} Z_F \iint \left|\vec{H}_t\right|^2 dF$$

attenuation: $\alpha = \frac{1}{2} \frac{P_d'}{P(z)}$

Resonant cavities

Rectangular cavity (TM-modes)

forward and backward traveling wave in the waveguide

$$E_x = \frac{k_{xm} k_z}{\omega \epsilon} D_{mn} \cos(k_{xm} x) \sin(k_{yn} y) [\mathrm{e}^{i k_z z} - r_{mn} \mathrm{e}^{-i k_z z}]$$

boundary conditions

$$E_x(z=0)=0 \quad \rightarrow \quad r_{mn}=1, \qquad E_x \sim \sin(k_z z)$$

$$E_x(z=l)=0 \quad \rightarrow \quad k_{zp} l = p\pi, \quad p=0,1,2,\ldots$$

$$H_x = 2k_{yn} D_{mn} \sin(k_{xm} x)\cos(k_{yn} y)\cos(k_{zp} z)$$

$$H_y = -2k_{xm} D_{mn} \cos(k_{xm} x)\sin(k_{yn} x)\cos(k_{zp} z), \qquad H_z = 0$$

$$E_x = i2 \frac{k_{xm} k_{zp}}{\omega \epsilon} D_{mn} \cos(k_{xm} x)\sin(k_{yn} y)\sin(k_{zp} z)$$

$$E_y = i2 \frac{k_{yn} k_{zp}}{\omega \epsilon} D_{mn} \sin(k_{xm} x)\cos(k_{yn} y)\sin(k_{zp} z)$$

$$E_z = 2 \frac{k_{cmn}^2}{i \omega \epsilon} D_{mn} \sin(k_{xm} x)\sin(k_{yn} y)\cos(k_{zp} z)$$

$$k_{xm} = m\frac{\pi}{a}, \quad k_{yn} = n\frac{\pi}{b}, \quad k_{zp} = p\frac{\pi}{l}, \quad k_{cmn} = \sqrt{\left(m\frac{\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)}$$

Example: TM_{110}-resonator (m=1, n=1, p=0)

$$H_x = 2\frac{\pi}{b} D_{mn} \sin(\pi\frac{x}{a})\cos(\frac{y}{b})$$

$$H_y = -2\frac{\pi}{a} D_{mn} \cos(\pi\frac{x}{a})\sin(\pi\frac{y}{b}), \quad H_z = 0$$

$$E_z = 2\frac{k_{c11}^2}{i\omega\epsilon} D_{mn} \sin(\pi\frac{x}{a})\sin(\pi\frac{y}{b}), \quad E_x = E_y = 0$$

resonance frequency $\quad k_{c11} = \frac{\omega_0}{c} = \sqrt{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2}$

stored energy

$$\bar{W} = \bar{W}_e + \bar{W}_m = 2\bar{W}_e = \frac{1}{2}\iiint \vec{E}\cdot\vec{D}^* dV = \frac{\epsilon}{2}\iiint |E_z|^2 dV$$

$$\bar{W}=\frac{\epsilon}{2}\left(\frac{2k_{c11}^2}{\omega\epsilon}\right)^2|D_{11}|^2\frac{a}{2}\frac{b}{2}l$$

dissipation per unit area

$$\bar{P}_d''=\frac{1}{2}\Re(Z_W)|J_S|^2=\frac{1}{2\kappa\delta_s}|\vec{H}_{\tan}|^2$$

$$\bar{P}_d=\frac{ab|D_{11}|^2}{\kappa\delta_s}\left[\left(1+2\frac{l}{a}\right)\left(\frac{\pi}{a}\right)^2+\left(1+2\frac{l}{b}\right)\left(\frac{\pi}{b}\right)^2\right]$$

Quality factor (Q-value)

$$Q_0=\frac{\omega_0\bar{W}}{\bar{P}_d}=\frac{1}{\delta_s}\frac{l(a^2+b^2)}{\left(1+2\frac{l}{a}\right)b^2+\left(1+2\frac{l}{b}\right)a^2}$$

For a=b:

$$\delta_s Q_0 = \frac{l}{1+2l/a} = 2\frac{V}{S} \quad \sim \frac{V}{S}$$

Q_0 gives the decay rate of the stored energy

$$-\frac{d\bar{W}}{dt} = \bar{P}_d = \frac{\omega_0}{Q_0}\bar{W} \quad \rightarrow \quad \bar{W} = \bar{W}_0 \mathrm{e}^{-(\omega_0/Q_0)t}$$

Resonance behaviour of a cavity mode

Instead of lossy walls assume lossy dielectric filling. That preserves the ideal mode but allows for studying losses.

