Joint
Universities
Accelerator
School

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SYNCHROTRON RADIATION

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LECTURE NOTES

Contents

1 Introduction to Electromagnetic Radiation

1.1 Units and Dimensions

In the following only **MKSA** units will be used. In this system the dimensions of the important physical quantities are

physical quantity	symbol	dimension
length	l	meter [m]
mass	m	kilogram [kg]
time	t	second [s]
current	I	Ampere [A]
velocity of light	c	$2.997925 \cdot 10^{8}$ m/s
charge	q	1 C = 1 A s
charge of an electron	e	$1.60203 \cdot 10^{-19}$ C
dielectric constant	ε_0	$8.85419 \cdot 10^{-12}$ As/Vm
permeability	μ_0	$4\pi \cdot 10^{-7}$ Vs/Am
voltage	V	1 volt [V]
electric field	E	V / m
magnetic field	B	1 tesla [T]

1.2 Rotating electric dipole

Before we start with the quantitative discussion of electromagnetic radiation, some simple examples may make something clear of the general physics behind. At first we will look at a static electrical dipole as shown in fig. 1.1. An observer notices a longer distance apart a field with downward direction. When the dipole is turned upside down within a very short time and turned back immediately after, only in the vicinity of the dipole the field follows the motion nearly without delay. At that time the observer don't notice any change of the electric field. Because of the limited velocity of the information (i.e. the velocity of light) it takes a certain time until this happens.

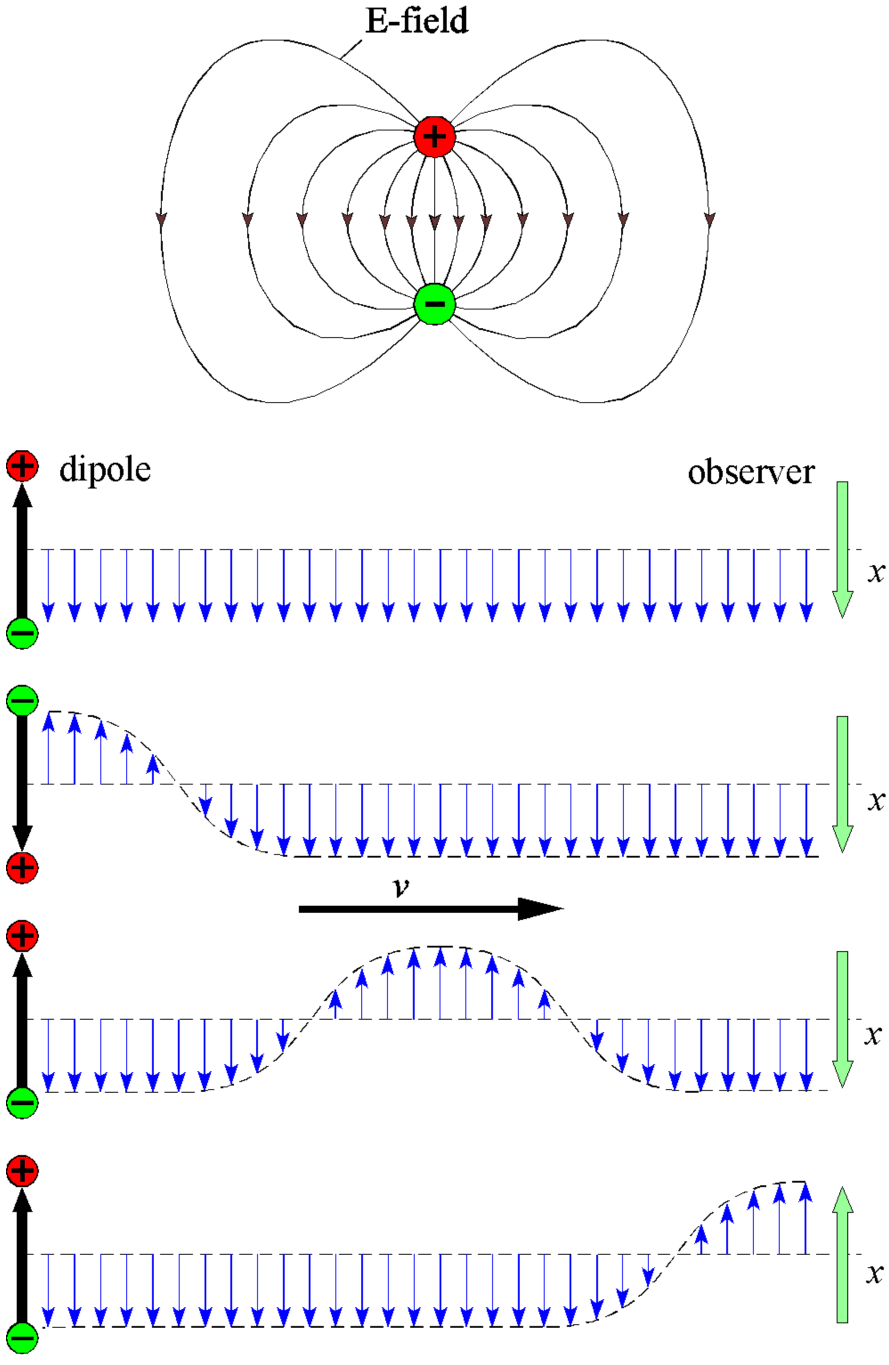


Fig. 1.1 Generation of electromagnetic waves by rotating a static dipole

One can see in fig. 1.1 that the delay (or "*retardation*") of the field spreading immediately leads to a wave of the electric field. According to "*Maxwell's equations*" this time dependent electric field generates also a corresponding magnetic field and we end up with an electromagetic wave.

1.3 Rotating magnetic dipole

In the first picture of fig. 1.2 the simplified field pattern of a magnetic dipole is sketched. When the magnet starts rotating around the axis perpendicular to the dipole axis the field distribution at a given time changes because of the limited velocity of the field spread. Fig. 1.2 shows three pattern with different rotation frequencies between 200 Hz and 10 kHz.

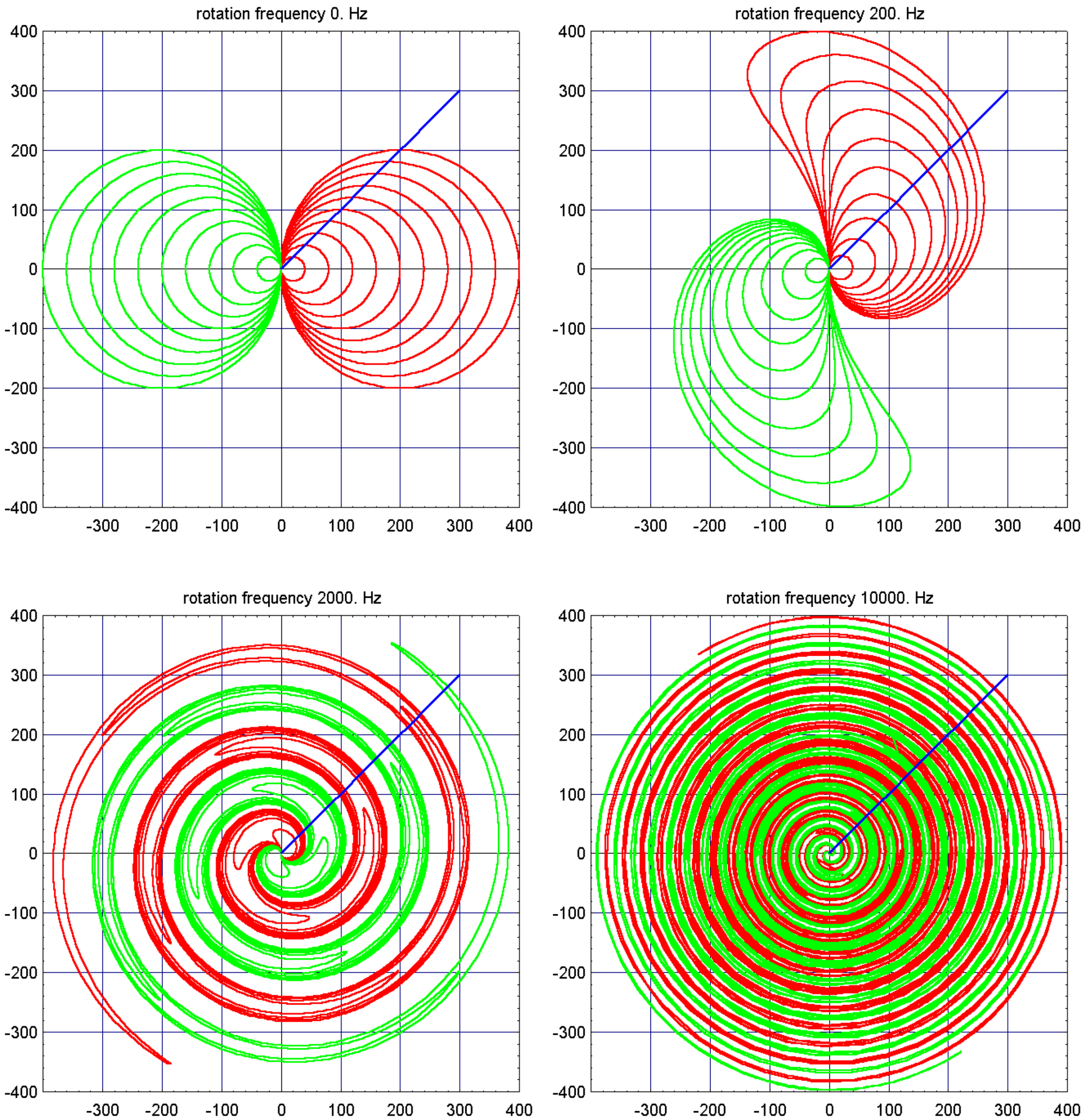


Fig. 1.2 Generation of spherical waves with a rotating magnetic dipole. The field is observed in an area of ±400 km. The rotating frequency varies between 0 Hz and 10 kHz.

At higher frequencies, one can directly see the generation of spherical waves traveling from the center to the outside. The information of the field strength produced by the dipole takes some time to reach the observation point far away from the origin. During this time the dipole position and the spatial field distribution in its vicinity has changed. Again the *retardation* of the time dependent field leads to electromagnetic radiation.

1.4 Relativistic charged particle traveling through a bending magnet

The last example is the radiation emitted by a charged particle moving with a velocity close to the velocity of light. Because of the relativistic contraction of length the field around such particles has not a spherical distribution as in the rest case but is contracted in the direction of motion. The electrical field is like a disk and its axis is identical with the particle trajectory as shown in fig. 1.3. In a bending magnet the particle trajectory follows a cycle. Consequently, the field pattern is rotated around the axis perpendicular to the plane of motion. Outside the cycle this rotation would require a field velocity larger than the velocity of light, which is according to elementary laws of relativity impossible. Therefore, the field is delayed (or "retarded") and finally it tears off the particle. Each particle produces a very short field pulse emitted into the forward direction. The corresponding frequency spectrum is very broad and covers the range between the visible light and X-rays.

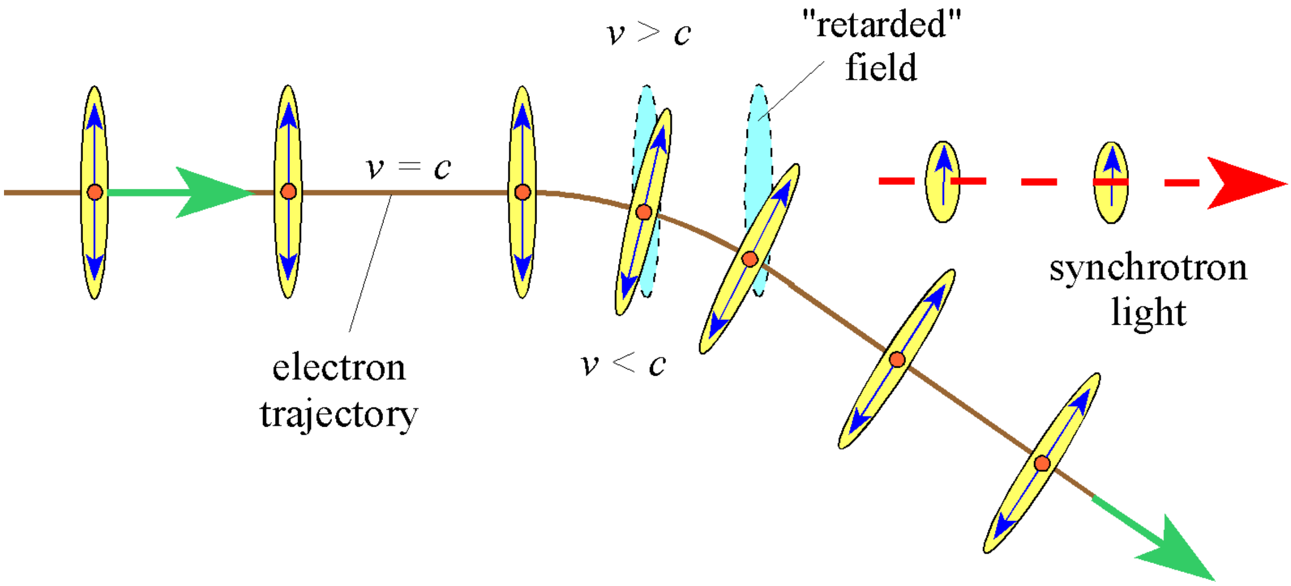


Fig. 1.3 Relativistic particle (electron) traveling through a field of a bending magnet.

It is easy to understand that this type of radiation is not be generated by slow moving nonrelativistic particles. In this case the field is almost spherical and the delay is negligible. This radiation occurs only at extremely relativistic velocities which are achievable with reasonable effort only with electrons. At the end of the forties this type of radiation has been observed the first time at the 70 MeV electron synchrotron built by General Electric. Therefore, this radiation is called today ***"synchrotron radiation"***.

In the following, this lecture will present the basics of electromagnetic radiation and in particular the physics of synchrotron radiation. There is a strong influence on the dynamic of the particle motion in circular electron machines as radiation damping, beam emittance and so on. Modern light sources produce synchrotron radiation by use of an extremely strong focused electron beam. This requires a very special magnet lattice.

2 Electromagnetic Waves

2.1 The wave equation

Oscillations are periodic changes of a physical quantity with time

$$S(t) = S_0 \exp i\omega t \tag{2.1}$$

It is the solution of the differential equation

$$\ddot{S}(t) + \omega^2 S(t) = 0 \tag{2.2}$$

A wave describes a periodic change with time and space

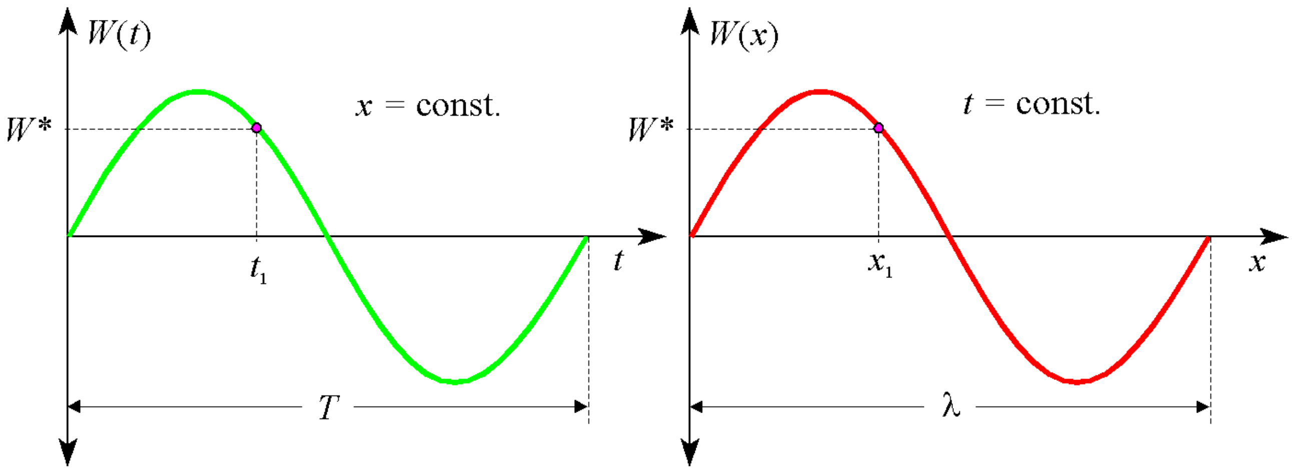


Fig. 2.1 Time and spatial dependence of an periodic physical quantity

The differential equations are

$$\ddot{W}(t) + \omega^2 W(t) = 0 \tag{2.3}$$

$$\omega = \frac{2\pi}{T} \quad \text{(frequency)}$$

$$\frac{\partial^2 W(x)}{\partial x^2} + k^2 W(x) = 0 \tag{2.4}$$

$$k = \frac{2\pi}{\lambda} \quad \text{(wave number)}$$

or more general for all 3 dimensions

$$\Delta W(\vec{r}) + \vec{k}^2 W(\vec{r}) = 0 \tag{2.5}$$

$$\vec{k} = (k_x, k_y, k_z)$$

At the time t_1 the wave has at the point x_1 the value W^*. At the time t_2 the wave point has moved to the point x_2

$$\begin{aligned} W^*(x,t) &= W_0 \exp i(\omega t_1 - k x_1) = W_0 \exp i(\omega t_2 - k x_2) \\ &\Rightarrow \quad \omega t_1 - k x_1 = \omega t_2 - k x_2 \\ &\Rightarrow \quad \omega (t_1 - t_2) = k (x_1 - x_2) \end{aligned} \tag{2.6}$$

The wave velocity (phase velocity) becomes

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\omega}{k} \tag{2.7}$$

From (2.3) we get

$$\ddot{W}(x,t) + \omega^2 W(x,t) = 0 \qquad \Rightarrow \qquad W(x,t) = -\frac{1}{\omega^2}\ddot{W}(x,t) \tag{2.8}$$

Inserting this result into (2.4) we get

$$\frac{\partial^2 W(x,t)}{\partial x^2} + k^2 W(x,t) = 0 \qquad \Rightarrow \qquad \frac{\partial^2 W(x,t)}{\partial x^2} - \frac{k^2}{\omega^2}\ddot{W}(x,t) = 0 \tag{2.9}$$

With the phase velocity (2.7) we find the one dimensional wave equation

$$\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{1}{v^2}\ddot{W}(x,t) = 0 \tag{2.10}$$

The general tree dimensional wave equation has then the form

$$\boxed{\Delta W(\vec{r},t) - \frac{1}{v^2}\ddot{W}(\vec{r},t) = 0} \tag{2.11}$$

with the *Laplace operator* $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \nabla^2$. The operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the so called *nabla operator*.

2.2 Maxwell's equations

The electromagnetic radiation is based on the Maxwell's equations. In MKSA units these equations have the form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Coulomb's law)} \tag{2.12}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.13}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.14}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(Ampere's law)} \tag{2.15}$$

One can easily show that time dependent electric or magnetic fields generates an electromagnetic wave. In the vacuum there is no current and therefore $\vec{j} = 0$. From (2.14) and (2.15) we get

$$\begin{aligned} \nabla \times \vec{E} &= -\dot{\vec{B}} & \Big| \ \frac{\partial}{\partial t} \\ \nabla \times \vec{B} &= \mu_0 \varepsilon_0 \dot{\vec{E}} & \Big| \ \nabla\times \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \nabla \times \dot{\vec{E}} &= -\ddot{\vec{B}} \\ \nabla \times \left(\nabla \times \vec{B}\right) &= \mu_0 \varepsilon_0 \nabla \times \dot{\vec{E}} \end{aligned} \tag{2.17}$$

Inserting the first equation into the second one we get

$$\nabla \times \left(\nabla \times \vec{B}\right) = -\mu_0 \varepsilon_0 \ddot{\vec{B}} \tag{2.18}$$

Using the vector relation $\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{B}$ and equation (2.13) we finally find

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \ddot{\vec{B}} = 0 \tag{2.19}$$

This is a wave equation of the form of (2.11). The phase velocity is

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997925 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}} \tag{2.20}$$

2.3 Wave equation of the vector and scalar potential

With the Maxwell equation $\nabla\vec{B}=0$ and the vector relation $\nabla(\nabla\times\vec{a})=0$ we can derive the magnetic field from a vector potential $\vec{A}$ as

$$\vec{B}=\nabla\times\vec{A}\,. \tag{2.21}$$

We insert this definition into the Maxwell equation (2.14) and get

$$\begin{aligned}\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}=-\nabla\times\left(\frac{\partial\vec{A}}{\partial t}\right)\\ \Rightarrow\quad \nabla\times&\left(\vec{E}+\frac{\partial A}{\partial t}\right)=0\end{aligned} \tag{2.22}$$

The expression $\left(\vec{E}+\partial\vec{A}/\partial t\right)$ can be written as a gradient of a scalar potential $\phi(\vec{r},t)$ in the form

$$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\nabla\phi \tag{2.23}$$

The electric field becomes

$$\vec{E}=-\left(\nabla\phi+\frac{\partial\vec{A}}{\partial t}\right). \tag{2.24}$$

With Coulomb's law (2.12) we find

$$\nabla\vec{E}=-\nabla\left(\nabla\phi+\frac{\partial\vec{A}}{\partial t}\right)=\frac{\rho}{\varepsilon_0} \tag{2.25}$$

or

$$\boxed{\nabla^2\phi+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)=-\frac{\rho}{\varepsilon_0}} \tag{2.26}$$

We take now the formula of Ampere's law (2.15) and insert the relations for the magnetic and electric field (2.21) and (2.24) and get

$$\begin{aligned}&\underbrace{\nabla\times\left(\nabla\times\vec{A}\right)}_{\nabla\cdot\left(\nabla\cdot\vec{A}\right)-\nabla^2\vec{A}}=\mu_0\vec{j}-\mu_0\varepsilon_0\left(\frac{\partial}{\partial t}\nabla\phi+\frac{\partial^2\vec{A}}{\partial t^2}\right)\\ &\nabla^2\vec{A}-\mu_0\varepsilon_0\left(\nabla\frac{\partial\phi}{\partial t}+\frac{\partial^2\vec{A}}{\partial t^2}\right)-\nabla\cdot\left(\nabla\cdot\vec{A}\right)=-\mu_0\vec{j}\end{aligned} \tag{2.27}$$

The relation becomes

$$\boxed{\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\cdot\left(\nabla\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right)=-\mu_0\vec{j}} \tag{2.28}$$

Equations (2.26) and (2.27) create a coupled system for the potentials $\vec{A}$ and ϕ. We define now the following gauge transformation

$$\begin{aligned}\vec{A}\quad&\rightarrow\quad\vec{A}'=\vec{A}+\nabla\Lambda\\ \phi\quad&\rightarrow\quad\phi'=\phi-\frac{\partial\Lambda}{\partial t}\end{aligned} \tag{2.29}$$

The free choice of $\Lambda(\vec{r},t)$ provides a set of potentials satisfying the *Lorentz condition*

$$\nabla\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0 \tag{2.30}$$

With the gauge transformation we get

$$\begin{gathered}\nabla\left(\vec{A}+\nabla\Lambda\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi-\frac{\partial\Lambda}{\partial t}\right)= \\ \underbrace{\nabla\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}}_{=0\ \text{(Lorentz condition)}}+\nabla(\nabla\Lambda)-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0\end{gathered} \tag{2.31}$$

If the function $\Lambda(\vec{r},t)$ is a solution of the wave equation

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0 \tag{2.34}$$

the Lorentz condition is fulfilled. In (2.26) we replace $\nabla\vec{A}$ by $-\dot{\phi}/c^2$ and get

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0} \tag{2.35}$$

With $c^2=1/\mu_0\varepsilon_0$ the expression (2.28) becomes

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\cdot\underbrace{\left(\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)}_{=0\ \ \text{(Lorentz condition)}}=-\mu_0\vec{j} \tag{2.36}$$

The result is then

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j} \tag{2.37}$$

The two expressions (2.35) and (2.37) are the decoupled equations for the potentials $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$. These ***inhomogeneous wave equations*** are the basis of all kind of electromagnetic radiation.