The cavity is driven by a current J passing through. J splits into a conduction current J_c=κE, responsible for the losses in the dielectric, and in an enforced current J_0 as driving term.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\vec{\nabla}^2\vec{E}=-\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H}=$$

$$=-\mu\frac{\partial}{\partial t}(\vec{J}_0+\kappa\vec{E}+\epsilon\frac{\partial\vec{E}}{\partial t})$$

$$\vec{\nabla}^2\vec{E}-\mu\kappa\frac{\partial\vec{E}}{\partial t}-\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}=\mu\frac{\partial\vec{J}_0}{\partial t} \qquad (1)$$

We expand E in modes

$$\vec{E} = \sum_n a_n(t)\,\vec{e}_n(x,y,z) \qquad (2)$$

$$\text{where} \quad \vec{\nabla}^2 \vec{e}_n + k_n^2 \vec{e}_n = 0$$

$$\vec{\nabla} \cdot \vec{e}_n = 0 \ \text{ in volume,} \qquad \vec{n} \times \vec{e}_n = 0 \ \text{ on walls}$$

$$\iiint \vec{e}_n \cdot \vec{e}_m \, dV = \delta_m^n$$

Substituting (2) in (1)

$$\sum_n \left[\frac{\partial^2 a_n}{\partial t^2} + \frac{\kappa}{\epsilon} \frac{\partial a_n}{\partial t} + \frac{k_n^2}{\mu \epsilon} a_n \right] \vec{e}_n = -\frac{1}{\epsilon} \frac{\partial \vec{J}_0}{\partial t} \qquad (3)$$

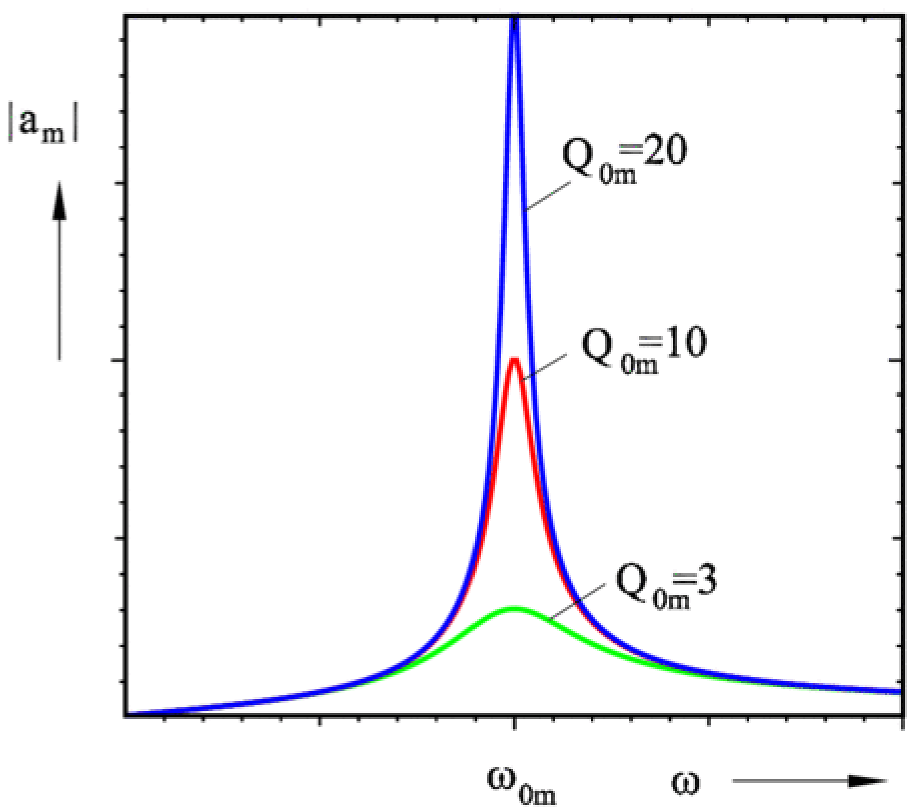
Multiplying (3) with e_m and integrating over V