2.4 The solution of the inhomogeneous wave equations

We have now to find the solution of the inhomogeneous wave equations (2.35) and (2.37). We start assuming a point charge in the origin of the coordinate system of the form

$$dq=\rho(\vec{r},t)\,\delta^3(\vec{r})\,dV \tag{2.38}$$

Outside the origin, i.e. $|\vec{r}|\neq 0$ the charge density ρ vanishes. The wave equations of the potential becomes

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0 \tag{2.39}$$

The potential has now a spherical symmetry as

$$\phi(\vec{r},t)=\phi(|\vec{r}|,t)=\phi(r,t) \tag{2.40}$$

We have now to evaluate the expression $\nabla^2\phi(r)$ for a point charge. A straight forward calculation yields

$$\nabla^2\phi(r) = \nabla\cdot\nabla\phi(r) = \nabla\left(\frac{\vec{r}}{r}\frac{\partial\phi}{\partial r}\right) = \left(\nabla\frac{\vec{r}}{r}\right)\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} = \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} \tag{2.41}$$

On the other hand we find the relation

$$\frac{\partial^2}{\partial r^2}\left(r\,\phi\right) = \frac{\partial}{\partial r}\left(\phi + r\frac{\partial\phi}{\partial r}\right) = 2\frac{\partial\phi}{\partial r} + r\frac{\partial^2\phi}{\partial r^2} \tag{2.42}$$

Combining these two expressions we get the wave equation in the form

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{1}{r}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(r\,\phi\right) = 0 \tag{2.43}$$

with the general solution

$$\phi(r,t) = \frac{1}{r}\,f_1(r-ct) + \frac{1}{r}\,f_2(r+ct) \tag{2.44}$$

The second term on the right hand side represents a reflected wave, which doesn't exist in this case. Therefore, the solution is reduced to

$$\phi(r,t) = \frac{1}{r}\,f(r-ct) \tag{2.45}$$

In order to evaluate the function $f(r-ct)$ one has to calculate the potential $\phi(r,t)$ in the origin of the coordinate system. The problem is that

$$r \to 0 \quad \Rightarrow \quad \phi(r,t) = \frac{f(r-ct)}{r} \to \infty \tag{2.46}$$

A better way is to compare the first and second derivatives of the potential. For $r \to 0$ we get

$$\frac{\partial\phi}{\partial r} \propto \frac{f(-ct)}{\underline{\underline{r^2}}} \quad \gg \quad \frac{\partial\phi}{\partial t} \propto \frac{1}{\underline{\underline{r}}}\frac{\partial f(-ct)}{\partial t} \tag{2.47}$$

The ratio of the second spatial derivative to the second time derivative is even much larger

$$\frac{\partial^2\phi}{\partial r^2} >> \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \qquad \text{for} \qquad r \to 0 \tag{2.48}$$

and we can simplify the wave equation (2.35) to

$$\nabla^2\phi(r,t) = -\frac{\rho}{\varepsilon_0} \qquad (r \to 0) \tag{2.49}$$

This is the well known *Poisson equation* for a static point charge. For $r \to 0$ the potential $\phi(r,t)$ approaches the Coulomb potential. Therefore, we can write

$$\phi(r,t) = \frac{1}{r}\,f(r-ct) \quad \xrightarrow{r\to 0} \quad \frac{1}{r}\,f(-ct) = \frac{1}{4\pi\varepsilon_0}\frac{\rho(0,t)}{r}\Delta V \tag{2.50}$$

Because of the limited velocity c of the electromagnetic fields, at a point r outside the origin the time dependent potential is delayed by

$$\Delta t = \frac{r}{c} \qquad \Rightarrow \qquad t \to t - \frac{r}{c} \tag{2.51}$$

At this point we have the *"retarded"* potential

$$d\phi(r,t) = \frac{1}{4\pi\varepsilon_0} \frac{\rho\left(0, t - \frac{r}{c}\right)}{r} dV \tag{2.52}$$

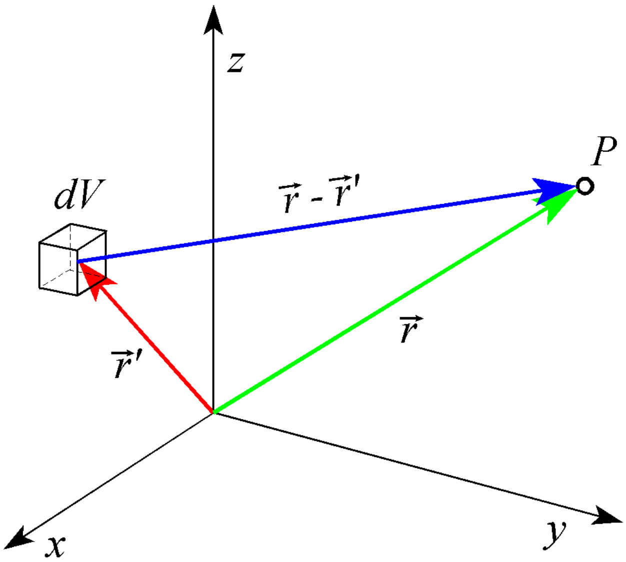


Fig. 2.2 Position of the charge element and the observer

In general the charge is not in the origin but at any point $\vec{r}\,'$ in a Volume dV. For this case the potential gets the form

$$d\phi(r,t) = \frac{1}{4\pi\varepsilon_0} \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{|\vec{r} - \vec{r}\,'|} dV\,. \tag{2.53}$$

It is retarded by the time $\Delta t = \frac{|\vec{r} - \vec{r}\,'|}{c}$. Since under real conditions one do not has a point charge the potential must be integrated over a finite volume containing the charge distribution. The result is then

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{|\vec{r} - \vec{r}\,'|} dV \tag{2.54}$$

The vector potential $\vec{A}(\vec{r},t)$ can according to (2.35) and (2.37) easily evaluated by replacing the expression $\frac{\rho}{\varepsilon_0}$ by $\mu_0 \vec{j}$. In this way we find

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{j}\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{|\vec{r} - \vec{r}\,'|} dV \tag{2.55}$$

These solutions of the two wave equations are called ***Liénard-Wiechert potentials***. An effect on the electromagnetic field at the point $\vec{r}$ and the time t is caused by ρ and $\vec{j}$ at the point $\vec{r}\,'$ and the earlier time $t' = t - |\vec{r} - \vec{r}\,'|/c$.

2.5 Liénard-Wiechert potentials of a moving charge

The calculation of the electromagnetic radiation emitted by a moving charged particle needs a careful integration over the charge, even in the case of point charges. We now replace the distance between the charge and the observer by

$$\vec{R} = \vec{r}\,' - \vec{r} \tag{2.56}$$

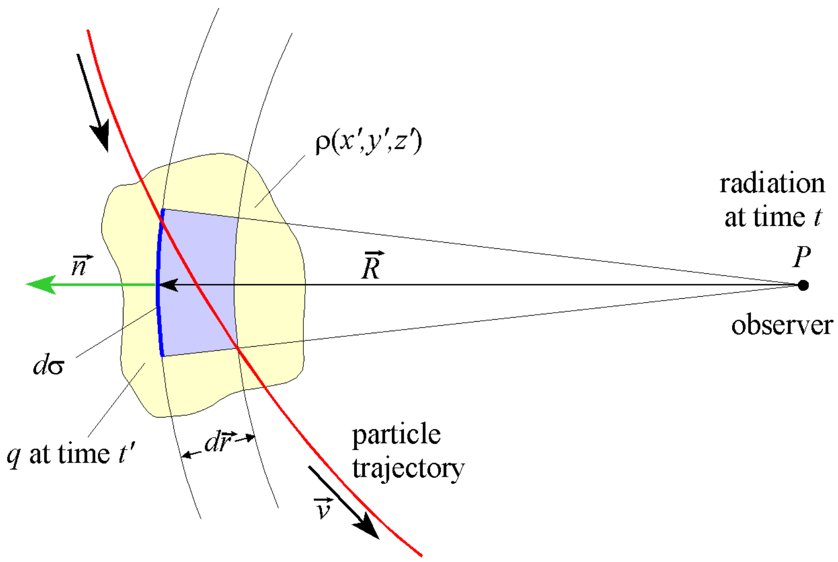


Fig. 2.3 Radiation from a moving charge

Radiation observed at the point P comes from all charges within a spherical shell with the center P, the radius $|\vec{R}|$ and the thickness $|d\vec{r}|$. If $d\sigma$ is the surface element of the shell the volume element is

$$dV = d\sigma\, dr \tag{2.57}$$

The retarded time for radiation from the outer surface of the shell is

$$t' = t - \frac{|\vec{R}|}{c} \tag{2.58}$$

and from the inner surface

$$t'' = t' - \frac{|d\vec{r}|}{c} \tag{2.59}$$

The electromagnetic field at P at the time t is generated by the charge within the volume element dV. The charge in this volume element is with $dr = |d\vec{r}|$

$$dq_1 = \rho\, d\sigma\, dr \tag{2.60}$$

For charges moving with the velocity $\vec{v}$ one has to add all charge that penetrate the inner shell surface during the time $dt = dr/c$, i.e.

$$dq_2 = \rho\, \vec{v}\, \vec{n}\, dt\, d\sigma \tag{2.61}$$

with the vector $\vec{n}$ normal to the outer surface defined by

$$\vec{n} = \frac{\vec{R}}{|\vec{R}|} \tag{2.62}$$

The total effective charge element is then

$$dq = dq_1 + dq_2 = \rho\, d\sigma\left(dr + \vec{v}\,\vec{n}\,dt\right) = \rho\, d\sigma\left(dr + \vec{v}\,\vec{n}\,\frac{dr}{c}\right) = \rho\left(1+\vec{n}\,\vec{\beta}\right) dr\, d\sigma \qquad (2.63)$$

With this relation we can write

$$\rho\, dr\, d\sigma = \rho\, dV = \frac{dq}{1+\vec{n}\,\vec{\beta}} \qquad (2.64)$$

Insertion into equation (2.54) gives

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{R\left(1+\vec{n}\vec{\beta}\right)} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\frac{1}{\left(1+\vec{n}\vec{\beta}\right)}\Bigg|_{t'} \qquad (2.65)$$

The current density an be written as

$$\vec{j} = \rho\,\vec{v} \qquad (2.66)$$

and the vector potential (2.55) becomes with (2.64)

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{v}\,dq}{R\left(1+\vec{n}\vec{\beta}\right)} = \frac{c\mu_0}{4\pi}\frac{q}{R}\frac{\vec{\beta}}{\left(1+\vec{n}\vec{\beta}\right)}\Bigg|_{t'} \qquad (2.67)$$

It is important to notice that the parameter in the expression on the right hand side must be taken at the retarded time *t'*. The equations (2.65) and (2.67) are the *Liénard-Wiechert* potentials for a moving point charge.

2.6 The electric field of a moving charged particle

Using the formula (2.23) we can derive the electric field at the point *P* by inserting the potentials as

$$\vec{E} = -\left(\nabla'\phi + \frac{\partial\vec{A}}{\partial t}\right) = -\frac{q}{4\pi\varepsilon_0}\nabla'\frac{1}{R\left(1+\vec{n}\,\vec{\beta}\right)} - \frac{c\,\mu_0 q}{4\pi}\frac{\partial}{\partial t}\frac{\vec{\beta}}{R\left(1+\vec{n}\,\vec{\beta}\right)} \qquad (2.68)$$

In order to simplify the calculations we define

$$a := R\left(1+\vec{n}\,\vec{\beta}\right) \qquad (2.69)$$

and set

$$\frac{c\mu_0 q}{4\pi} = \frac{c^2\mu_0 q}{4\pi c} = \frac{\mu_0 q}{4\pi\mu_0\varepsilon_0 c} = \frac{q}{4\pi\varepsilon_0}\frac{1}{c} \qquad (2.70)$$

The electrical field is then

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left[\nabla'\frac{1}{a} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\vec{\beta}}{a}\right)\right] \qquad (2.71)$$

Notice that all expressions concerning the moving charge must be evaluated at the retarded time *t'*. To indicate the calculation at the retarded time we will add a ' to the symbol (i.e. t', ∇', etc.). With

$$\nabla'\frac{1}{a} = -\frac{1}{a^2}\nabla' a \qquad \text{and} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{dt'}{dt} \qquad (2.72)$$

we get

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a^2}\nabla' a - \frac{1}{c}\frac{dt'}{dt}\frac{\partial}{\partial t'}\left(\frac{\vec{\beta}}{a}\right)\right] \tag{2.73}$$

This formula needs the knowledge of $\nabla' a$, dt'/dt and $\partial\left(\vec{\beta}/a\right)/\partial t'$. The detailed calculation provides the following results.

The variation of the distance $\left|\vec{R}\right| = R$ is

$$dR = \vec{v}\,\vec{n}\,dt' \qquad \boxed{\frac{dR}{dt'} = \vec{v}\,\vec{n}} \quad (2.74) \qquad \text{and} \qquad \boxed{\frac{d\vec{R}}{dt'} = \vec{v}} \quad (2.75)$$

The retarded time is

$$t' = t - \frac{R}{c} \tag{2.76}$$

and we find

$$\frac{dt'}{dt} = 1 - \frac{1}{c}\frac{dR}{dt'}\frac{dt'}{dt} = 1 - \frac{\vec{v}\,\vec{n}}{c}\frac{dt'}{dt} = 1 - \vec{n}\,\vec{\beta}\frac{dt'}{dt} \qquad \boxed{\frac{dt'}{dt} = \frac{1}{1+\vec{n}\vec{\beta}} = \frac{R}{a}} \tag{2.77}$$

The nabla operator for the retarded time is defined as

$$\begin{aligned}\nabla' &= \left(\frac{\partial}{\partial x} + \frac{\partial t'}{\partial x'}\frac{\partial}{\partial t'}, \frac{\partial}{\partial y} + \frac{\partial t'}{\partial y'}\frac{\partial}{\partial t'}, \frac{\partial}{\partial z} + \frac{\partial t'}{\partial z'}\frac{\partial}{\partial t'}\right) \\ &= \nabla + \nabla' t'\frac{\partial}{\partial t'}\end{aligned} \tag{2.78}$$

With this relations we find with $\nabla R = -\vec{n}$

$$\nabla' R = \nabla R + \nabla' t'\frac{\partial R}{\partial t'} \qquad \nabla' R = -\vec{n} + \nabla' t'\frac{\partial R}{\partial t'} \tag{2.79}$$

The gradient of the retarded time becomes

$$\begin{aligned}\nabla' t' &= \nabla'\left(t - \frac{1}{c}R\right) = -\frac{1}{c}\nabla' R = -\frac{1}{c}\left(-\vec{n} + \nabla' t'\frac{\partial R}{\partial t'}\right) \\ &= -\frac{1}{c}\left(-\vec{n} + \vec{n}\,\vec{v}\,\nabla' t'\right) = \frac{\vec{n}}{c} - \vec{n}\,\vec{\beta}\,\nabla' t' \\ \Rightarrow &\quad \nabla' t'\left(1 + \vec{n}\,\vec{\beta}\right) = \frac{\vec{n}}{c}\end{aligned} \qquad \boxed{\nabla' t' = \frac{\vec{n}}{c\left(1+\vec{n}\,\vec{\beta}\right)} = \frac{\vec{R}}{c\,a}} \tag{2.80}$$

With this result we can finally write $\nabla' R$ in the form

$$\nabla' R = -\vec{n} + \frac{\vec{n}R}{ca}\left(\vec{v}\,\vec{n}\right) \qquad \boxed{\nabla' R = \vec{n}\left(\frac{R}{a}\left(\vec{\beta}\,\vec{n}\right) - 1\right)} \tag{2.81}$$

For further calculations we need $\nabla'(\vec{R}\vec{\beta})$. Since the velocity of the particle does not depend on the position of P we have $\nabla\vec{\beta} = 0$. With $\nabla\vec{R} = -1$ we can calculate

$$\nabla'(\vec{R}\vec{\beta}) = \nabla(\vec{R}\vec{\beta}) + \nabla' t' \frac{\partial(\vec{R}\vec{\beta})}{\partial t'}$$
$$= (\nabla\vec{R})\vec{\beta} + \vec{R}\nabla\vec{\beta} + \left(\vec{\beta}\frac{\partial\vec{R}}{\partial t'} + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)\nabla' t'$$
$$= -\vec{\beta} + \left(\vec{\beta}\frac{\partial\vec{R}}{\partial t'} + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)\frac{\vec{R}}{ca}$$

$$\boxed{\nabla'(\vec{R}\vec{\beta}) = -\vec{\beta} + \frac{\vec{R}}{ca}\left(\vec{\beta}\frac{\partial\vec{R}}{\partial t'} + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)} \qquad (2.82)$$

For the time derivative of a we get

$$\frac{\partial a}{\partial t'} = \frac{\partial}{\partial t'}R(1+\vec{n}\vec{\beta}) = \frac{\partial R}{\partial t'} + \frac{\partial}{\partial t'}(\vec{R}\vec{\beta})$$
$$= \vec{v}\,\vec{n} + \frac{\partial\vec{R}}{\partial t'}\vec{\beta} + \vec{R}\frac{\partial\vec{\beta}}{\partial t'} = \vec{v}\,\vec{n} + \vec{v}\,\vec{\beta} + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}$$

$$\boxed{\frac{\partial a}{\partial t'} = c\vec{\beta}\vec{n} + c\vec{\beta}^2 + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}} \qquad (2.83)$$

With (2.81) and (2.82) we find the first required expression

$$\nabla' a = \nabla' R + \nabla'(\vec{R}\vec{\beta}) = \vec{n}\frac{R}{a}(\vec{\beta}\vec{n}) - \vec{n} - \vec{\beta} + \frac{\vec{R}}{ca}(\vec{\beta}\vec{v}) + \frac{\vec{R}}{ca}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)$$

$$\boxed{\nabla' a = -\vec{n} - \vec{\beta} + \frac{\vec{R}}{a}\left(\vec{n}\vec{\beta} + \vec{\beta}^2 + \frac{\vec{R}}{c}\frac{\partial\vec{\beta}}{\partial t'}\right)} \qquad (2.84)$$

The time derivative of the ratio $\vec{\beta}/a$ becomes with (2.83)

$$\frac{\partial}{\partial t'}\left(\frac{\vec{\beta}}{a}\right) = \frac{1}{a}\frac{\partial\vec{\beta}}{\partial t'} - \frac{\vec{\beta}}{a^2}\frac{\partial a}{\partial t'} = \frac{1}{a}\frac{\partial\vec{\beta}}{\partial t'} - \frac{\vec{\beta}}{a^2}\left(c\vec{\beta}\vec{n} + c\vec{\beta}^2 + \vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)$$
$$= \frac{1}{a}\frac{\partial\vec{\beta}}{\partial t'} - \frac{c}{a^2}\vec{\beta}(\vec{\beta}\vec{n}) - \frac{c}{a^2}\vec{\beta}^3 - \frac{\vec{\beta}}{a^2}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)$$

$$\boxed{\frac{\partial}{\partial t'}\left(\frac{\vec{\beta}}{a}\right) = \frac{1}{a}\frac{\partial\vec{\beta}}{\partial t'} - \frac{\vec{\beta}}{a^2}\left\{c(\vec{\beta}\vec{n}) + \vec{R}\frac{\partial\vec{\beta}}{\partial t'} + c\vec{\beta}^2\right\}} \qquad (2.85)$$

Now we insert the relations (2.77), (2.84) and (2.85) into the equation (2.73). The result is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a^2}\left\{-\vec{n} - \vec{\beta} + \frac{\vec{R}}{a}\left(\vec{n}\vec{\beta} + \vec{\beta}^2 + \frac{\vec{R}}{c}\frac{\partial\vec{\beta}}{\partial t'}\right)\right\} - \frac{R}{ca}\left\{\frac{1}{a}\frac{\partial\vec{\beta}}{\partial t'} - \frac{\vec{\beta}}{a^2}\left(c(\vec{\beta}\vec{n}) + \vec{R}\frac{\partial\vec{\beta}}{\partial t'} + c\vec{\beta}^2\right)\right\}\right]$$
$$= \frac{q}{4\pi\varepsilon_0 a^3}\left[-a\vec{n} - a\vec{\beta} + \vec{R}(\vec{n}\vec{\beta}) + \vec{R}\vec{\beta}^2 + \frac{1}{c}\vec{R}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right) - \frac{Ra}{c}\frac{\partial\vec{\beta}}{\partial t'} + R\vec{\beta}(\vec{\beta}\vec{n}) + \frac{R}{c}\vec{\beta}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right) + R\vec{\beta}^3\right] \qquad (2.86)$$
$$= \frac{q}{4\pi\varepsilon_0 a^3}\left\{\left[-a\vec{n} - a\vec{\beta} + \vec{R}(\vec{n}\vec{\beta}) + \vec{R}\vec{\beta}^2 + R\vec{\beta}(\vec{\beta}\vec{n}) + R\vec{\beta}^3\right]_1 + \frac{1}{c}\left[\vec{R}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right) - Ra\frac{\partial\vec{\beta}}{\partial t'} + R\vec{\beta}\left(\vec{R}\frac{\partial\vec{\beta}}{\partial t'}\right)\right]_2\right\}$$

With the definition of a we can manipulate the expression in the first bracket $[\cdots]_1$ in the following way

$$[\cdots]_1 = -R(1+\vec{n}\vec{\beta})(\vec{n}+\vec{\beta}) + \vec{R}(\vec{n}\vec{\beta}) + \vec{R}\vec{\beta}^2 + R\vec{\beta}(\vec{n}\vec{\beta}) + R\vec{\beta}^3$$
$$= -R\vec{n} - R\vec{\beta} + \vec{R}\vec{\beta}^2 + R\vec{\beta}^3 \qquad (2.87)$$
$$= \vec{R}(\vec{\beta}^2 - 1) + R\vec{\beta}(\vec{\beta}^2 - 1) = -(1-\vec{\beta}^2)(\vec{R} + \vec{\beta}R)$$

The second bracket $[\cdots]_2$ becomes with $\dot{\vec{\beta}} = \partial\vec{\beta}/\partial t'$

$$\begin{aligned}[\cdots]_2 &= \vec{R}\left(\vec{R}\dot{\vec{\beta}}\right) - R^2\dot{\vec{\beta}} - R^2\left(\vec{n}\vec{\beta}\right)\dot{\vec{\beta}} + R\vec{\beta}\left(\vec{R}\dot{\vec{\beta}}\right) \\ &= \left(\vec{R} + R\vec{\beta}\right)\left(\vec{R}\dot{\vec{\beta}}\right) - \left(R^2 + R\left(\vec{R}\vec{\beta}\right)\right)\dot{\vec{\beta}} \\ &= \left(\vec{R} + R\vec{\beta}\right)\left(R\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\left[\vec{R}\left(\vec{R} + R\vec{\beta}\right)\right]\end{aligned} \tag{2.88}$$

Using the vector relation

$$\vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}\cdot\left(\vec{a}\cdot\vec{b}\right) = \vec{a}\times\left(\vec{b}\times\vec{c}\right) \tag{2.89}$$

we get

$$[\cdots]_2 = \vec{R}\times\left[\left(\vec{R} + R\vec{\beta}\right)\times\dot{\vec{\beta}}\right] \tag{2.90}$$

Now we replace the two brackets in equation (2.86) by the expressions (2.87) and (2.90) and get the electric field in the final form

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{-\frac{1-\vec{\beta}^2}{a^3}\left(\vec{R}+\vec{\beta}R\right) + \frac{1}{ca^3}\vec{R}\times\left[\left(\vec{R}+\vec{\beta}R\right)\times\dot{\vec{\beta}}\right]\right\} \tag{2.91}$$

We can write this equation in a slightly different way, namely

$$\begin{aligned}\vec{E} &= \frac{q}{4\pi\varepsilon_0}\left\{-\frac{1-\vec{\beta}^2}{R^3\left(1+\vec{n}\vec{\beta}\right)^3}\left(\vec{n}R + \vec{\beta}\,R\right) + \frac{1}{cR^3\left(1+\vec{n}\vec{\beta}\right)^3}(\vec{n}R)\times\left[\left(\vec{n}R+\vec{\beta}\,R\right)\times\dot{\vec{\beta}}\right]\right\} \\ &= \frac{q}{4\pi\varepsilon_0}\left\{-\frac{1-\vec{\beta}^2}{R^2\left(1+\vec{n}\vec{\beta}\right)^3}\left(\vec{n}+\vec{\beta}\,\right) + \frac{1}{cR\left(1+\vec{n}\vec{\beta}\right)^3}\,\vec{n}\times\left[\left(\vec{n}+\vec{\beta}\,\right)\times\dot{\vec{\beta}}\right]\right\}\end{aligned} \tag{2.92}$$

The first term drops down with $1/R^2$ and vanishes at longer distances. The second term, however, reduces only inversely proportional to the distance R. It determines the radiation far away from the source charge. For further discussions of the synchrotron radiation, we are only interested in the long distance field. Therefore, we can we can neglect the first term in (2.19) and get

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{ca^3}\left\{\vec{R}\times\left[\left(\vec{R}+\vec{\beta}R\right)\times\dot{\vec{\beta}}\right]\right\} \tag{2.93}$$

Since $\vec{R}$ points into the direction opposite to the direction of the radiation, one can directly derive from (2.93) that the electric field is polarized orthogonal to direction of radiation.

2.7 The magnetic field of a moving charged particle

With the relations (2.21) and (2.67) we can calculate the magnetic field of a moving charged particle and we find

$$\vec{B} = \nabla'\times\vec{A} = \frac{c\mu_0 q}{4\pi}\nabla'\times\left(\frac{\vec{\beta}}{a}\right) = \frac{c\mu_0 q}{4\pi}\left(\frac{1}{a}\nabla'\times\vec{\beta} - \frac{1}{a^2}(\nabla' a)\times\vec{\beta}\right) \tag{2.94}$$

With $\vec{\beta} = (\beta_x, \beta_y, \beta_z)$ we use the "retarded" curl operation

$$\nabla' \times \vec{\beta} = \begin{vmatrix} e_x & e_x & e_z \\ \frac{\partial}{\partial x} + \frac{\partial t'}{\partial x'}\frac{\partial}{\partial t'} & \frac{\partial}{\partial y} + \frac{\partial t'}{\partial y'}\frac{\partial}{\partial t'} & \frac{\partial}{\partial z} + \frac{\partial t'}{\partial z'}\frac{\partial}{\partial t'} \\ \beta_x & \beta_y & \beta_z \end{vmatrix} \tag{2.95}$$

The evaluation of this operation provides

$$\begin{aligned} \nabla' \times \vec{\beta} &= \begin{pmatrix} \left(\frac{\partial}{\partial y} + \frac{\partial t'}{\partial y'}\frac{\partial}{\partial t'}\right)\beta_z - \left(\frac{\partial}{\partial z} + \frac{\partial t'}{\partial z'}\frac{\partial}{\partial t'}\right)\beta_y \\ \left(\frac{\partial}{\partial z} + \frac{\partial t'}{\partial z'}\frac{\partial}{\partial t'}\right)\beta_x - \left(\frac{\partial}{\partial x} + \frac{\partial t'}{\partial x'}\frac{\partial}{\partial t'}\right)\beta_z \\ \left(\frac{\partial}{\partial x} + \frac{\partial t'}{\partial x'}\frac{\partial}{\partial t'}\right)\beta_y - \left(\frac{\partial}{\partial y} + \frac{\partial t'}{\partial y'}\frac{\partial}{\partial t'}\right)\beta_x \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \beta_z}{\partial y} - \frac{\partial \beta_y}{\partial z} \\ \frac{\partial \beta_x}{\partial x} - \frac{\partial \beta_z}{\partial z} \\ \frac{\partial \beta_y}{\partial x} - \frac{\partial \beta_x}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial t'}{\partial y'}\frac{\partial \beta_z}{\partial t'} - \frac{\partial t'}{\partial z'}\frac{\partial \beta_y}{\partial t'} \\ \frac{\partial t'}{\partial z'}\frac{\partial \beta_x}{\partial t'} - \frac{\partial t'}{\partial x'}\frac{\partial \beta_z}{\partial t'} \\ \frac{\partial t'}{\partial x'}\frac{\partial \beta_y}{\partial t'} - \frac{\partial t'}{\partial y'}\frac{\partial \beta_x}{\partial t'} \end{pmatrix} = \nabla \times \vec{\beta} + \left(\nabla t' \times \dot{\vec{\beta}}\right) \end{aligned} \tag{2.96}$$

Since $\vec{v}$ is independent of the position P of the observer we have $\nabla \times \vec{\beta} = 0$. The gradient of the retarded time has been derived in equation (2.80). The result is

$$\nabla' \times \vec{\beta} = \frac{1}{ca}\left(\vec{R} \times \dot{\vec{\beta}}\right) \tag{2.97}$$

The second expression needed in (2.94) is

$$\nabla' a = -\vec{n} - \vec{\beta} + b\,\vec{R} \qquad \text{with} \qquad b := \frac{1}{a}\left(\vec{n}\,\vec{\beta} + \vec{\beta}^2 + \frac{\vec{R}}{c}\dot{\vec{\beta}}\right) \tag{2.98}$$

With this relation we get

$$\nabla' a \times \vec{\beta} = \left(-\vec{n} - \vec{\beta} + b\,\vec{R}\right) \times \vec{\beta} = -\left[\vec{n} \times \vec{\beta}\right] - \underbrace{\left[\vec{\beta} \times \vec{\beta}\right]}_{=0} + b\left[\vec{R} \times \vec{\beta}\right] = -\left[\vec{n} \times \vec{\beta}\right] + b\left[\vec{R} \times \vec{\beta}\right] \tag{2.99}$$

Now we insert the relations (2.97) and (2.99) into the field equation (2.94) and find

$$\begin{aligned} \vec{B} &= \frac{c\mu_0 q}{4\pi}\left(\frac{1}{ca^2}\left[\vec{R} \times \dot{\vec{\beta}}\right] + \frac{1}{a^2}\left[\vec{n} \times \vec{\beta}\right] - \frac{Rb}{a^2}\left[\vec{n} \times \vec{\beta}\right]\right) \\ &= \frac{c\mu_0 q}{4\pi}\left(-\frac{\left[\vec{\beta} \times \vec{n}\right]}{a^2} - \frac{R}{ca^2}\left[\dot{\vec{\beta}} \times \vec{n}\right] + \frac{R}{a^3}\left(\vec{n}\vec{\beta} + \vec{\beta}^2 + \frac{\vec{R}}{c}\dot{\vec{\beta}}\right)\left[\vec{\beta} \times \vec{n}\right]\right) \end{aligned} \tag{2.100}$$

As for the electric field we are also for the magnetic field only interested in the contributions at far away places. Therefore, we reduce the formula (2.100) in a way that it only contains termes proportional to $1/R$. The result of this approximation for long distance fields is then

$$\vec{B} = \frac{c\mu_0 q}{4\pi}\left(-\frac{\left[\dot{\vec{\beta}}\times\vec{n}\right]}{cR\left(1+\vec{n}\vec{\beta}\right)^2}+\frac{\left(\dot{\vec{\beta}}\vec{n}\right)\left[\vec{\beta}\times\vec{n}\right]}{cR\left(1+\vec{n}\vec{\beta}\right)^3}\right) \tag{2.101}$$

There is an important relationship between the electric and magnetic field emitted by a moving charged particle. To find this relationship we modify the formula (2.86) in the following way

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left[-\vec{n}-\vec{\beta}+b\,\vec{R}\right]-\frac{R}{c\,a^2}\dot{\vec{\beta}}+\frac{R\vec{\beta}}{a^2}b\right\} \tag{2.102}$$

The vector multiplication oft this equation with the unit vector $\vec{n}$ gives

$$\begin{aligned}\left[\vec{E}\times\vec{n}\right] &= \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left[-\vec{n}-\vec{\beta}+b\,\vec{R}\right]-\frac{R}{c\,a^2}\dot{\vec{\beta}}+\frac{R\vec{\beta}}{a^2}b\right\}\times\vec{n}\\ &= \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left(-\underbrace{\left[\vec{n}\times\vec{n}\right]}_{=0}-\left[\vec{\beta}\times n\right]+b\underbrace{\left[\vec{R}\times\vec{n}\right]}_{=0}\right)-\frac{R}{ca^2}\left[\dot{\vec{\beta}}\times\vec{n}\right]+\frac{Rb}{a^2}\left[\vec{\beta}\times\vec{n}\right]\right\}\\ &= \frac{q}{4\pi\varepsilon_0}\left\{-\frac{\left[\vec{\beta}\times\vec{n}\right]}{a^2}-\frac{R}{ca^2}\left[\dot{\vec{\beta}}\times\vec{n}\right]+\frac{R}{a^3}\left(\vec{n}\vec{\beta}+\vec{\beta}^2+\frac{\vec{R}}{c}\dot{\vec{\beta}}\right)\left[\vec{\beta}\times\vec{n}\right]\right\}\end{aligned} \tag{2.103}$$

Comparison with the equation (2.100) for the magnetic field leads directly to the following simple relation between the magnetic and electric field

$$\vec{B} = \frac{1}{c}\left[\vec{E}\times\vec{n}\right] \tag{2.104}$$

One can directly see that the magnetic field is perpendicular to the electric field and the polarisation of both fields is perpendicular to the direction of radiation. We can now state the *Poynting vector* of the radiation in the form

$$\vec{S} = \frac{1}{\mu_0}\left[\vec{E}\times\vec{B}\right] = \frac{1}{c\,\mu_0}\left[\vec{E}\times\left(\vec{E}\times\vec{n}\right)\right] \tag{2.105}$$

We apply again the vector relation $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}\left(\vec{a}\,\vec{c}\right)-\vec{c}\left(\vec{a}\,\vec{b}\right)$ and get

$$\vec{E}\times\left(\vec{E}\times\vec{n}\right)=\vec{E}\left(\vec{E}\,\vec{n}\right)-\vec{n}\,\vec{E}^2=-\vec{n}\,\vec{E}^2 \tag{2.106}$$

The Poynting vector finally becomes

$$\vec{S} = -\frac{1}{c\mu_0}\vec{E}^2\vec{n} \tag{2.107}$$

This is the power density of the radiation parallel to $\vec{n}$ observed at the point P per unit cross section. For some calculations it is also helpful to evaluate the Poynting vector at the retarded time t'. With the relation (2.77) we find

$$\vec{S}' = \vec{S}\frac{dt}{dt'} = -\frac{1}{c\,\mu_0}\vec{E}^2\,\vec{n}\frac{dt}{dt'} = -\frac{1}{c\,\mu_0}\vec{E}^2\frac{a}{R}\vec{n} \tag{2.108}$$

or

$$\vec{S}' = -\frac{1}{c\mu_0}\vec{E}^2\left(1+\vec{n}\vec{\beta}\right)\vec{n} \tag{2.109}$$

3 Synchrotron Radiation

3.1 Radiation power and energy loss

Now we choose a coordinate system K^* which moves with the particle of the charge $q = e$. In this reference frame the particle velocity vanishes and the charge oscillates about a fixed point. We get

$$\vec{v}^* = 0 \quad \rightarrow \quad \vec{\beta}^* = 0 \quad \rightarrow a = R \tag{3.1}$$

It is important to notice that $\dot{\vec{\beta}}^* \neq 0$! The expression (2.93) is then modified to

$$\vec{E}^* = \frac{e}{4\pi\varepsilon_0}\frac{1}{cR^3}\left(\vec{R}\times\left[\vec{R}\times\dot{\vec{\beta}}^*\right]\right) = \frac{e}{4\pi\varepsilon_0}\frac{1}{cR}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right) \tag{3.2}$$

The radiated power per unit solid angle at the distance R from the generating charge is

$$\begin{aligned}\frac{dP}{d\Omega} &= -\vec{n}\,\vec{S}\,R^2 = \frac{1}{c\,\mu_0}\frac{e^2}{(4\pi\varepsilon_0)^2}\frac{1}{c^2}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 \\ &= \frac{e^2}{(4\pi)^2 c\,\varepsilon_0}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2\end{aligned} \tag{3.3}$$

With the vector relation $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}\left(\vec{a}\,\vec{c}\right) - \vec{c}\left(\vec{a}\,\vec{b}\right)$ and $\vec{n}\,\vec{n} = \vec{n}^2 = 1$ we find

$$\begin{aligned}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 &= \left(\vec{n}\left(\vec{n}\,\dot{\vec{\beta}}^*\right) - \dot{\vec{\beta}}^*\left(\vec{n}\,\vec{n}\right)\right)^2 = \vec{n}^2\left(\vec{n}\,\dot{\vec{\beta}}^*\right)^2 - 2\vec{n}\left(\vec{n}\,\dot{\vec{\beta}}^*\right)\dot{\vec{\beta}}^* + \dot{\vec{\beta}}^{*2} \\ &= \dot{\vec{\beta}}^{*2} - \left(\vec{n}\,\dot{\vec{\beta}}^*\right)^2\end{aligned} \tag{3.4}$$

Since $\vec{n}\,\dot{\vec{\beta}}^* = |\vec{n}|\left|\dot{\vec{\beta}}^*\right|\cos\Theta = \left|\dot{\vec{\beta}}^*\right|\cos\Theta$ where Θ is the angle between the direction of the particle acceleration and the direction of observation the relation (3.4) becomes

$$\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 = \dot{\vec{\beta}}^{*2} - \dot{\vec{\beta}}^{*2}\cos^2\Theta = \dot{\vec{\beta}}^{*2}\left(1-\cos^2\Theta\right) = \dot{\vec{\beta}}^{*2}\sin^2\Theta \tag{3.5}$$

The power per unit solid angle is then

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 c\varepsilon_0}\dot{\vec{\beta}}^{*2}\sin^2\Theta \tag{3.5}$$

The spatial power distribution corresponds to the power distribution of a *Hertz' dipole*. It is shown in fig. 3.1. The total power radiated by the charged particle can be achieved by integrating (3.5) over all solid angle. With

$$d\Omega = \sin\Theta'\,d\Theta'\,d\phi \tag{3.6}$$

we can write

$$P = \frac{e^2}{(4\pi)^2 c\,\varepsilon_0}\dot{\vec{\beta}}^{*2}\int_0^{2\pi}\int_0^{\pi}\sin^3\Theta\,d\Theta\,d\phi \qquad (3.7)$$

where ϕ is the azimuth angle with respect to the direction of the acceleration. The result of the integrals is simply 4/3 and the total power becomes

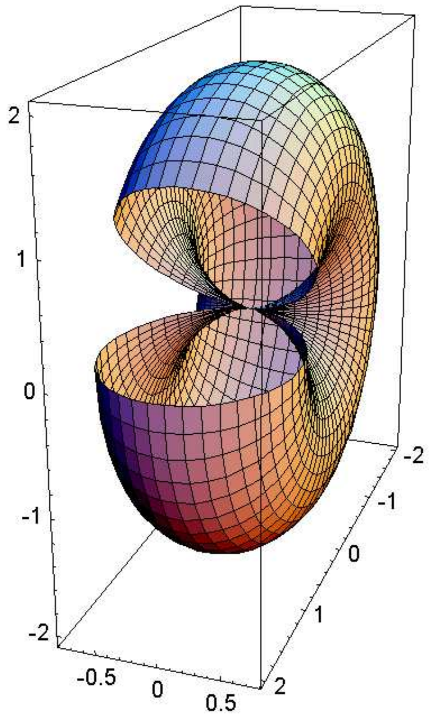


Fig. 3.1 Power distribution of an oscillating charged particle in the reference frame K^* ($v = 0$)

$$\boxed{P = \frac{e^2}{6\pi\varepsilon_0 c}\dot{\vec{\beta}}^{*2}} \qquad (3.8)$$

This result was first found by *Lamor*. One can directly see that radiation only occurs while the charged particle is accelerated. With the modification

$$\dot{\vec{\beta}}^{*} = \frac{\dot{\vec{v}}^{*}}{c} = \frac{m\dot{\vec{v}}^{*}}{c}\,\frac{\dot{\vec{p}}}{mc} \qquad (3.9)$$

we get

$$\boxed{P = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{d\vec{p}}{dt}\right)^2} \qquad (3.10)$$

This is the radiation of a non-relativistic particle. To get an expression for extreme relativistic particles we have to replace the time t by the Lorentz-invariant time $d\tau = dt/\gamma$ and the momentum $\vec{p}$ by the 4-momentum P_μ

$$dt \;\rightarrow\; d\tau = \frac{1}{\gamma}dt \quad \text{with} \quad \gamma = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1-\beta^2}}$$
$$\vec{p} \;\rightarrow\; P_\mu \quad (4\text{-momentum}) \qquad (3.11)$$

or

$$\left(\frac{d\vec{p}}{dt}\right)^2 \;\rightarrow\; \left(\frac{dP_\mu}{d\tau}\right)^2 = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 \qquad (3.12)$$

With this modification we get the radiated power in the relativistic invariant form

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right] \tag{3.13}$$

The radiation power depends mainly on the angle between the direction of particle motion $\vec{v}$ and the direction of the acceleration $d\vec{v}/d\tau$. There are two different cases:

1. linear acceleration: $\frac{d\vec{v}}{d\tau} \parallel \vec{v}$
2. circular acceleration: $\frac{d\vec{v}}{d\tau} \perp \vec{v}$

3.1.1 Linear acceleration

The particle energy is

$$E^2 = (m_0 c^2)^2 + p^2 c^2 . \tag{3.14}$$

After differentiating we get

$$E\frac{dE}{d\tau} = c^2 p \frac{dp}{d\tau} \tag{3.15}$$

Using $E = \gamma m_0 c^2$ and $p = \gamma m_0 v$ we have

$$\frac{dE}{d\tau} = v\frac{dp}{d\tau} \tag{3.16}$$

Insertion into the radiation formula (3.13) gives

$$\begin{aligned} P_s &= \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}\left[\left(\frac{dp}{d\tau}\right)^2 - \left(\frac{v}{c}\right)^2\left(\frac{dp}{d\tau}\right)^2\right] \\ &= \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}(1-\beta^2)\left(\frac{dp}{d\tau}\right)^2 \end{aligned} \tag{3.17}$$

With $1-\beta^2 = 1/\gamma^2$ we can write

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}\left(\frac{dp}{\gamma\, d\tau}\right)^2 = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}\left(\frac{dp}{dt}\right)^2 \tag{3.18}$$

For linear acceleration holds

$$dp/dt = (c\, dp)/(c\, dt) = dE/dx$$

and we get

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^2}\left(\frac{dE}{dx}\right)^2 . \tag{3.19}$$

Today in most of the modern electron linacs one can achieve

$$\frac{dE}{dx} \approx 15\frac{\text{MeV}}{\text{m}}$$

and gets the radiation power

$$P_s = 4\cdot 10^{-17}\ \text{Watt (!)}$$

which is completely negligible. In a linac synchrotron radiation has not to be taken into account independent of the particle energy. Therefore, at extremely high energies linear collider are the favorite machine type rather than circular accelerators.

3.1.2 Circular acceleration

Completely different is the situation when the acceleration is perpendicular to the direction of particle motion. In this case the particle energy stays constant. Erquation (3.13) reduces to

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{d\tau}\right)^2 = \frac{e^2 c\,\gamma^2}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{dt}\right)^2 \tag{3.20}$$

On a circular trajectory with the radius ρ a change of the orbit angle $d\alpha$ causes a momentum variation

$$dp = p\, d\alpha \tag{3.21}$$

With $v = c$ and $E = pc$ follows

$$\frac{dp}{dt} = p\omega = \frac{p\,v}{R} = \frac{E}{\rho} \tag{3.22}$$

We insert this result in (3.20) and get with $\gamma = E/m_0c^2$

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4}\frac{E^4}{\rho^2} \tag{3.23}$$

Comparison of the radiation from an electron and a proton with the same energy gives

$$m_e c^2 = 0.511\,\text{MeV}$$

$$m_p c^2 = 938.19\,\text{MeV}$$

$$\frac{P_{\text{s,e}}}{P_{\text{s,p}}} = \left(\frac{m_p c^2}{m_e c^2}\right)^4 = 1.13\cdot 10^{13}\ (!)$$

This radiation is therefore observed in most of the cases from electrons. Only at extremely high energies of $E > 1$ TeV also for protons the synchrotron radiation starts playing a certain role.

In a circular accelerator the energy loss per turn is

$$\Delta E = \oint P_s\, dt = P_s\, t_b = P_s \frac{2\pi\rho}{c} \tag{3.24}$$

The time t_b is the duration a particle needs to travel through the bending magnets. In straight sections no radiation is emitted.

We insert (3.23) into (3.24) and get

$$\Delta E = \frac{e^2}{3\varepsilon_0 \left(m_0 c^2\right)^4}\frac{E^4}{\rho} \tag{3.25}$$

For electrons one can reduce this formula to a very simple expression

$$\Delta E[\text{keV}] = 88.5 \frac{E^4[\text{GeV}^4]}{\rho[\text{m}]} \tag{3.26}$$

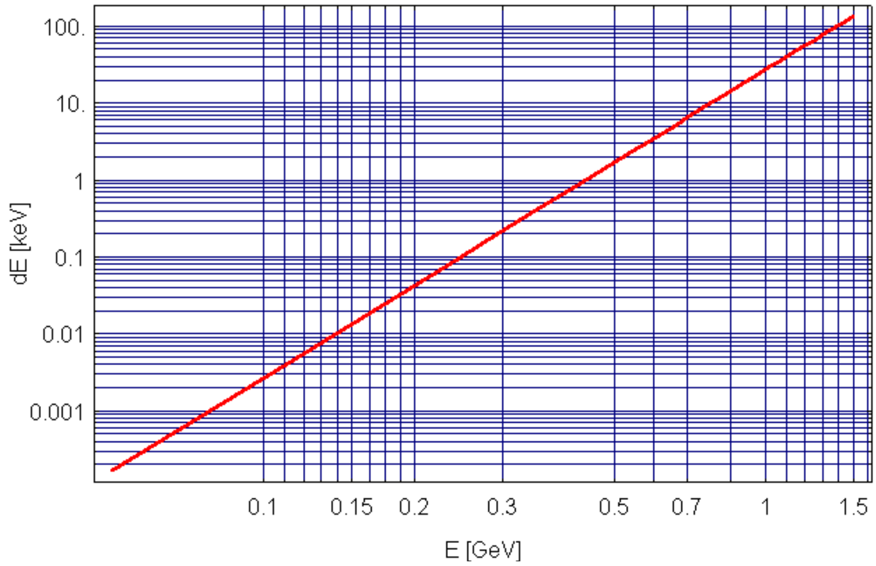


Fig. 3.2 Energy loss per revolution in the storage ring DELTA at the University of Dortmund as a function of the particle energy

The synchrotron radiation was investigated the first time by *Liénard* at the end of the last century. It was observed almost 50 years later at the 70 GeV-synchrotron of General Electric in the USA.