$$\frac{\partial^2 a_m}{\partial t^2} + \frac{\kappa}{\epsilon}\frac{\partial a_m}{\partial t} + \frac{k_n^2}{\mu\epsilon} a_m = -\frac{1}{\epsilon}\frac{\iiint \partial \vec{J}_0}{\partial t} \cdot \vec{e}_m dV = \frac{\partial f_m}{\partial t}$$

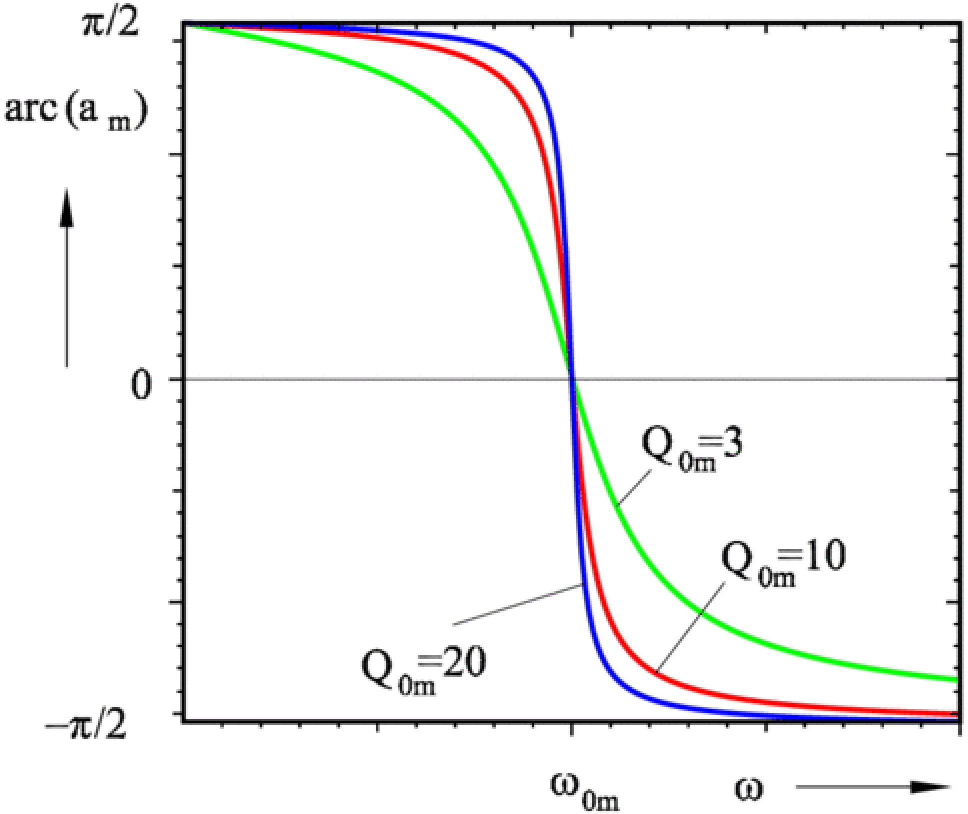
$$[-\omega^2 + i\frac{\kappa}{\epsilon}\omega + \frac{k_m^2}{\mu\epsilon}] a_m = i\omega f_m$$

$$a_m = \frac{Q_{0m}}{\omega_{0m}} \frac{f_m}{1 + iQ_{0m}[\frac{\omega}{\omega_{0m}} - \frac{\omega_{0m}}{\omega}]}$$