At high electron energies the bending radius of the magnets has to increase with higher power of the energy because of the relation

$$\Delta E \propto \frac{E^4}{\rho} \tag{3.27}$$

Table 3.1 Parameter of a few circular electron accelerators

	L [m]	E [GeV]	ρ [m]	B [T]	ΔE [keV]
BESSY I	62.4	0.80	1.78	1.500	20.3
DELTA	115	1.50	3.34	1.500	134.1
DORIS	288	5.00	12.2	1.370	$4.53 \cdot 10^3$
ESRF	844	6.00	23.4	0.855	$4.90 \cdot 10^3$
PETRA	2304	23.50	195.0	0.400	$1.38 \cdot 10^5$
LEP	$27 \cdot 10^3$	70.00	3000	0.078	$7.08 \cdot 10^5$

3.2 Spatial distribution of the radiation from a relativistic particle

The power per unit solid angle was given in (3.5) as

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 c\,\varepsilon_0} \dot{\beta}^{*2} \sin^2\Theta$$

for the radiation of a charged particle in the reference frame K^*. The angular distribution corresponds to that of the Hertz' dipole as shown in fig. 3.1. For relativistic particles the radiation pattern is significantly different. The radiation is focused forward into a narrow cone with the opening angle of approximately $1/\gamma$.

The radiation power per unit solid angle is according to (3.3)

$$\frac{dP}{d\Omega} = -\vec{n}\,\vec{S}'\,R^2 \tag{3.28}$$

With the relation (2.109) for the Poynting vector at the radiated time we get

$$\frac{dP}{d\Omega} = \frac{1}{c\,\mu_0}\,\vec{E}^2\left(1+\vec{n}\,\vec{\beta}\right)R^2 \tag{3.29}$$

Inserting the electrical field (2.93) and with the charge of an electron $q = e$ we find

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{c\,\mu_0}\,\frac{e^2}{\left(4\pi\varepsilon_0\right)^2}\,\frac{1}{c^2 a^6}\left\{\vec{R}\times\left[\left(\vec{R}+\vec{\beta}\,R\right)\times\dot{\vec{\beta}}\right]\right\}^2\left(1+\vec{n}\,\vec{\beta}\right)R^2\\ &= \frac{1}{c\,\mu_0}\,\frac{e^2}{\left(4\pi\varepsilon_0\right)^2}\,\frac{R^5}{c^2 a^5}\left\{\vec{n}\times\left[\left(\vec{n}+\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right\}^2\end{aligned} \tag{3.30}$$

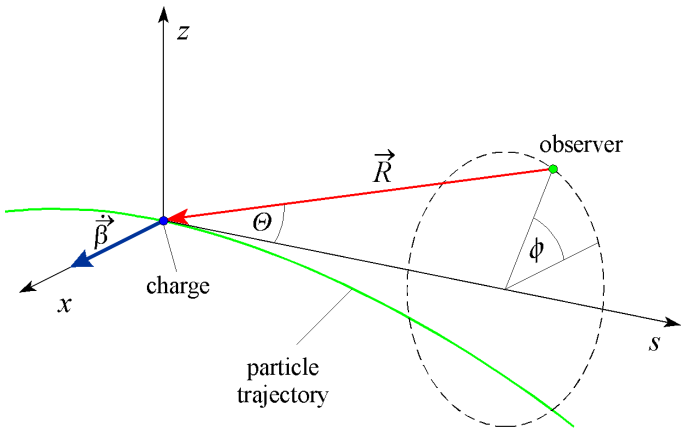


Fig. 3.3 The coordinate system of the moving charged particle

The vector $\vec{R}$ pointing from the observer to the moving particle is (see fig. 3.3)

$$\vec{R} = -R\begin{pmatrix}\sin\Theta\cos\phi\\ \sin\Theta\sin\phi\\ \cos\Theta\end{pmatrix} \tag{3.31}$$

and the correlated unit vector

$$\vec{n} = \begin{pmatrix}-\sin\Theta\cos\phi\\ -\sin\Theta\sin\phi\\ -\cos\Theta\end{pmatrix} \tag{3.32}$$

The *Lorentz force* of an electron traveling along a trajectory in a magnet is

$$\vec{F} = -e\,\vec{v}\times\vec{B} = -e\begin{pmatrix}-v\,B_z\\ 0\\ 0\end{pmatrix} = \gamma\,m_0\,\dot{\vec{v}} \tag{3.33}$$

with

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \qquad \dot{\vec{v}} = \begin{pmatrix} \dot{v}_x \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{B} = \begin{pmatrix} 0 \\ B_z \\ 0 \end{pmatrix} \tag{3.34}$$

A straight forward calculation yields

$$\gamma\, m_0\, \dot{v}_x = e\, v\, B_z = e\, c\, \beta\, B_z \tag{3.35}$$

On the other hand the bending radius of a trajectory in a magnet can be evaluated according to

$$\frac{1}{\rho} = \frac{e}{p} B_z = \frac{e\, B_z}{\gamma\, m_0 v} \qquad \Rightarrow \qquad B_z = \frac{\gamma\, m_0 v}{e \rho} \tag{3.36}$$

The transverse acceleration of the particle can now be written in the form

$$\dot{v}_x = \frac{c^2 \beta^2}{\rho} \tag{3.37}$$

With (3.34) and (3.37) we get

$$\vec{\beta} = \frac{\vec{v}}{c} = \begin{pmatrix} 0 \\ 0 \\ v/c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \tag{3.38}$$

and

$$\dot{\vec{\beta}} = \begin{pmatrix} \dot{v}_x / c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \left(c\beta^2\right)/\rho \\ 0 \\ 0 \end{pmatrix} \tag{3.39}$$

Using again the vector relation $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\left(\vec{a}\,\vec{c}\right) - \vec{c}\left(\vec{a}\,\vec{b}\right)$ the double product in (3.30) becomes

$$\begin{aligned}
\left\{\vec{n} \times \left(\left[\vec{n} + \vec{\beta}\right] \times \dot{\vec{\beta}}\right)\right\} &= \left(\vec{n} + \vec{\beta}\right)\left(\vec{n}\,\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\left(\vec{n}\left[\vec{n} + \vec{\beta}\right]\right) \\
&= \left(\vec{n} + \vec{\beta}\right)\left(\vec{n}\,\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\left(1 + \vec{n}\,\vec{\beta}\right) \\
&= \begin{pmatrix} -\sin\Theta\cos\phi \\ -\sin\Theta\sin\phi \\ \beta - \cos\Theta \end{pmatrix} \left(-\sin\Theta\cos\phi \frac{c\beta^2}{\rho}\right) - \begin{pmatrix} \left(c\beta^2\right)/\rho \\ 0 \\ 0 \end{pmatrix} \left(1 - \beta\cos\Theta\right) \\
&= \frac{c\beta^2}{\rho} \left\{ \begin{pmatrix} \sin^2\Theta\cos^2\phi \\ \sin^2\Theta\sin\phi\cos\phi \\ -\left(\beta - \cos\Theta\right)\sin\Theta\cos\phi \end{pmatrix} - \begin{pmatrix} 1 - \beta\cos\Theta \\ 0 \\ 0 \end{pmatrix} \right\}
\end{aligned} \tag{3.40}$$

The square of this expression is

$$\{\cdots\}^2 = \frac{c^2\beta^4}{\rho^2}\left\{\begin{pmatrix} \sin^2\Theta\ \cos^2\phi \\ \sin^2\Theta\sin\phi\cos\phi \\ -(\beta-\cos\Theta)\sin\Theta\cos\phi \end{pmatrix}^2 - 2\sin^2\Theta\cos^2\phi(1-\beta\cos\Theta) + (1-\beta\cos\Theta)^2\right\}$$

$$= \frac{c^2\beta^4}{\rho^2}\left\{\sin^4\Theta\cos^4\phi + \sin^4\Theta\sin^2\phi\cos^2\phi + (\beta-\cos\Theta)^2\sin^2\Theta\cos^2\phi - \right.$$

$$\left. - 2\sin^2\Theta\cos^2\phi(1-\beta\cos\Theta) + (1-\beta\cos\Theta)^2\right\} \qquad (3.41)$$

$$= \frac{c^2\beta^4}{\rho^2}\left\{\sin^4\Theta\cos^4\phi + \sin^4\Theta\sin^2\phi\cos^2\phi + \beta^2\sin^2\Theta\cos^2\phi - 2\beta\cos\Theta\sin^2\Theta\cos^2\phi + \right.$$

$$\left. + \cos^2\Theta\sin^2\Theta\cos^2\phi - 2\sin^2\Theta\cos^2\phi + 2\beta\cos\Theta\sin^2\Theta\cos^2\phi + (1-\beta\cos\Theta)^2\right\}$$

and

$$\{\cdots\}^2 = \frac{c^2\beta^4}{\rho^2}\left\{\sin^4\Theta\cos^2\phi\underbrace{(\cos^2\phi+\sin^2\phi-1)}_{=0} + (\beta^2-1)\sin^2\Theta\cos^2\phi + (1-\beta\cos\Theta)^2\right\}$$

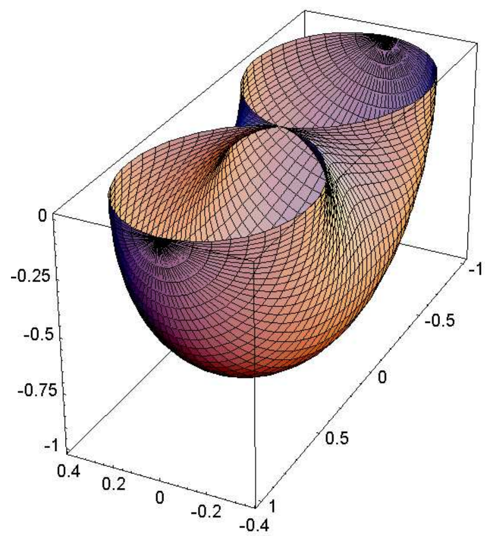
$$= \frac{c^2\beta^4}{\rho^2}\left\{(\beta^2-1)\sin^2\Theta\cos^2\phi + (1-\beta\cos\Theta)^2\right\} \qquad (3.42)$$

From the definition (2.69) we derive with (3.32) and (3.38)

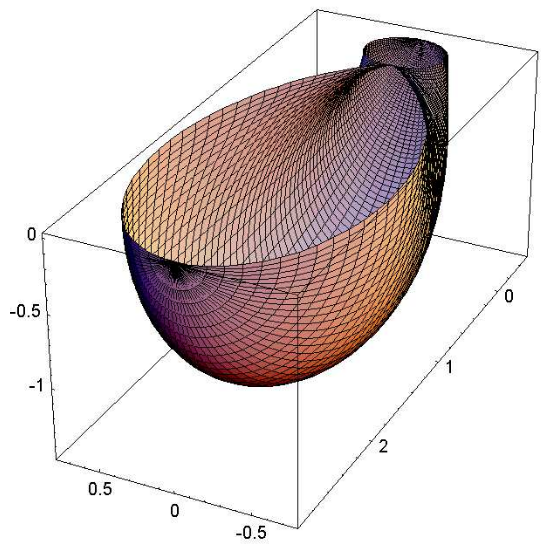
$$a = R(1+\vec{n}\vec{\beta}) = R(1-\beta\cos\Theta) \qquad (3.43)$$

We insert (3.42) and (3.43) into (3.30) and the radiated power per unit solid angle becomes

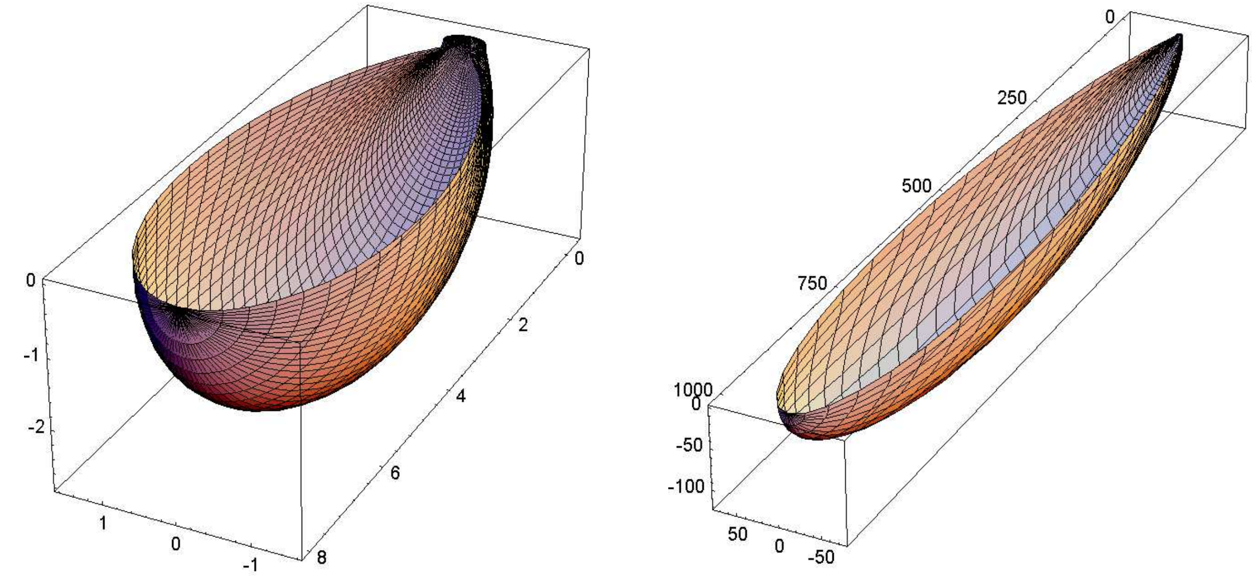
$$\boxed{\frac{dP}{d\Omega} = \frac{1}{c^3\mu_0}\frac{e^2}{(4\pi\varepsilon_0)^2}\frac{\beta^4}{\rho^2}\frac{(\beta^2-1)\sin^2\Theta\cos^2\phi + (1-\beta\cos\Theta)^2}{(1-\beta\cos\Theta)^5}} \qquad (3.44)$$



β = 0



β = 0.3



β = 0.5 **β = 0.9**

Fig. 3.4 Radiation pattern for different particle velocities between β = 0 and β = 0.9.

With the dimensionless particle energy

$$\gamma = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1+\beta^2}} \tag{3.45}$$

we vary the angle Θ between the direction of particle motion and the direction of photon emission according to

$$\Theta = \frac{u}{\gamma} \qquad (u = \text{ dimensionless number}) \tag{3.46}$$

and calculate the photon intensity using equation (3.44). The result is shown in fig. 3.5.

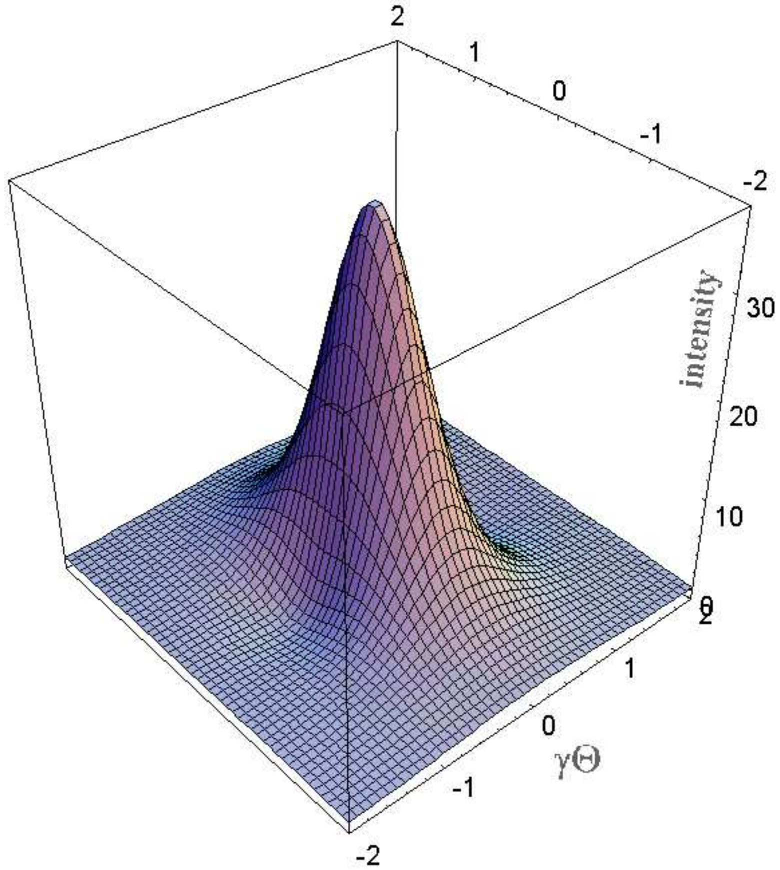


Fig. 3.5 Photon intensity of the synchrotron radiation as a function of the angle Θ in terms of $1/\gamma$

It is directly to see that the radiation is mainly concentrated within a cone of an opening angle of $\pm 1/\gamma$. In equation (3.44) we set $\phi = \pi/2$ and the fraction on the right hand side reduces to

$$w(\Theta) = \frac{1}{(1-\beta\cos\Theta)^3} \tag{3.47}$$

With the conditions $\gamma \gg 1$ and $\Theta \ll 1$ we find the approximations

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1-\frac{1}{2\gamma^2} \qquad \text{and} \qquad \cos\Theta \approx 1-\frac{\Theta^2}{2}$$

and we get from (3.47)

$$w(\Theta) \approx \left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\Theta^2}{2}\right)\right]^{-3} = \left[1-1+\frac{\Theta^2}{2}+\frac{1}{2\gamma^2}-\frac{\Theta^2}{4\gamma^2}\right]^{-3} \approx \left(\frac{\Theta^2}{2}+\frac{1}{2\gamma^2}\right)^{-3} \tag{3.48}$$

The peak intensity is at Θ = 0, i.e.

$$w(0) = \left(\frac{1}{2\gamma^2}\right)^{-3} \tag{3.49}$$

We chose now an angle of $\Theta = 1/\gamma$ and find the relation

$$\frac{w(1/\gamma)}{w(0)} = \frac{\left(\frac{1}{2\gamma^2}+\frac{1}{2\gamma^2}\right)^{-3}}{\left(\frac{1}{2\gamma^2}\right)^{-2}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \tag{3.50}$$

One can see that most of the radiation is emitted within the cone of $\Theta_s = 1/\gamma$. Therefore, the opening angle of the synchrotron radiation is given by this amount.

3.3 Time structure and radiation spectrum

In the following we will only present a phenomenological approach to the calculation of the photon spectrum of the synchrotron radiation. A detailed evaluation of the spectral functions can be derived in "J.D. Jackson, *Classical Electrodynamics*, Sect. 14" or in "H. Wiedemann, *Particle Accelerator Physics II*, chapter 7.4".

As shown above the synchrotron radiation is focused sharply within a cone of an opening angle $\Theta = 1/\gamma$. Therefore, an observer locking onto the particle trajectory while the electron passes a bending magnet (fig. 3.6) can see the radiation the first time when the electron has reached the point **A**.

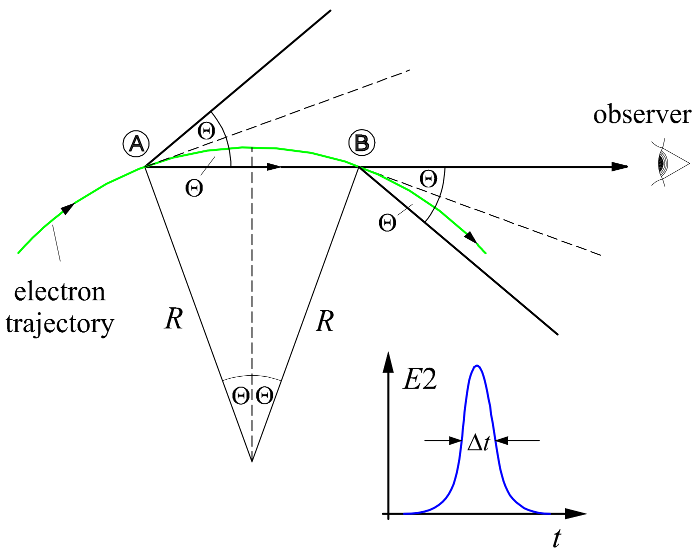


Fig. 3.6 Generation of a short flash of synchrotron light by an electron passing a bending magnet

The photons emitted at point **A** fly along a straight line directly to the observer with the velocity of light. The electron, however, takes the circular trajectory and its velocity is slightly less than the velocity of light. During this time the radiation cone strokes across the observer until the point **B** is reached. This is the last position from which radiation can be observed. The duration of the light flash is simply the difference of the time used by the electron and by the photon moving from the point A to point B

$$\Delta t = t_e - t_\gamma = \frac{2\rho\Theta}{c\beta} - \frac{2\rho\sin\Theta}{c} \tag{3.51}$$

or

$$\begin{aligned}\Delta t &= \frac{2\rho}{c}\left(\frac{\Theta}{\beta} - \Theta + \frac{\Theta^3}{3!} - \cdots\right)\\ &= \frac{2\rho}{c}\left(\frac{1}{\gamma - 1/2\gamma} - \frac{1}{\gamma} + \frac{1}{6\gamma^3}\right)\end{aligned} \tag{3.52}$$

With

$$\frac{1}{\gamma - 1/2\gamma} = \frac{1}{\gamma}\,\frac{1}{1-1/2\gamma^2} \approx \frac{1}{\gamma}\left(1+\frac{1}{2\gamma^2}\right) = \frac{1}{\gamma}+\frac{1}{2\gamma^3} \tag{3.53}$$

we get

$$\Delta t \approx \frac{2\rho}{c}\left(\frac{1}{\gamma}+\frac{1}{2\gamma^3}-\frac{1}{\gamma}+\frac{1}{6\gamma^3}\right) = \frac{4\rho}{3c\gamma^3} \tag{3.54}$$

In order to calculate the pulse length we assume a bending radius of ρ = 3.3 m and a beam energy of E = 1.5 GeV, i.e. γ = 2935. With this parameters the pulse length becomes

$$\Delta t = 5.8\cdot 10^{-19}\,\mathrm{sec} \tag{3.55}$$

This extremely short pulse causes a broad frequency spectrum with the *typical frequency*

$$\omega_{\mathrm{typ}} = \frac{2\pi}{\Delta t} = \frac{3\pi c\gamma^3}{2\rho}\,. \tag{3.56}$$

More often the *critical frequency*

$$\omega_{\mathrm{c}} = \frac{\omega_{\mathrm{typ}}}{\pi} = \frac{3c\gamma^3}{2\rho} \tag{3.57}$$

is used. The exact calculation of the radiation spectrum has been carried out the first time by *Schwinger*. He found

$$\frac{d\dot{N}}{d\varepsilon/\varepsilon} = \frac{P_0}{\omega_{\mathrm{c}}\hbar}\,S_{\mathrm{s}}\!\left(\frac{\omega}{\omega_{\mathrm{c}}}\right) \tag{3.58}$$

With the radiation power given in (3.23)

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0\left(m_0c^2\right)^4}\,\frac{E^4}{\rho^2}$$

the total power radiated by N electrons is

$$P_0 = \frac{e^2c\gamma^4}{6\pi\varepsilon_0\rho^2}\,N = \frac{e\gamma^4}{3\varepsilon_0\rho}\,I_b \tag{3.59}$$

with the beam current

$$I_b = \frac{N\,e\,c}{2\pi\,\rho} \tag{3.60}$$

The spectral function in (3.58) has the form

$$S_{\mathrm{s}}(\xi) = \frac{9\sqrt{3}}{8\pi}\,\xi\int_{\xi}^{\infty} K_{5/3}(\xi)\,d\xi \tag{3.61}$$

where $K_{5/3}(\xi)$ is the modified Bessel function and $\xi = \omega/\omega_{\mathrm{c}}$.

Because of energy conservation the spectral function satisfies the condition

$$\int_0^{\infty} S_s(\xi)\,d\xi = 1 \tag{3.62}$$

Integrating until the upper limit ξ = 1, i.e. $\omega = \omega_c$, gives

$$\int_0^1 S_s(\xi)\,d\xi = \frac{1}{2} \tag{3.63}$$

This result shows that the critical frequency ω_c divides the spectrum into two parts of identical radiation power. An example of a spectrum radiated from a bending magnet is shown in fig. 3.7.

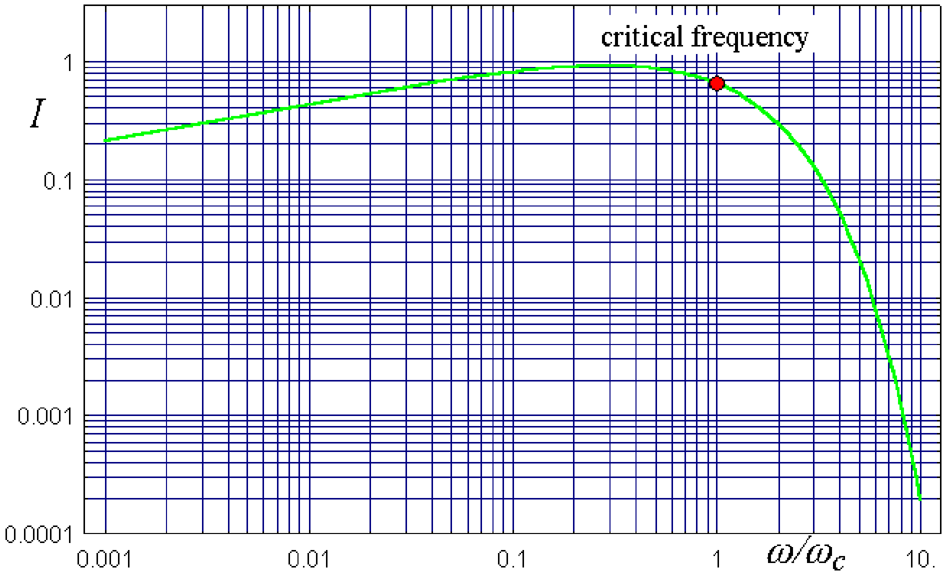


Fig. 3.7 Spectrum of the synchrotron radiation emitted by electrons with a kinetic energy of E = 1.5 GeV and a bending radius of ρ = 3.3 m

The radiation from a bending magnet is emitted within a horizontal fan as shown in fig. 3.8.

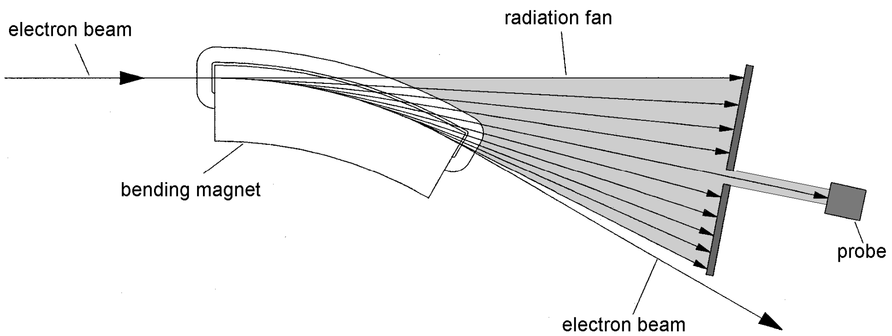


Fig. 3.8 Synchrotron radiation from a bending magnet

The broad spectrum emits in the visible regime almost white light as to be seen in fig. 3.9. Above the critical frequency the spectral intensity drops down rapidly.

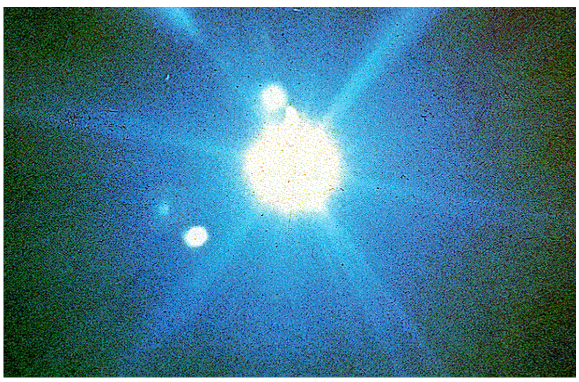


Fig. 3.9 The visible light emitted by relativistic electrons

4 Electron Dynamics with Radiation

4.1 The particles as harmonic oscillators

Because of the longitudinal and transverse focusing, the particles oscillate with respect to the reference phase of the rf-cavity or the beam orbit defined by the magnet structure. In a good approximation we can investigate the oscillations like a harmonic oscillator.

4.1.1 Synchrotron oscillation

In a circular accelerator as a synchrotron or a storage ring it is necessary to compensate the energy loss during the revolutions by a rf-cavity. Averaged over many revolutions the compensation must be perfect. Therefore, the so called "phase focusing" takes care of a stable phase of the particled with respect to the rf-voltage.

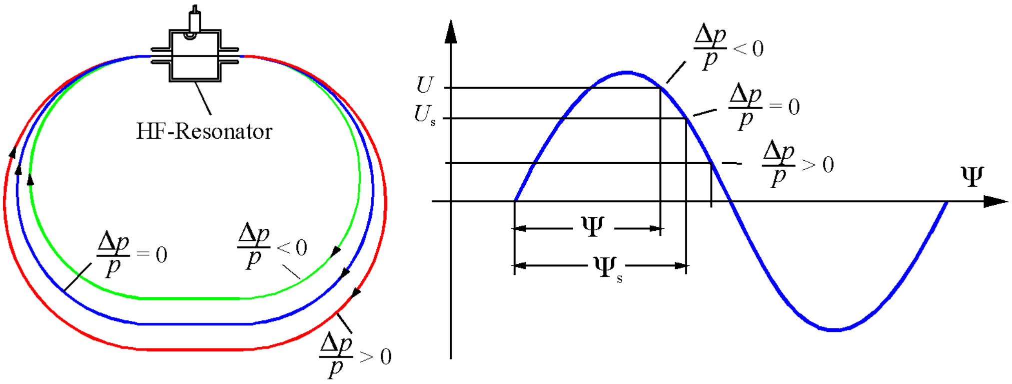


Fig. 4.1 Principal of the phase focusing in a cyclic machine

For an on-momentum particle ($\Delta p/p = 0$) the energy change per revolution is

$$E_0 = eU_0 \sin\Psi_s - W_0 \tag{4.1}$$

with the reference phase Ψ_s , the peak voltage U_0 and the energy loss W_0 due to synchrotron radiation. For any particle with a phase deviation $\Delta\Psi$ we find

$$E = eU_0 \sin(\Psi_s + \Delta\Psi) - W \tag{4.2}$$

The energy loss can be expanded as

$$W = W_0 + \frac{dW}{dE}\Delta E \tag{4.3}$$

The difference between (4.1) and (4.2) is

$$\Delta E = E - E_0 = eU_0\left[\sin(\Psi_s + \Delta\Psi) - \sin\Psi_s\right] - \frac{dW}{dE}\Delta E \tag{4.4}$$

Since the frequency of the phase oscillations is very low compared to the revolution frequency $f_u = 1/T_0$ the time derivative of (4.4) can therefore be written as

$$\Delta\dot{E} = \frac{\Delta E}{T_0} = \frac{eU_0}{T_0}\left[\sin(\Psi_s + \Delta\Psi) - \sin\Psi_s\right] - \frac{dW}{dE}\frac{\Delta E}{T_0} \tag{4.5}$$

The phase difference $\Delta\Psi$ is caused by the different revolution time of the particles with energy deviation. The time difference for relativistic particles is

$$\Delta T = T_0 \frac{\Delta L}{L_0} = T_0\,\alpha\,\frac{\Delta E}{E} \tag{4.6}$$

Here we have used the *momentum-compaction-factor* α defined as

$$\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p} \tag{4.7}$$

With the period of the rf-voltage T_{rf} the phase shift becomes

$$\Delta\Psi = 2\pi \frac{\Delta T}{T_{\mathrm{rf}}} = \omega_{\mathrm{rf}}\, \Delta T \tag{4.8}$$

The ratio of the rf-frequency and the revolution frequency must be an integer number

$$q = \frac{\omega_{\mathrm{rf}}}{\omega_{\mathrm{u}}} \qquad \text{with} \qquad q = \text{integer} \tag{4.9}$$

q is often called the harmonic number. Combining (4.6) and (4.8) we get

$$\Delta\Psi = q\,\omega_{\mathrm{u}} \Delta T = 2\pi\, q \frac{\Delta T}{T_0} = 2\pi\, q\, \alpha \frac{\Delta E}{E} \tag{4.10}$$

and after differentation

$$\Delta\dot{\Psi} = \frac{\Delta\Psi}{T_0} = \frac{2\pi\, q\, \alpha}{T_0} \frac{\Delta E}{E} \tag{4.11}$$

First we discuss only the case with small phase oscillations, i.e. $\Delta\Psi \ll \Psi_{\mathrm{s}}$. Then we can write

$$\begin{aligned} &\sin(\Psi_{\mathrm{s}} + \Delta\Psi) - \sin\Psi_{\mathrm{s}} \\ &= \sin\Psi_{\mathrm{s}} \cos\Delta\Psi + \cos\Psi_{\mathrm{s}} \sin\Delta\Psi - \sin\Psi_{\mathrm{s}} \\ &\approx \Delta\Psi \cos\Psi_{\mathrm{s}} \end{aligned} \tag{4.12}$$

With this appriximation equation (4.5) reduces to

$$\Delta\dot{E} = \frac{eU_0}{T_0} \Delta\Psi \cos\Psi_{\mathrm{s}} - \frac{dW}{dE} \frac{\Delta E}{T_0} \tag{4.13}$$

A second differentiation provides

$$\Delta\ddot{E} = \frac{eU_0}{T_0} \Delta\dot{\Psi} \cos\Psi_{\mathrm{s}} - \frac{dW}{dE} \frac{\Delta\dot{E}}{T_0} \tag{4.14}$$

Insertion of (4.11) gives

$$\Delta\ddot{E} + \frac{1}{T_0} \frac{dW}{dE} \Delta\dot{E} - \frac{2\pi\, q\, e\, \alpha\, U_0 \cos\Psi_{\mathrm{s}}}{T_0^2\, E} \Delta E = 0 \tag{4.15}$$

or

$$\boxed{\Delta\ddot{E} + 2a_{\mathrm{s}} \Delta\dot{E} + \Omega^2 \Delta E = 0} \tag{4.16}$$

with

$$a_{\mathrm{s}} = \frac{1}{2T_0} \frac{dW}{dE} \tag{4.17}$$

and

$$\Omega = \omega_{\mathrm{u}} \sqrt{-\frac{e\, U_0\, q\, \alpha \cos\Psi_{\mathrm{s}}}{2\pi\, E}} \tag{4.18}$$

The equation (4.16) can be solved by the ansatz

$$\Delta E(t) = \Delta E_0 \exp\omega\, t \tag{4.19}$$

Then we get

$$\omega = -a_s \pm \sqrt{a_s^2 - \Omega^2} \tag{4.20}$$

Since the damping is very weak ($a_s \ll \Omega$) the energy oscillation can be written in the form

$$\Delta E(t) = \Delta E_0 \exp\left(-a_s\, t\right) \exp\left(i\,\Omega t\right) \tag{4.21}$$

We have a damped harmonic oscillation with the frequency Ω. This oscillation is called the *synchrotron oscillation*.

4.1.2 Betatron oscillation

The motion of a charged particle through the magnet lattice of a cyclic accelerator can be expressed in linear approximation by the fundamental equations

$$\begin{gathered} x''(s) + \left(\frac{1}{\rho^2(s)} - k(s)\right) x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p} \\ z''(s) + k(s)\, z(s) = 0 \end{gathered} \tag{4.22}$$

where $\rho(s)$ and $k(s)$ give the bending radius and the quadrupole strength of the magnet lattice. Here only on-momentum particles are interesting and with $K(s) = 1/\rho^2(s) - k(s)$ we find for the horizontal plane

$$x''(s) + K(s)\, x(s) = 0 \tag{4.23}$$

In the vertical plane a similar equation holds. According to *Floquet's theorem* we find the solution

$$x(s) = \sqrt{\varepsilon}\, \sqrt{\beta(s)} \cos\left[\Psi(s) + \phi\right] \tag{4.24}$$

with the constant beam emittance ε and the variable but periodic betafunction $\beta(s)$. The phase also varies with the place along the orbit and can be expressed as

$$\Psi(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)} \tag{4.25}$$

The solution (4.24) is a transverse spatial particle oscillation with respect to the beam orbit. For ultra relativistic particles with $v = c$ there is a strong correlation between the position s at the orbit and the time t

$$s(t) = s_0 + c\, t \tag{4.26}$$

With this relation one can also understand the spatial oscillation (4.24) as a time dependent oscillation within the magnet structure. This transverse periodic particle motion is called *betatron oscillation*. The formalism in (4.22) contains no damping, it is only valid for particles without radiation. This is true for all particles of very low energies or for particles with high mass (see eq. (3.23)). In the case of high energy electrons we have a damping of the betatron oscillation. This damping will be introduced below.

4.2 Radiation damping

The damping needs under all circumstances an energy loss depending on the oscillation amplitude. The mechanism of the damping of particle oscillations is based on the emission of synchrotron radiation. This will be discussed in the following.

4.2.1 Damping of synchrotron radiation

The radiated power of the synchrotron radiation is (3.23)

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0} \frac{1}{\left(m_0 c^2\right)^4} \frac{E^4}{\rho^2} \tag{4.27}$$

The bending radius is

$$\frac{1}{\rho} = \frac{e}{p} B = \frac{ec}{E} B \quad \Rightarrow \quad \frac{E^2}{\rho^2} = e^2 c^2 B^2 \tag{4.28}$$

With this expression we can write the radiated power in the form

$$P_s = C E^2 B^2 \quad \text{with} \quad C = \frac{e^4 c^3}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4} \tag{4.29}$$

In order to evaluate the radiation damping of the synchrotron oscillation we use the equation (4.16)

$$\Delta\ddot{E} + 2a_s \Delta\dot{E} + \Omega^2 \Delta E = 0$$

with the damping constant (4.17)

$$a_s = \frac{1}{2T_0} \frac{dW}{dE}$$

It is necessary to calculate the ration dW/dE . For this purpose we estimate the energy loss along a dispersion trajectory with the element

$$ds' = \left(1 + \frac{\Delta x}{\rho}\right) ds \tag{4.30}$$

Using $ds'/dt = c$ we get the energy loss per revolution

$$W = \int_0^{T_0} P_s \, dt = \oint P_s \frac{ds'}{c} = \frac{1}{c} \oint P_s \left(1 + \frac{\Delta x}{\rho}\right) ds \tag{4.31}$$

The displacement Δx is caused by an energy deviation according to

$$\Delta x = D \frac{\Delta E}{E} \tag{4.32}$$

With this relation the energy loss becomes

$$W = \frac{1}{c} \oint P_s \left(1 + \frac{D}{\rho} \frac{\Delta E}{E}\right) ds \tag{4.33}$$

Differentiating gives

$$\frac{dW}{dE} = \frac{1}{c} \oint \left[\frac{dP_s}{dE} + \frac{D}{\rho}\left(\frac{dP_s}{dE} \frac{\Delta E}{E} + P_s \frac{1}{E}\right)\right] ds \tag{4.34}$$

The energy deviation ΔE performs periodic vibrations about the reference energy. After averaging over a long time the influence of the energy deviation vanishes

$$\left\langle \frac{\Delta E}{E} \right\rangle = 0 \tag{4.35}$$

Equation (4.34) becomes

$$\frac{dW}{dE} = \frac{1}{c} \oint \left[\frac{dP_s}{dE} + \frac{D\,P_s}{\rho\,E}\right] ds \tag{4.36}$$

For further calculations we need an expression for dP_s/dE. We use the radiation formula (4.29) and get

$$\frac{dP_s}{dE} = 2CEB^2 + 2CE^2B\frac{dB}{dE} = 2P_s\left(\frac{1}{E} + \frac{1}{B}\frac{dB}{dE}\right) \tag{4.37}$$

In quadrupoles with non vanishing dispersion the field variation with the particle energy is

$$\frac{dB}{dE} = \frac{dB}{dx}\frac{dx}{dE} = \frac{dB}{dx}\frac{D}{E} \tag{4.38}$$

It is put into the expression (4.37) and we get from (4.36)

$$\begin{aligned}\frac{dW}{dE} &= \frac{1}{c}\oint\left[2P_s\left(\frac{1}{E} + \frac{D}{BE}\frac{dB}{dx}\right) + P_s\frac{D}{\rho E}\right]ds \\ &= \underbrace{\frac{2}{cE}\oint P_s\,ds}_{\frac{2W_0}{E}} + \frac{1}{cE}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx} + \frac{1}{\rho}\right)ds\end{aligned} \tag{4.39}$$

With (4.17) the damping constant is then

$$\boxed{\alpha_s = \frac{1}{2T_0}\frac{dW}{dE} = \frac{W_0}{2T_0E}\left[2 + \frac{1}{cW_0}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx} + \frac{1}{\rho}\right)ds\right]} \tag{4.40}$$

or

$$\boxed{\alpha_s = \frac{W_0}{2T_0E}(2 + \mathcal{D})} \tag{4.41}$$

with

$$\mathcal{D} = \frac{1}{cW_0}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx} + \frac{1}{\rho}\right)ds \tag{4.42}$$

For practical use it is more convenient to apply the bending radius ρ and the quadrupole strength k rather than the magnetic field and its gradient. From the definition of the magnet parameter we can derive

$$\left.\begin{aligned} k = \frac{ec}{E}\frac{dB}{dx} &\rightarrow \frac{dB}{dx} = \frac{kE}{ec} \\ \frac{1}{\rho} = \frac{ec}{E}B &\rightarrow \frac{1}{B} = \frac{ec}{E}\rho \end{aligned}\right\} \Rightarrow \frac{1}{B}\frac{dB}{dx} = k\rho \tag{4.43}$$

In addition we write the radiation power in the form

$$P_s = \frac{C}{e^2c^2}\frac{E^4}{\rho^2} \tag{4.44}$$

Then the integral (4.42) becomes

$$\begin{aligned}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx} + \frac{1}{\rho}\right)ds &= \frac{CE^4}{e^2c^2}\oint\frac{D}{\rho^2}\left(2k\rho + \frac{1}{\rho}\right)ds \\ &= \frac{CE^4}{e^2c^2}\oint\frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right)ds\end{aligned} \tag{4.45}$$

The energy radiated by an on-momentum particle is

$$W_0 = \int_0^{T_0} P_s dt = \frac{1}{c}\oint P_s ds = \frac{CE^4}{e^2c^3}\oint \frac{ds}{\rho^2} \tag{4.46}$$

and we modify the damping constant for the synchrotron oscillation as

$$a_s = \frac{W_0}{2T_0 E}(2 + \mathcal{D}) \quad \text{with} \quad \mathcal{D} = \frac{\oint \frac{D}{R}\left(2k + \frac{1}{\rho^2}\right) ds}{\oint \frac{ds}{\rho^2}} \tag{4.47}$$

It is important to mention that the damping only depends on the magnet structure of the machine. It is possible to change the damping by varying the function $\mathcal{D}$. In particular for $\mathcal{D} < -2$ the synchrotron damping is disappeared and the beam is unstable. This would happen using an alternating gradient synchrotron (*combined function magnets*) as a storage ring with constant fields. In existing synchrotrons with combined function magnets antidamping is compensated by the adiabatic damping during the acceleration.