with $\omega_{0m} = c k_m, \quad Q_{0m} = \epsilon\,\omega_{0m}/\kappa$

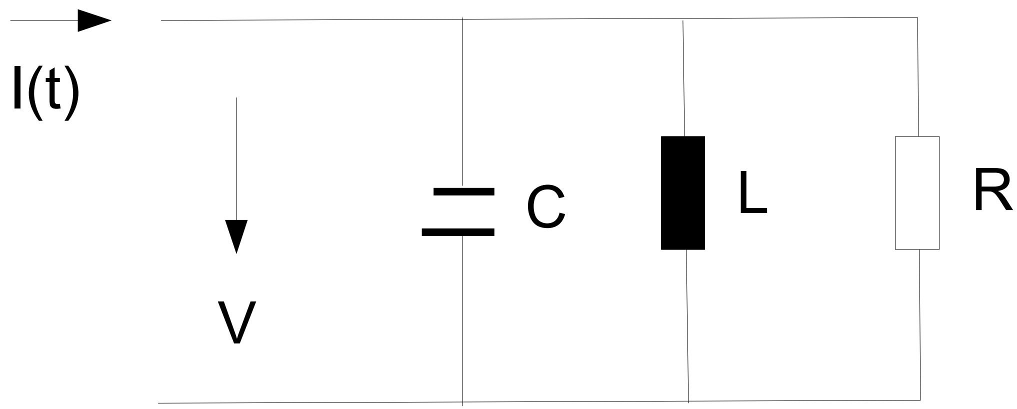


|a_m|
Q_0m=20
Q_0m=10
Q_0m=3
ω_0m
ω



π/2
arc (a_m)
0
−π/2
Q_0m=3
Q_0m=10
Q_0m=20
ω_0m
ω

Well seperated modes can be represented by a resonator



$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad Q_0 = \frac{\omega_0 W}{P_d} = \omega_0 RC$$

Bandwidth $$B = \frac{(\omega_0 + \delta\omega) - (\omega_0 - \delta\omega)}{\omega_0} = 2\frac{\delta\omega}{\omega_0}$$

Filling time $$T_f = 2\frac{Q_0}{\omega_0}$$

Accelerating voltage for a particle passing the cavity on-axis

$$V_m = \left| \int_0^g a_m \vec{E}_m \cdot \vec{e}_z \, \mathrm{e}^{i\omega t} \, dz \right|, \qquad z = vt$$

Shunt impedance (amplitude independent)

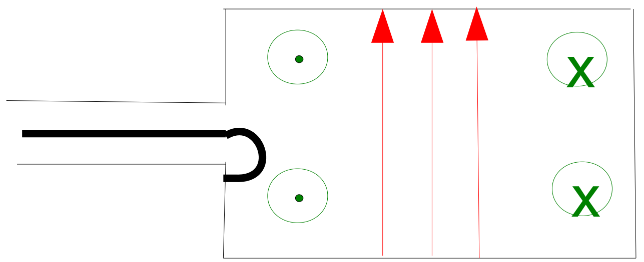
$$R_{shm} = \frac{V_m^2}{P_{dm}} = 2\mathrm{R}_m$$

R-upon-Q (accelerating voltage for a given stored energy)

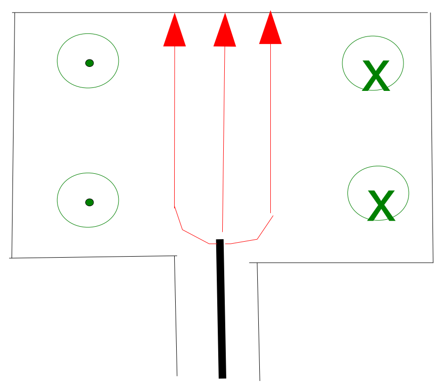
$$\frac{R_{shm}}{Q_{0\mathrm{m}}} = \frac{V_m^2}{\omega_{0\mathrm{m}} W_m} = \frac{2}{\omega_{0\mathrm{m}} C_m}$$

Coupling to a cavity

Loop / magnetic coupling



Probe / electric coupling



Electromagnetic coupling