4.2.2 Damping of betatron oscillations

We will now discuss the damping of the transverse particle oscillations. Following *Floquet's transformation* we can write

$$\left.\begin{aligned} z &= b\sqrt{\beta(s)}\cos\phi \\ z' &= -\frac{b}{\sqrt{\beta(s)}}\sin\phi \end{aligned}\right\} \quad \underset{\Rightarrow}{A := b\sqrt{\beta(s)}} \quad \left\{\begin{aligned} z &= A\cos\phi \\ z' &= -\frac{A}{\beta(s)}\sin\phi \end{aligned}\right. \tag{4.48}$$

Then we can calculate the amplitude *A* using the trajectory parameter *z* and *z'*.

$$A^2 = A^2\cos^2\phi + A^2\sin^2\phi = z^2 + \left[\beta(s)z'\right]^2 \tag{4.49}$$

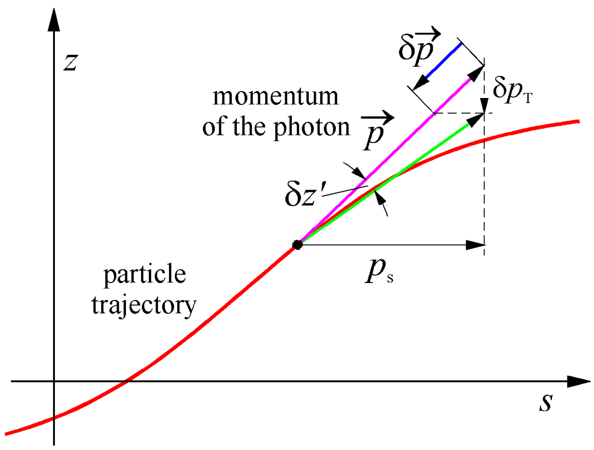


Fig. 4.2 The damping of the transversal particle oscillations

A photon is emitted in the direction of particle motion and the particle momentum $\vec{p}$ is reduced by $\delta\vec{p}$. The electron momentum is then

$$\vec{p}^{\,*} = \vec{p} - \delta\vec{p} \tag{4.50}$$

The longitudinal component p_s of the particle momentum is restored by the rf-cavity, the transverse component, however, stays reduced. Accordingly, the angle *z'* is reduced by the amount

$$\delta z' = -\frac{\delta p_\perp}{|\vec{p}|} \tag{4.51}$$

The energy variation of the ultra-relativistic electron is then

$$\delta E = \frac{c^2}{v}\delta p_\perp \tag{4.52}$$

or using $v = z'c$

$$\delta E = \frac{c}{z'}\delta p_\perp\,. \tag{4.53}$$

With the relation $E = c|\vec{p}|$ follows

$$\delta z' = -\frac{\delta E}{E}z' \tag{4.54}$$

From (4.49) we get the variation (z does not change (!))

$$\delta\left(A^2\right) = \underbrace{\delta\left(z^2\right)}_{=0} + \delta\left(z'^2\beta^2(s)\right) = \beta^2(s)\,\delta\left(z'^2\right) \tag{4.55}$$

and we find

$$2A\delta A = 2\beta^2(s)\,z'\,\delta z' \quad \Rightarrow \quad A\delta A = \beta^2(s)\,z'\delta z' \tag{4.56}$$

After insertion of (4.54) we get

$$A\delta A = -\beta^2(s)\,z'^2\frac{\delta E}{E} \tag{4.57}$$

Now one has to average over z'^2. Taking the formula (4.48) gives

$$\left\langle z'^2\right\rangle = \frac{A^2}{2\pi\beta^2(s)}\int_0^{2\pi}\sin^2\phi\,d\phi = \frac{A^2}{2\beta^2(s)} \tag{4.58}$$

In this way we find with the relation (4.57)

$$A\langle\delta A\rangle = -\frac{A^2}{2\beta^2(s)}\beta^2(s)\frac{\delta E}{E} = -\frac{A^2}{2}\frac{\delta E}{E} \tag{4.59}$$

After a full revolution the energy losses δE have accumulated to the total loss W_0. The average amplitude variation per revolution is then

$$\Delta A = \sum\langle\delta A\rangle \tag{4.60}$$

From equation (4.59) we get

$$\frac{\Delta A}{A} = -\frac{W_0}{2E} \tag{4.61}$$

Obviously the amplitude decreases, i.e. we have a damping of the betatron oscillation. The damping constant can be evaluated according to

$$\frac{dA}{A} = -a_z\,dt \tag{4.62}$$

With the revolution time $\Delta t = T_0$ we finally find

$$\boxed{a_z = -\frac{\Delta A}{A \Delta t} = \frac{W_0}{2ET_0}} \tag{4.63}$$

A similar calculations which includes the dispersion function provides the expression for the horizontal damping constant

$$\boxed{a_x = \frac{W_0}{2ET_0}(1-\mathbf{D})} \tag{4.64}$$

4.3 The Robinson theorem

With the equations (4.47), (4.63) and (4.64) we have derived the damping constants for the longitudinal synchrotron oscillation and the both transverse betatron oscillations:

$$\begin{aligned} a_s &= \frac{W_0}{2T_0 E}(2+\mathbf{D}) = \frac{W_0}{2T_0 E} J_s \\ a_z &= \frac{W_0}{2T_0 E} = \frac{W_0}{2T_0 E} J_z \\ a_x &= \frac{W_0}{2T_0 E}(1-\mathbf{D}) = \frac{W_0}{2T_0 E} J_x \end{aligned} \tag{4.65}$$

with

$$\begin{aligned} J_s &= 2+\mathbf{D} \\ J_z &= 1 \\ J_x &= 1-\mathbf{D} \end{aligned} \tag{4.66}$$

From these relations we can directly derive the *Robinson criteria*

$$\boxed{J_x + J_z + J_s = 4} \tag{4.67}$$

The total damping is constant. The change of the damping partition is possible by varying the quantity

$$\mathbf{D} = \frac{\oint \frac{D}{R}\left(2k + \frac{1}{R^2}\right) ds}{\oint \frac{ds}{R^2}} \tag{4.68}$$

In most of the cases and in particular we have $\mathbf{D} \ll 1$. This condition is called the "natural damping partition". In strong focusing machines it is possible to shift the particles onto a dispersion trajectory by variation of the particle energy. With this measure one can change the value of $\mathbf{D}$ within larger limits. The trajectory circumference L depends on the rf-frequency f as

$$L = q\lambda = q\frac{c}{f} \quad \Rightarrow \quad dL = -qc\frac{df}{f^2} \tag{4.69}$$

We get

$$\frac{\Delta L}{L} = -\frac{qc}{L}\frac{\Delta f}{f^2} = -\frac{\Delta f}{f} \tag{4.70}$$

With the momentum compaction factor we get

$$\frac{\Delta L}{L} = \alpha \frac{\Delta E}{E} \quad \Rightarrow \quad \frac{\Delta E}{E} = \frac{1}{\alpha}\frac{\Delta L}{L} = -\frac{1}{\alpha}\frac{\Delta f}{f} \tag{4.71}$$

The variation of the rf-frequency f shifts the beam onto the dispersion trajectory

$$x_{\mathrm{D}}(s) = -D(s)\frac{1}{\alpha}\frac{\Delta f}{f} \tag{4.72}$$

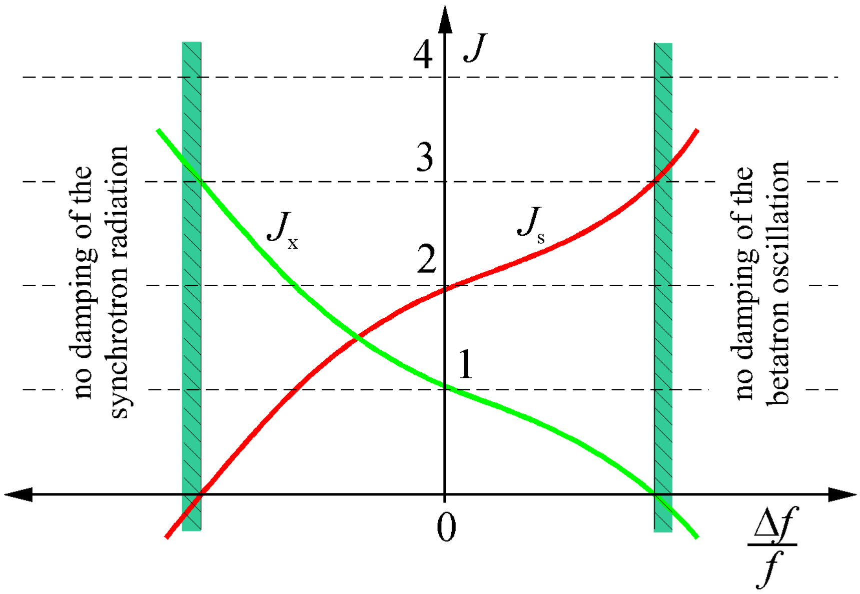


Fig. 4.3 Variation of the damping partition by changing the rf.frequency

Particles traveling along a dispersion trajectory pass through a quadrupole off-axis. Then the quads act like a combined function magnet and the amount of **D** increases or decreases depending on the frequency shift. The result is a change of the damping partition as shown in Fig. 4.3.

5 Particle distribution in the transversal phase space

5.1 Transversal beam emittance

The natural beam emittance is determined by the emission of synchrotron radiation. This happens only in the bending magnets and therefore only effects in the bending magnet have to be taken into account.

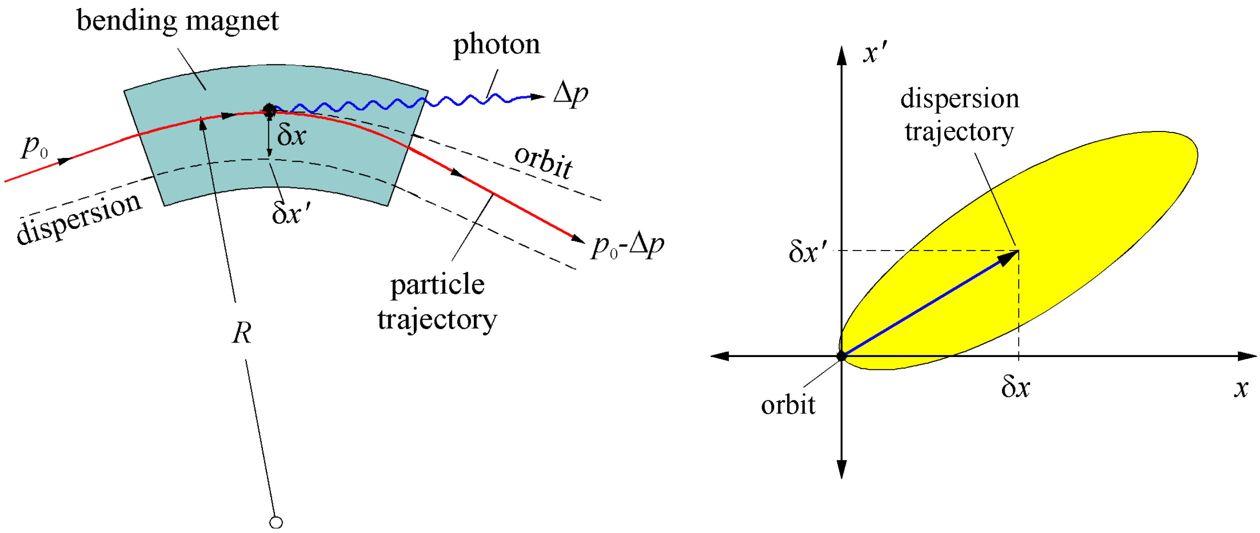


Fig. 5.1 Generation of betatron oscillations by emission of a photon

We start with an electron traveling along the ideal orbit with the reference momentum p_0. The emittance is then $\varepsilon_i = 0$. In the dipole the particle emits a photon with the momentum Δp and continues the flight with the momentum $p_0 - \Delta p$. It now belongs to a dispersion trajectory with the displacement and angle

$$\delta x = D\frac{\Delta p}{p} \qquad \text{and} \qquad \delta x' = D'\frac{\Delta p}{p} \tag{5.1}$$

with respect to the orbit. As a consequence it starts oscillating after the emission of a photon and has therefore a finite emittance. It can be calculated using the ellipse relation.

$$\begin{aligned}\varepsilon_i &= \gamma\delta x^2 + 2\alpha\delta x\delta x' + \beta\delta x'^2 \\ &= \left(\frac{dp}{p}\right)^2\left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right) \\ &= \left(\frac{dp}{p}\right)^2 \mathbf{H}\,(s)\end{aligned} \tag{5.2}$$

This relation is correct only for one certain single electron. To get the beam emittance one had to integrate over all particles in the beam, or, with other words, over the energy distribution of the electrons. For relativistic particles is

$$\frac{\Delta p}{p} = \frac{\Delta E}{E} \tag{5.3}$$

A similar calculation as for the bunch length gives the natural beam emittance in the form

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar c}{m_0 c^2} \gamma^2 \frac{\left\langle \frac{1}{R^3} \mathcal{H}(s) \right\rangle}{J_x \left\langle \frac{1}{R^2} \right\rangle} \tag{5.4}$$

The damping is represented by the amount $J_x = 1 - \mathcal{D}$. The averaging $\langle \dots \rangle$ has to be done only in the bending magnets. If all bending magnets are equal, i.e. they have the same bending radius R and the same length l, we get with $J_x \approx 1$ the simplified expression

$$\varepsilon_x = 1.47 \cdot 10^{-6} \frac{E^2}{Rl} \int_0^l \mathcal{H}(s) ds \tag{5.5}$$

In this formula we have E in [GeV], R in [m] and ε_x in [m rad]. One can directly see that because of

$$\mathcal{H}(s) = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2 \right) \tag{5.6}$$

the emittance is small whenever the betafunction and the dispersion is small inside a bending magnet. Circular electron machines for low emittance beams need therefore a magnet focusing providing small waists for the optical functions.

5.2 Examples

Increasing the quadrupole strength decreases within a usual range the betafunctions and the dispersion. This consequently reduces the function $\mathcal{H}(s)$ and produces a lower beam emittance. We can demonstrate such behavior taking a cyclic machine with a simple so called "FODO-lattice". In this case we have a succession of a focussing quadrupole (F), a drift space with a bending magnet (0), a defocusing quadrupole (D) and again a drift space with a bending magnet (0). This explains the name FODO-lattice. An example for a machine with such magnet structure is shown in fig. 5.2.

The chosen parameters of the cell allow the variation of both quadrupole strengths within a range from k = 0.4 m^{-2} to k = 1.6 m^{-2}. Values between this limits give stable optics. In fig. 6.3 the beam emittance is shown as a function of the quadrupole strengths. Here for simplicity the gradients for both quadrupole families have been set always to the same value. Variation of k from 0.4 m^{-2} to ≈1.5 m^{-2} reduces the emittance almost by two orders of magnitude !

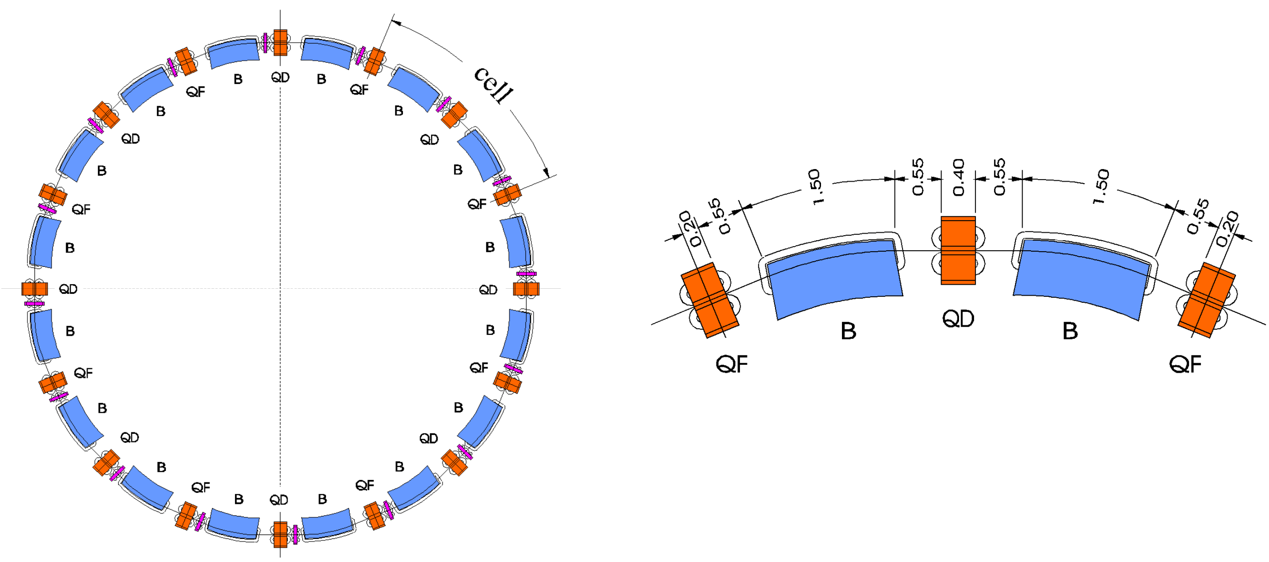


Fig. 5.2 A simple ring with FODO-structure. On the right hand side one cell of the lattice is drawn

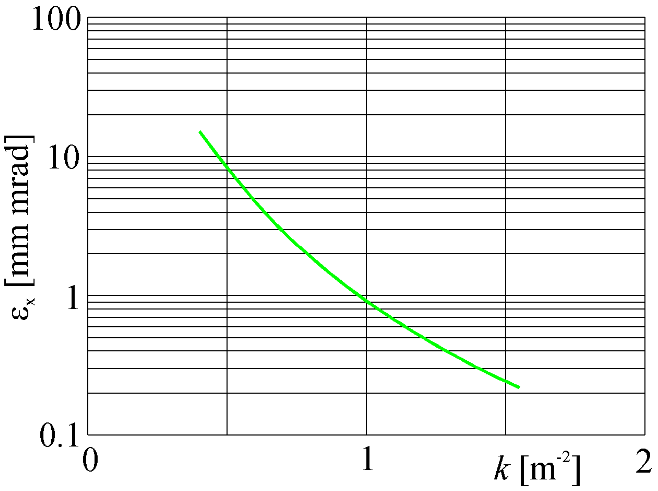


Fig. 5.3 Beam emittance as a function of the quadrupole strengths.

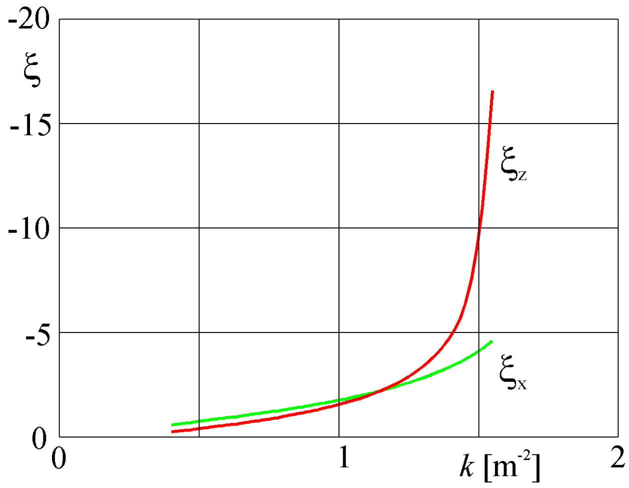


Fig. 5.4 Chromaticity as a function of the quadrupole strengths.

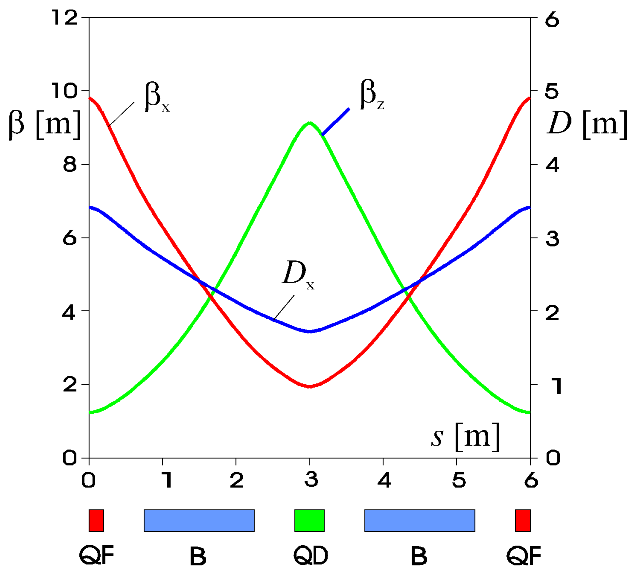


Fig. 5.5 Optical functions of the FODO lattice

The strong focusing with the low beam emittance has, however, a significant disadvantage. With increasing quadrupole strength, the chromaticity increases rapidly as shown in fig. 5.4. Machines with extremely low beam emittances, i.e. dedicated synchrotron light sources, need a very effective sextupole structure for chromaticity compensation. The main problem is the reduction of the dynamic aperture by the strong nonlinear magnetic fields.

The betafunction and the dispersion have in the bending magnet not the minimum value. Therefore, the FODO lattice provides for given bending magnets not necessarily extremely low emittances.

Much lower beam emittances are available with the "triplett-structure", as shown in fig. 5.6. In this case between the bending magnets three quadrupoles are arranged, namely QD-QF-QD. The resulting optical functions have inside the bending magnet a waist. The smallest values of the horizontal optical functions are now in the bending magnet, which gives low amounts for $H(s)$.

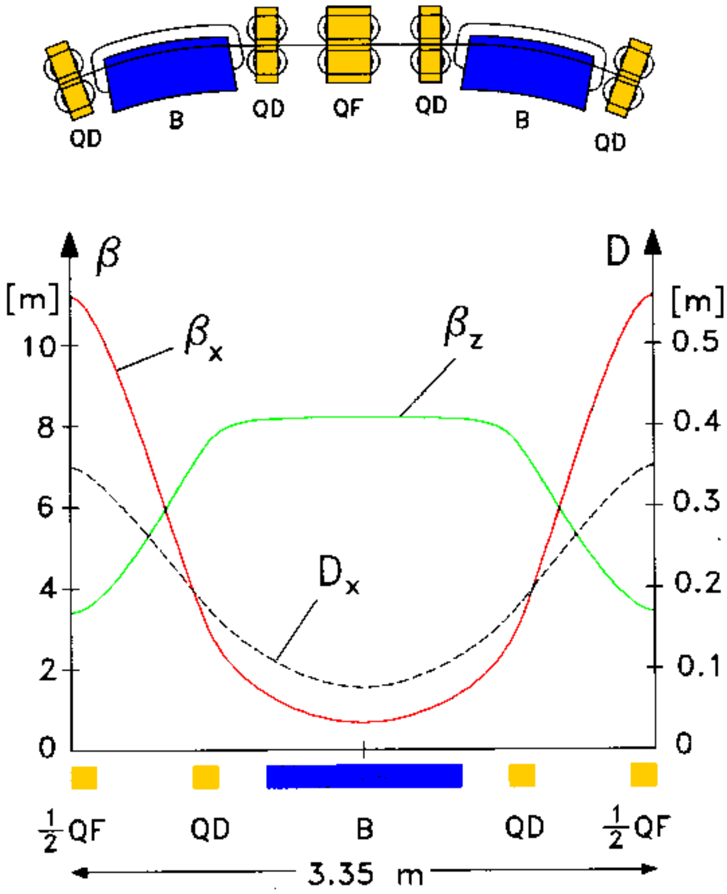


Fig. 5.6 Triplet structure and its optical functions

This structure has been used for the electron storage ring DELTA at the University of Dortmund. The emittance at an beam energy of E = 1.5 GeV is $\varepsilon_x = 7{\cdot}10^{-9}$ m rad. This is state of the art in modern synchrotron radiation sources.

6 Low emittance lattices

6.1 Basic idea of low emittance lattices

What is the lowest possible beam emittance ?

In electron storage rings optimized as dedicated synchrotron radiation sources long straight sections for wiggler and undulator magnets are required. This straight sections have usually no dispersion, i.e. $D \equiv 0$. Therefore, at the beginning of the bending magnet next to the insertion the dispersion has the initial value

$$\begin{pmatrix} D_0 \\ D_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.1}$$

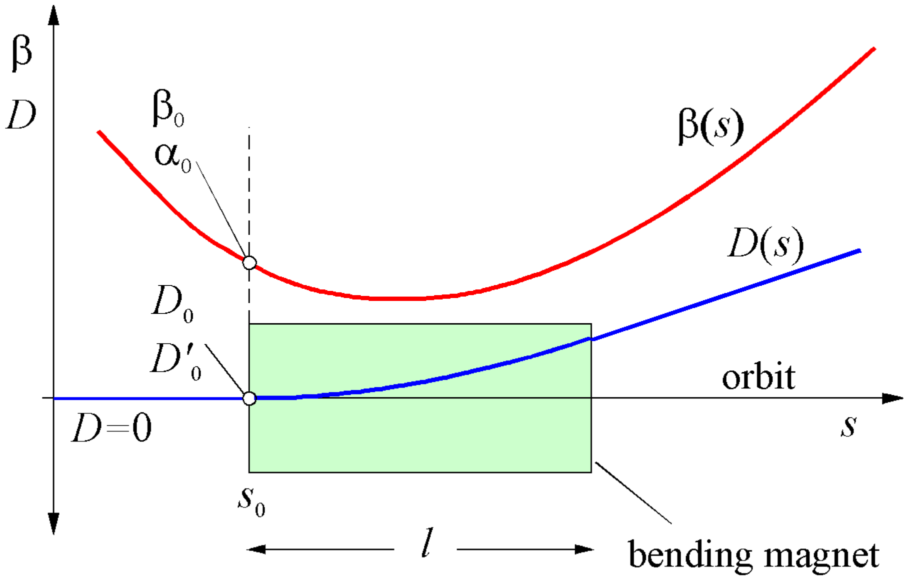


Fig. 6.1 Optical functions at in a bending magnet for minimum possible emittance

With this initial condition the dispersion in the bending magnet is well defined. With $s/R < 1$ we get

$$\begin{aligned} D(s) &= R\left(1-\cos\frac{s}{R}\right) \approx \frac{s^2}{2R} \\ D'(s) &= \sin\frac{s}{R} \approx \frac{s}{R} \end{aligned} \tag{6.2}$$

Under these conditions the emittance can only be changed by varying the initial values β_0 and α_0 of the betafunction. These functions can be transformed in the bending magnet as

$$\begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \tag{6.3}$$

and after straight forward calculations

$$\begin{aligned} \beta(s) &= \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) &= \alpha_0 - \gamma_0 s \\ \gamma(s) &= \gamma_0 = \text{const.} \end{aligned} \tag{6.4}$$

With this results we can write the function $\mathbf{H}(s)$ in the form

$$\begin{aligned} \mathcal{H}(s) &= \gamma(s)D^2(s) + 2\alpha(s)D(s)D'(s) + \beta(s)D'^2(s) \\ &= \frac{1}{R^2}\left(\frac{\gamma_0}{4}s^4 - \alpha_0 s^3 + \beta_0 s^2\right) \end{aligned} \tag{6.5}$$

For identical bending magnets and with $J_x = 1$ we get from (5.4) or (5.5)

$$\begin{aligned} \varepsilon_x &= C_\gamma \frac{\gamma^2}{Rl}\int_0^l \mathcal{H}(s)ds \\ &= C_\gamma \gamma^2 \left(\frac{l}{R}\right)^3 \left(\frac{\gamma_0 l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right) \end{aligned} \tag{6.6}$$

with

$$C_\gamma = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_0 c} = 3.832\cdot 10^{-13}\,\mathrm{m} \tag{6.7}$$

The relation

$$\frac{l}{R} = \Theta \tag{6.8}$$

is the bending angle of the magnet. With this expression we can write

$$\boxed{\varepsilon_x = C_\gamma \gamma^2 \Theta^3 \left(\frac{\gamma_0 l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right)} \tag{6.9}$$

Since the emittance grows with Θ^3 one should use many short bending magnets rather than a few long ones to get beams with low emittances.

In order to get the minimum possible emittance we have to vary the initial conditions β_0 and α_0 in (6.9) until the minimum is found. This is the case if

$$\begin{aligned} \frac{\partial \varepsilon_x}{\partial \alpha_0} &= \mathsf{A}\frac{\partial}{\partial \alpha_0}\left(\frac{1+\alpha_0^2}{\beta_0}\frac{l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right) \\ &= \mathsf{A}\left(\frac{\alpha_0}{\beta_0}\frac{l}{10} - \frac{1}{4}\right) = 0 \end{aligned} \tag{6.10}$$

and

$$\frac{\partial \varepsilon_x}{\partial \beta_0} = \mathsf{A}\left(-\frac{1+\alpha_0^2}{\beta_0^2}\frac{l}{20} + \frac{1}{3}\right) = 0 \tag{6.11}$$

with $\mathsf{A} = C_\gamma \gamma^2 \Theta^3$. With the two equations (6.10) and (6.11) we can calculate the unknown initial conditions β_0 and α_0. We get

$$\boxed{\begin{aligned} \beta_{0,\min} &= 2\sqrt{\frac{3}{5}}\,l = 1.549l \\ \alpha_{0,\min} &= \sqrt{15} = 3.873 \end{aligned}} \tag{6.12}$$

The minimum possible emittance is therefore determined only by the magnet length l.

This principle is used by the *Chasman Green lattice*, as shown in fig. 6.2. Looking into the details one will find that the optical functions do not exactly fit the conditions (6.12). In particular we have in realistic beam optics $\alpha_0 < \sqrt{15}$. The reason is the extremely high chromaticity caused by the ideal initial conditions (6.12). The reduction of the dynamic aperture would be too large.

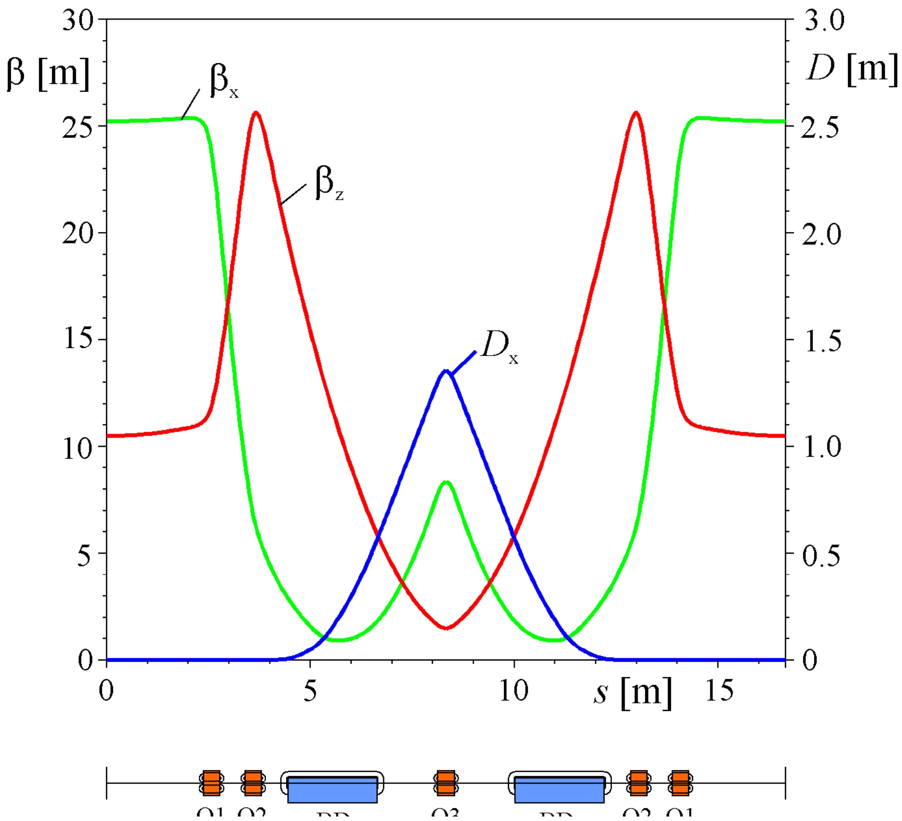


Fig. 6.2 An example of a *Chassman Green lattice* (HiSOR project, Japan)

The simple magnet structure in fig. 7.2 has no flexibility. Therefore, more quadrupole magnets are used in modern light sources as the ESRF in Grenoble (fig. 6.3 and 6.4)

Fig. 6.3 Site of the European Synchrotron Radiation Facility ESRF in Grenoble

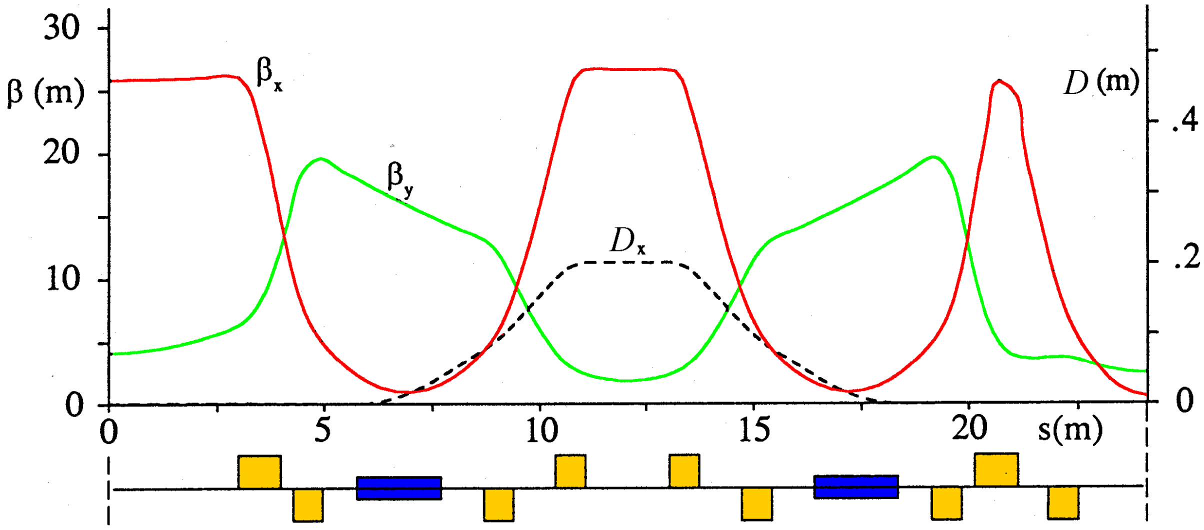


Fig. 6.4 The optical functions of one cell of the ESRF lattice.

Magnet structures of this type are often called "*double bend achromat lattice*" (DBA). Another modification of this optical principle is the "*triple bend achromat lattice*" (TBA), as applied in the storage ring BESSY II in Berlin (fig. 6.5).

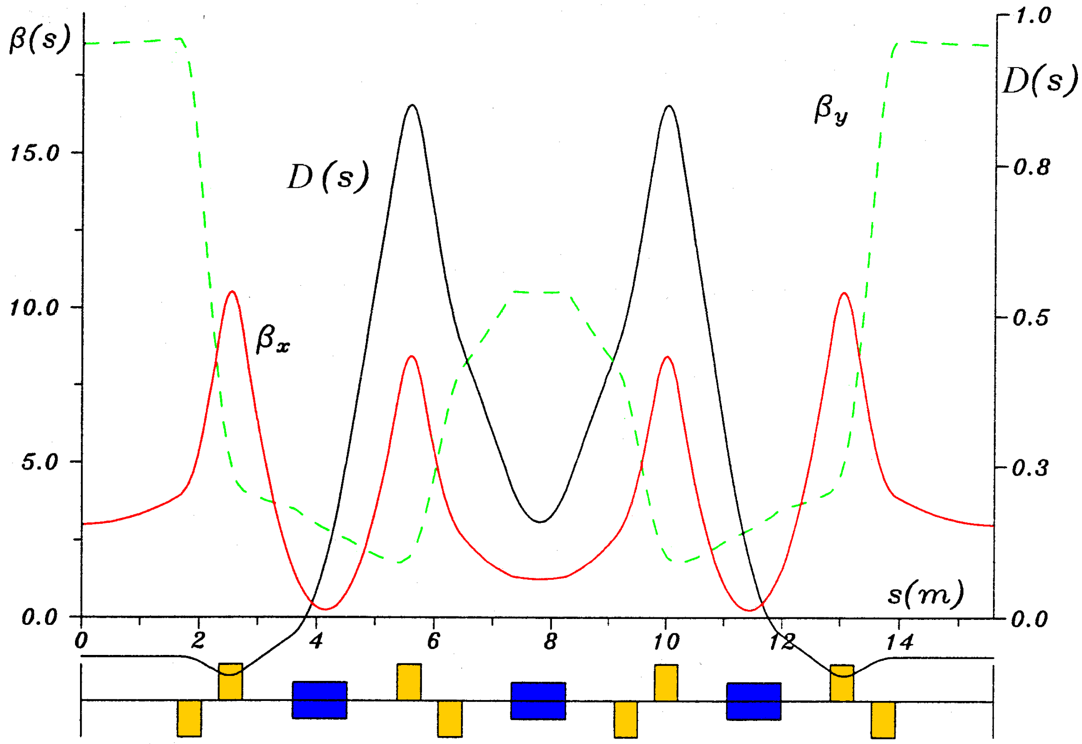


Fig. 6.5 The optical functions of one cell of BESSY II in Berlin

7 Appendix A: Undulator radiation

Synchrotron radiation is nowadays mostly generated by use of undulators (or „insertion devices“).

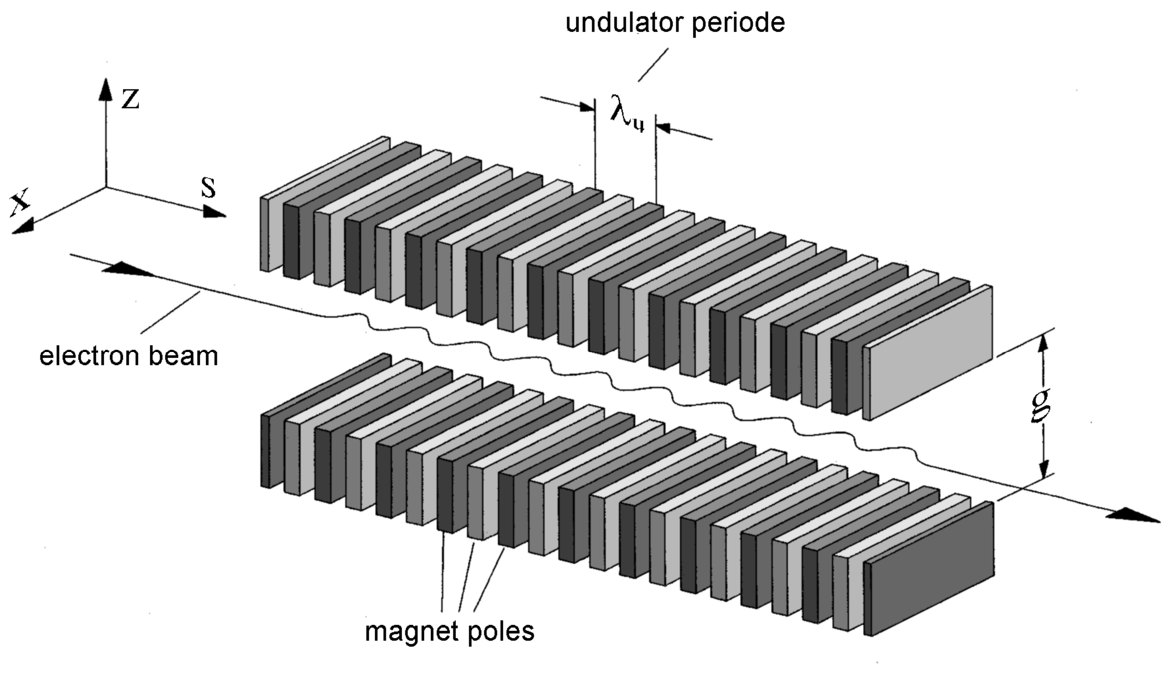


Fig. 7.1 Principal of an wiggler or undulator manet

This is a magnet with a larger number of short dipoles with alternating polarity. The difference between „wiggler“ and „undulators“ is mainly given by the magnet strength and will be defined later. First we will call it W/U-magnet.

7.1 The field of a wiggler or undulator

Along the orbit one has a periodic field with the period length λ_u. The potential is

$$\varphi(s,z) = f(z)\cos\left(2\pi\frac{s}{\lambda_u}\right) = f(z)\cos(k_u s). \tag{7.1}$$

In *x*-direction the magnet is assumed to be unlimited.

The function *f*(*z*) gives the vertical field pattern. With the *Laplace equation*

$$\nabla^2\varphi(s,z) = 0 \tag{7.2}$$

we get

$$\frac{d^2 f(z)}{dz^2} - f(z)k_u^2 = 0 \tag{7.3}$$

and find the solution

$$f(z) = A\sinh(k_u z) \tag{7.4}$$

Inserting into (7.1) the potential becomes

$$\varphi(s,z) = A\sinh(k_u z)\cos(k_u s) \tag{7.5}$$

and the vertical field component

$$B_z(s,z) = \frac{\partial\varphi}{\partial z} = k_u A \cosh(k_u z)\cos(k_u s)\,. \tag{7.6}$$

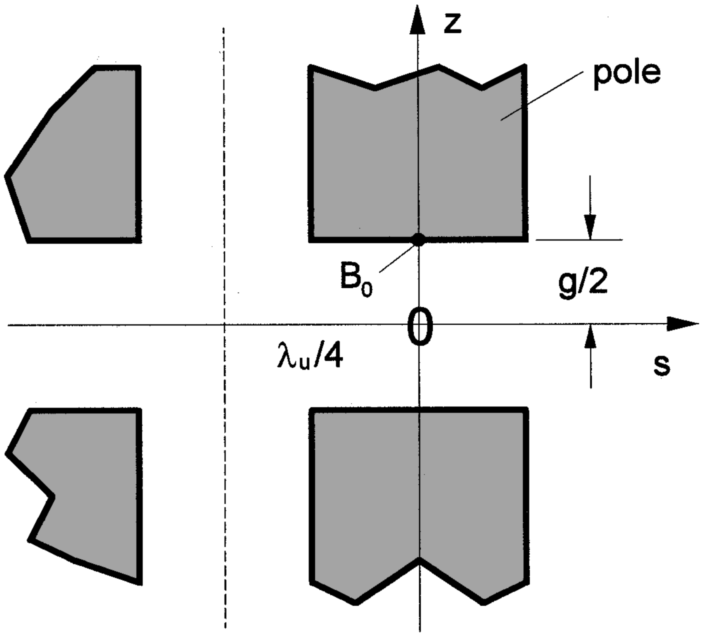


Fig. 7.2 Definition of the poletip field

In order to get the integration constant A we take the pole tip field B_0 at $\{s,z\} = \{0, g/2\}$. With (7.6) we get

$$B_0 = B_z\left(0, \frac{g}{2}\right) = k_u A \cosh\left(k_u \frac{g}{2}\right) = k_u A \cosh\left(\pi \frac{g}{\lambda_u}\right) \tag{7.7}$$

and

$$A = \frac{B_0}{k_u \cosh\left(\pi \frac{g}{\lambda_u}\right)} \tag{7.8}$$

Insertion into (7.6) provides

$$B_z(s,z) = \frac{B_0}{\cosh\left(\pi \frac{g}{\lambda_u}\right)} \cosh(k_u z)\cos(k_u s) \tag{7.9}$$

and

$$B_z(s,z) = \frac{B_0}{\cosh\left(\pi \frac{g}{\lambda_u}\right)} \sinh(k_u z)\sin(k_u s) \tag{7.10}$$

At the orbit the periodic field has the maximum value

$$\tilde{B} = \frac{B_0}{\cosh\left(\pi \frac{g}{\lambda_u}\right)} \tag{7.11}$$

For given period length the λ_u the field decreases with increasing gap height g. Short periods require therefore small pole distances.

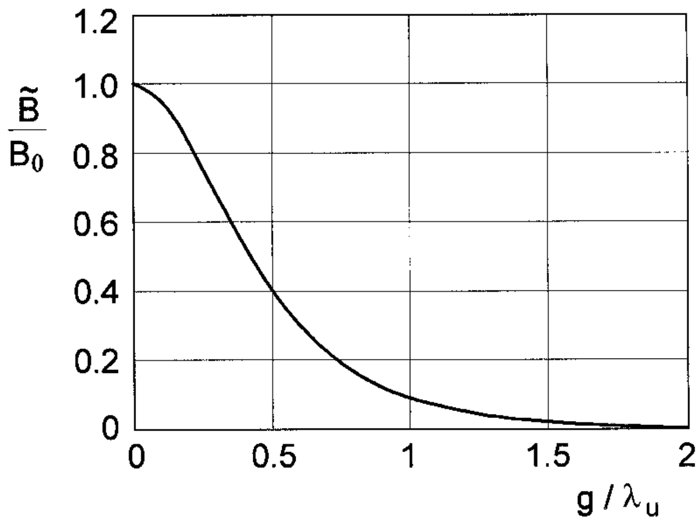


Fig. 7.3 Peak field at the orbit as a function of the relation between gap height and period length

At the beam the periodic field is

$$B_z(s,z) = \tilde{B}\sin(k_u s)\,. \qquad (7.12)$$

The most simple design is an electromagnet

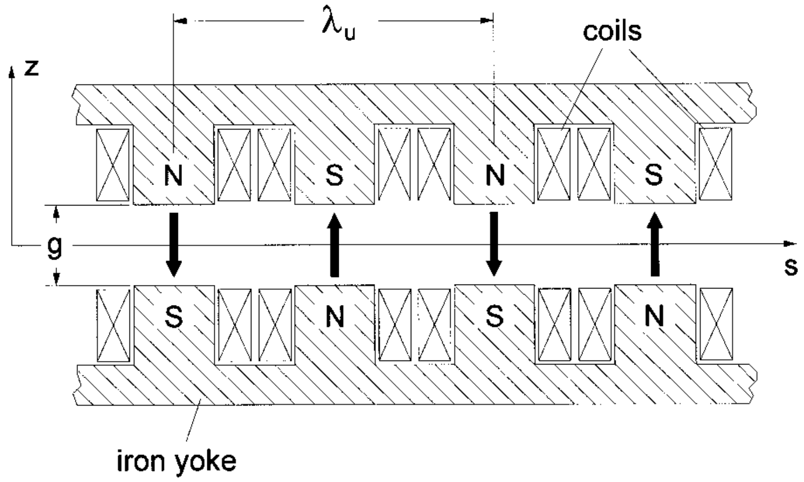


Fig. 7.4 Design as an electromagnet

Shorter period length down to a few cm are possible by use of permanent magnets. The field variation is made by changing the gap height.

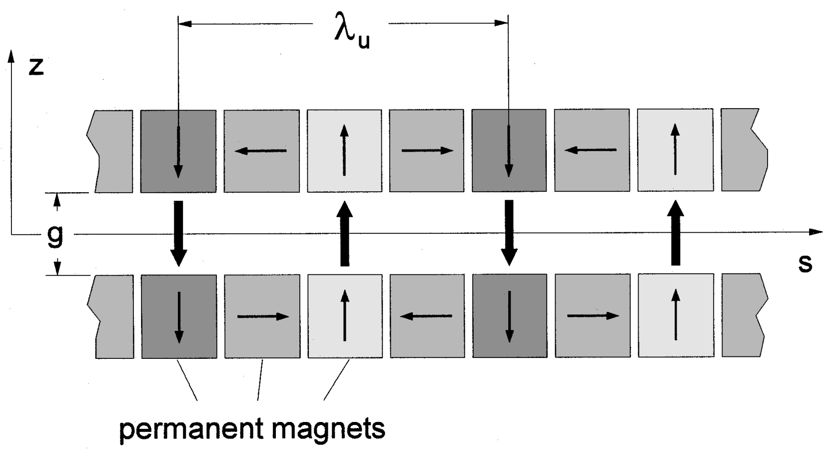


Fig. 7.5 Undulator using permanent magnets

A hybrid magnet consists of permanent magnets and iron poles.

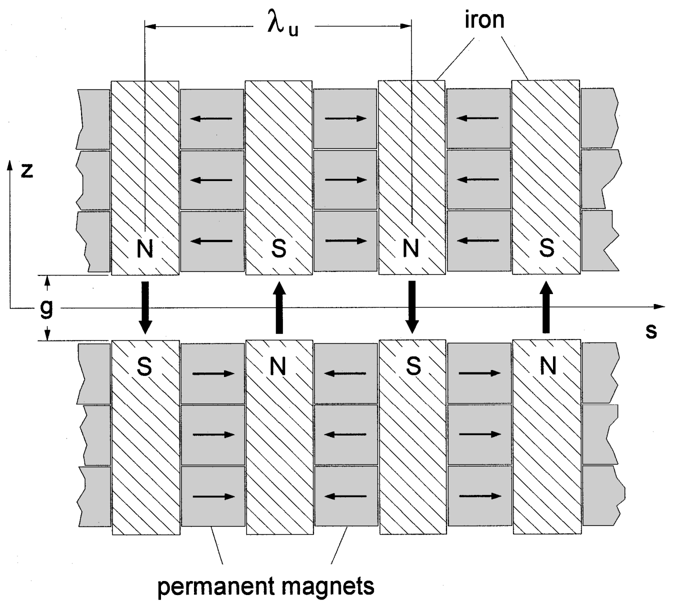


Fig. 7.6 Principal of a hybrid magnet

W/U-magnets have maximum fields at the beam about 1 T. The minimum wave length is limited because of

$$\lambda_c = \frac{4\pi}{3}\frac{R}{\gamma^3} = \frac{4\pi c\left(m_0 c^2\right)^3}{3eE^2}\frac{1}{\tilde{B}} \tag{7.13}$$

Shorter wave lengths are possible with superconductive wigglers with fields of $\tilde{B} > 5\ \mathrm{T}$

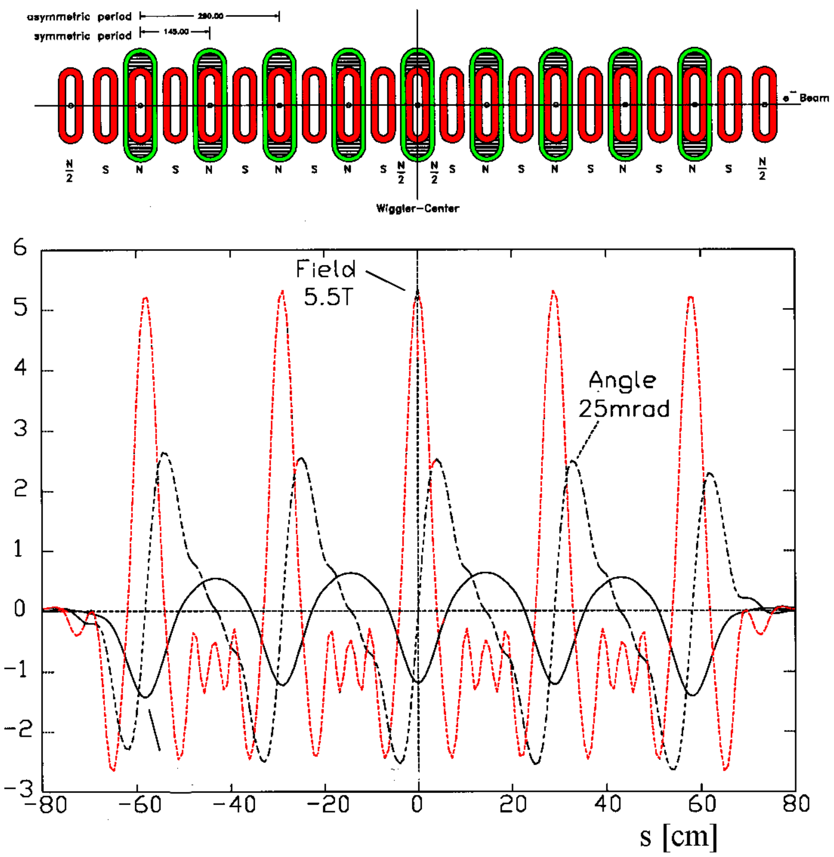


Fig. 7.7 Example of a superconductive wiggler

The W/U-field has to be matched that the total bending angle is zero.

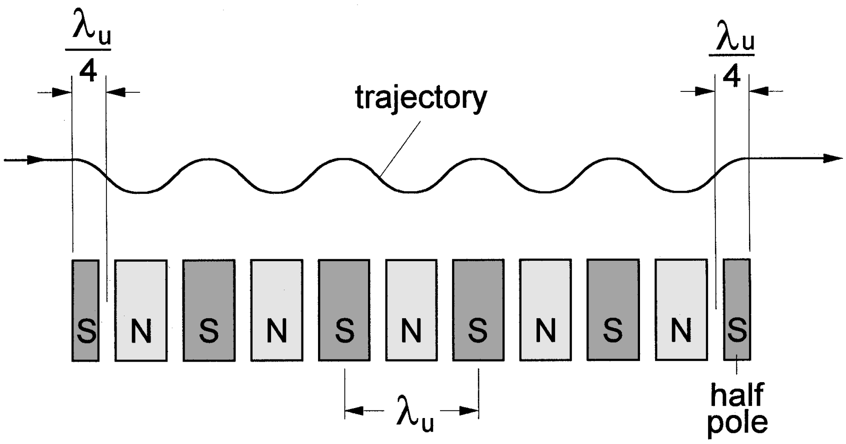


Fig. 7.8 Matched undulator trajectory

We have then

$$\int_{W/U} B_z(s)\,ds = \tilde{B}\int_{s_1}^{s_2}\cos(k_u s)\,ds = 0 \tag{7.14}$$

This condition is fulfilled if

$$s_1 = 0 \qquad \text{and} \qquad s_2 = n\lambda_u + \frac{\lambda_u}{2} \tag{7.15}$$

with n = 1,2, … . It is possible to utilize at both ends short magnet pieces of half pole length. In addition one has to shim the single poles to compensate the unavoidable tolerances.

7.2 Equation of motion in an W/U-magnet

In a W/U-magnet we have the Lorentz force

$$\vec{F} = \dot{\vec{p}} = m_0\gamma\,\dot{\vec{v}} = e\,\vec{v}\times\vec{B} \tag{7.16}$$

With the approximation

$$\vec{B} = \begin{pmatrix} 0 \\ B_z \\ B_s \end{pmatrix} \qquad \text{und} \qquad \vec{v} = \begin{pmatrix} v_x \\ 0 \\ v_s \end{pmatrix} \tag{7.17}$$

we get

$$\dot{\vec{v}} = \frac{e}{m_0\gamma}\begin{pmatrix} -v_s B_z \\ -v_x B_s \\ v_x B_z \end{pmatrix} \tag{7.18}$$

The velocity component in z-direction is very small and can be neglected. With $\dot{x} = v_x$ and $\dot{s} = v_s$ we have the motion in the s-x-plane

$$\begin{aligned} \ddot{x} &= -\dot{s}\frac{e}{m_0\gamma}B_z(s) \\ \ddot{s} &= \dot{x}\frac{e}{m_0\gamma}B_z(s) \end{aligned} \tag{7.19}$$

This is a coupled set of equations. The influence of the horizontal motion on the longitudinal velocity is very small

$$\dot{x} = v_x << c \qquad \text{and} \qquad \dot{s} = v_s = \beta c = \text{const.} \tag{7.20}$$

In this case only the first equation of (7.19) is important and we get

$$\ddot{x} = -\frac{\beta c e \tilde{B}}{m_0 \gamma} \cos(k_u s) \tag{7.21}$$

We replace with

$$\dot{x} = x' \beta c \qquad \text{and} \qquad \ddot{x} = x'' \beta^2 c^2$$

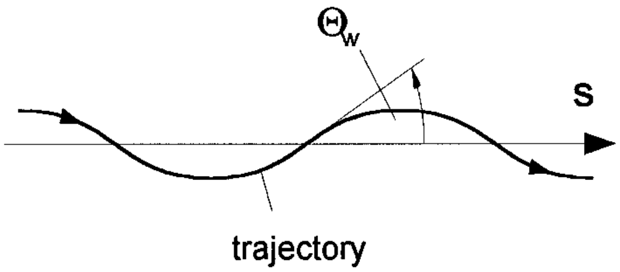
the time derivative by a spatial one and get

$$x'' = -\frac{e\tilde{B}}{m_0 \beta c \gamma} \cos(k_u s) = -\frac{e\tilde{B}}{m_0 \beta c \gamma} \cos\left(2\pi \frac{s}{\lambda_u}\right) \tag{7.22}$$

With $\beta = 1$ we can write

$$\begin{aligned} x'(s) &= \frac{\lambda_u e \tilde{B}}{2\pi m_0 \gamma c} \sin(k_u s) \\ x(s) &= \frac{\lambda_u^2 e \tilde{B}}{4\pi^2 m_0 \gamma c} \cos(k_u s) \end{aligned} \tag{7.23}$$

The maximum angle is at $\sin(k_u s) = 1$



$$\Theta_w = x'_{max} = \frac{1}{\gamma} \frac{\lambda_u e \tilde{B}}{2\pi m_0 c} \tag{7.24}$$

The dimensionless parameter

$$K = \frac{\lambda_u e \tilde{B}}{2\pi m_0 c} \tag{7.25}$$

is called wiggler or undulator parameter. The maximum trajectory angle is then

$$\Theta_w = \frac{K}{\gamma} \tag{7.26}$$

This is the natural opening angle of the synchrotron radiation. With the parameter K we can now distinguish between wiggler and undulator:

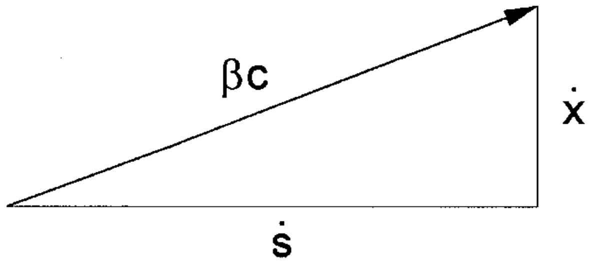
$$\begin{aligned} &\textbf{undulator} \quad \text{if} \quad K \le 1 \quad \text{i.e.} \quad \Theta_w \le 1/\gamma \\ &\textbf{wiggler} \quad \text{if} \quad K > 1 \quad \text{i.e.} \quad \Theta_w > 1/\gamma \end{aligned} \tag{7.27}$$

Now we go back to the system of coupled equations (7.19). We assume that the horizontal motion is only determined by a constant average velocity $\bar{v}_s = \langle \dot{s} \rangle$. From (7.23) and (7.25) we get

$$x'(s) = \frac{K}{\gamma} \sin(k_u s) = \Theta_w \sin(k_u s) \tag{7.28}$$

With $\dot{x} = \beta c x'$, $s = \beta c t$ and $\omega_u = k_u \beta c$ one can write

$$\dot{x}(t) = \beta c \Theta_w \sin(\omega_u t) = \beta c \frac{K}{\gamma} \sin(\omega_u t) \tag{7.29}$$



For the velocity holds

$$\dot{s}^2 = (\beta c)^2 - \dot{x}^2$$

and with $\beta^2 = 1 - \frac{1}{\gamma^2}$ we get

$$\dot{s}(t) = c\sqrt{1 - \left(\frac{1}{\gamma^2} + \frac{\dot{x}^2}{c^2}\right)} \tag{7.30}$$

Since the expression in the brackets is very small, the root can be expand in the way

$$\begin{aligned} \dot{s}(t) &= c\left[1 - \frac{1}{2}\left(\frac{1}{\gamma^2} + \frac{\dot{x}^2}{c^2}\right)\right] \\ &= c\left[1 - \frac{1}{2\gamma^2}\left(1 + \frac{\gamma^2}{c^2}\dot{x}^2\right)\right] \end{aligned} \tag{7.31}$$

Inserting the horizontal velocity (7.29) and using the relation

$$\sin^2(x) = (1 - \cos 2x)/2$$

we get

$$\dot{s}(t) = c\left\{1 - \frac{1}{2\gamma^2}\left[1 + \frac{\beta^2 K^2}{2}\left(1 - \cos(2\omega_u t)\right)\right]\right\} \tag{7.32}$$

This can be written in the form

$$\dot{s}(t) = \langle \dot{s} \rangle + \Delta\dot{s}(t)$$

with the average velocity

$$\langle \dot{s} \rangle = c\left\{1 - \frac{1}{2\gamma^2}\left[1 + \frac{\beta^2 K^2}{2}\right]\right\} \tag{7.33}$$

and the oscillation

$$\Delta\dot{s}(t) = \frac{c\beta^2 K^2}{4\gamma^2}\cos(2\omega_u t) \tag{7.34}$$

From (7.33) we derive the relative velocity with β =

$$\beta^* = \frac{\langle \dot{s} \rangle}{c} = 1 - \frac{1}{2\gamma^2}\left[1 + \frac{K^2}{2}\right] \tag{7.35}$$

With (7.29) and (7.33) to (7.35) we get

$$\begin{aligned} \dot{x}(t) &= \beta c \frac{K}{\gamma}\sin(\omega_u t) \\ \dot{s}(t) &= \beta^* c + \frac{c\beta^2 K^2}{4\gamma^2}\cos(2\omega_u t) \end{aligned} \tag{7.36}$$

Using $\omega_u = k_u \beta c$ and β = 1 one can evaluate the velocity simply by integration. In the laboratory frame we have

$$x(t) = -\frac{K}{k_u \gamma} \cos(\omega_u t)$$
$$s(t) = \beta^* c\, t + \frac{K^2}{8 k_u \gamma^2} \sin(2\omega_u t) \qquad (7.37)$$

We get an impressive form of motion in the center of mass system K^*, which moves with the velocity β^* with respect to the laboratory system. With the transformation

$$x^* = x \qquad \text{und} \qquad s^* = \gamma\left(s - \beta c\, t\right) \qquad (7.38)$$

we get

$$x(t) = -\frac{K}{k_u \gamma} \cos(\omega_u t)$$
$$s(t) = \beta^* c t + \frac{K^2}{8 k_u \gamma^2} \sin(2\omega_u t) \qquad (7.39)$$

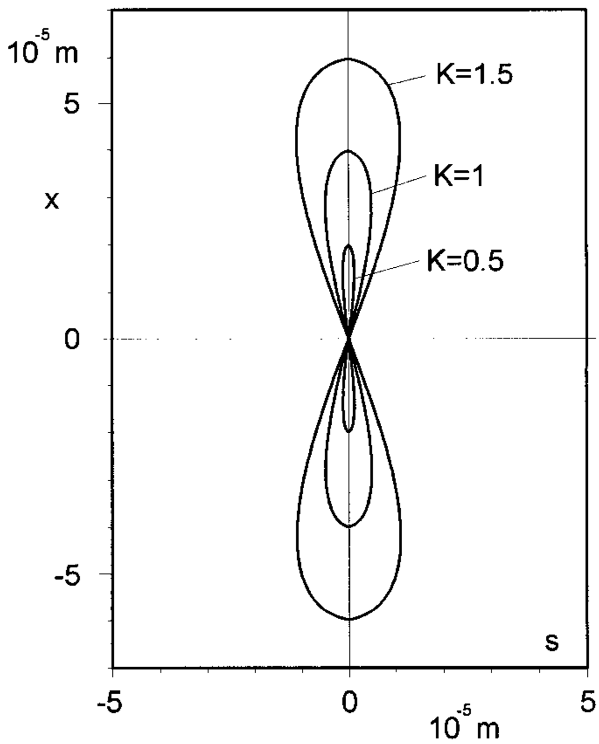


Fig. 7.9 Particle motion in the center of mass frame traveling through an undulator magnet

7.3 Undulatorradiation

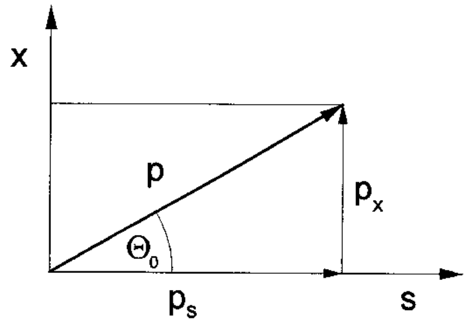
Because of the periodic motion in the undulator radiation is emitted in the laboratory frame with a well defined frequency

$$\Omega_w = \frac{2\pi}{T} = \frac{2\pi\beta c}{\lambda_u} = k_u \beta c \qquad (7.40)$$

In the moving frame with the average velocity β^* the frequency is transformed according to

$$\omega^* = \gamma^* \Omega_w \qquad (7.41)$$

The system emits monochromatic radiation. In order to transform a photon into the laboratory system we take a photon emitted under the angle Θ_0



Energy and momentum of the photon are

$$E = \hbar\omega$$
$$p = \frac{\hbar\omega}{c} \tag{7.42}$$

and the 4-vector becomes

$$P_\mu = \begin{pmatrix} E/c \\ p_x \\ p_z \\ p_s \end{pmatrix} = \begin{pmatrix} E/c \\ p\sin\Theta_0 \\ 0 \\ p\cos\Theta_0 \end{pmatrix} \tag{7.43}$$

Transformation into the System K^* is then

$$\begin{pmatrix} E^*/c \\ p_x^* \\ p_z^* \\ p_s^* \end{pmatrix} = \begin{pmatrix} \gamma^* & 0 & 0 & -\beta^*\gamma^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta^*\gamma^* & 0 & 0 & \gamma^* \end{pmatrix} \cdot \begin{pmatrix} E/c \\ p\sin\Theta_0 \\ 0 \\ p\cos\Theta_0 \end{pmatrix} \tag{7.44}$$

The energy of the photon becomes

$$\frac{E^*}{c} = \gamma^*\frac{E}{c} - \beta^*\gamma^* p\cos\Theta_0 = \gamma^{`*}\frac{\hbar\omega_w}{c}\left(1-\beta^*\cos\Theta_0\right) \tag{7.45}$$

With $E^* = \hbar\omega^*$ we get

$$\frac{\hbar\omega^*}{c} = \gamma^*\frac{\hbar\omega_w}{c}\left(1-\beta^*\cos\Theta_0\right) \tag{7.46}$$

and

$$\omega_w = \frac{\omega^*}{\gamma^*\left(1-\beta^*\cos\Theta_0\right)} \tag{7.47}$$

Using (7.41) we can write

$$\omega_w = \frac{\Omega_w}{1-\beta^*\cos\Theta_0}$$

and find

$$\frac{\omega_w}{\Omega_w} = \frac{\lambda_u}{\lambda_w} = \frac{1}{1-\beta^*\cos\Theta_0} \tag{7.48}$$

with

$$\lambda_w = \lambda_u\left(1-\beta^*\cos\Theta_0\right) \tag{7.49}$$

Now we replace β^* by (7.35) and expand

$$\cos\Theta_0 \approx 1 - \frac{\Theta_0^2}{2} \qquad \text{da} \qquad \Theta_0 \approx \frac{1}{\gamma} < 1$$

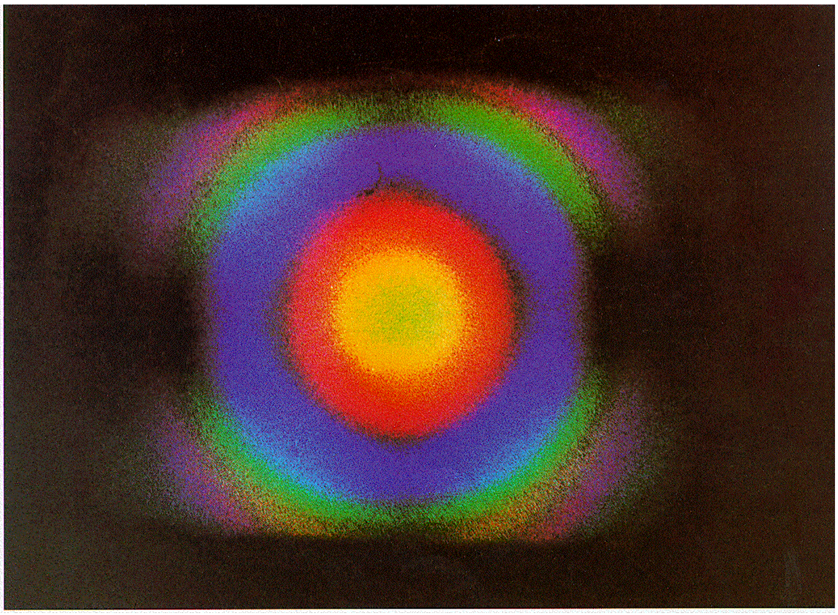
After this manipulations we find

$$\begin{aligned} \lambda_u\left(1-\beta^*\cos\Theta_0\right) &= \lambda_u\left[1-\left(1-\frac{1+K^2/2}{2\gamma^2}\right)\left(1-\frac{\Theta_0^2}{2}\right)\right] \\ &= \lambda_u\left[1-\left(1-\frac{\Theta_0^2}{2}-\frac{1+K^2/2}{2\gamma^2}\right)+\ldots\right] \\ &\approx \lambda_u\left(\frac{\Theta_0^2}{2}+\frac{1+K^2/2}{2\gamma^2}\right) \end{aligned} \tag{7.50}$$

This approximation is usually fulfilled with high precision. Using equation (7.49) we get the important "coherence condition for undulator radiation"

$$\lambda_w = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\Theta_0^2\right) \tag{7.51}$$

The wavelength of the radiation is mainly determined by λ_u, γ, and K. With increasing angle Θ_0 also the wavelength increases.

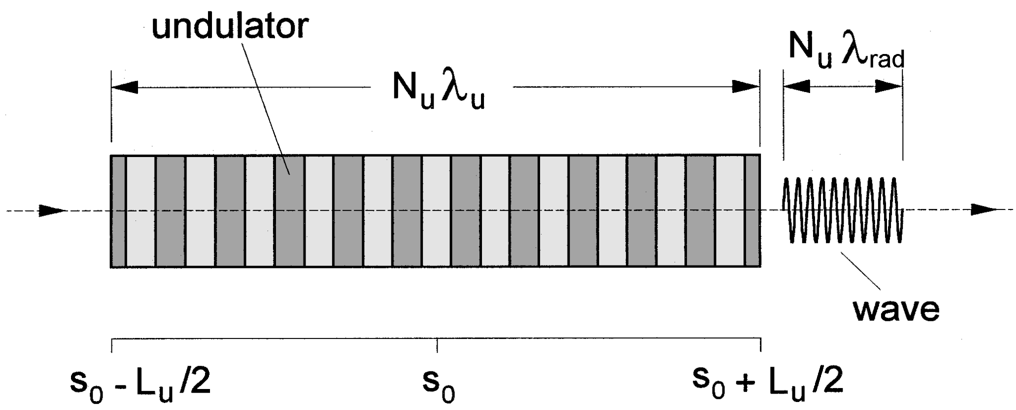


The total length of the undulator is

$$L_u = N_u\lambda_u \tag{7.52}$$

If s_0 marks the center of the undulator, the emitted wave has the time dependent function

$$u(\omega_w,t) = \begin{cases} a\exp i\omega_w t & \text{if} \quad -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \tag{7.53}$$



The wave has the duration

$$T = N_u \lambda_w / c \quad \Rightarrow \quad \omega_w T = 2\pi N_u \tag{7.54}$$

Such limited wave generates a continuous spectrum of partial waves. Their amplitudes are given by the Fourier integral

$$A(\omega) = \frac{1}{\sqrt{2\pi}\,T} \int_{-\infty}^{+\infty} u(\omega_w, t) \exp(-i\omega t)\, dt \tag{7.55}$$

Insertion into (7.53) gives

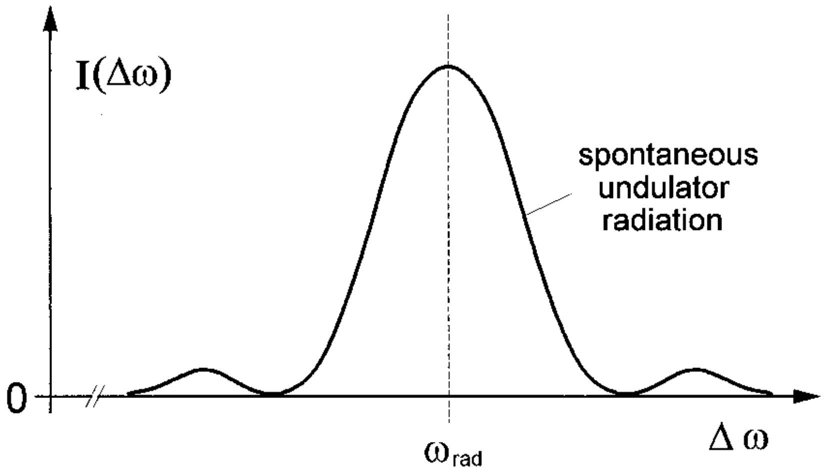
$$\begin{aligned} A(\omega) &= \frac{a}{\sqrt{2\pi}\,T} \int_{-T/2}^{+T/2} \exp\left[-i\left(\omega - \omega_w\right) t\right] dt \\ &= \frac{2a}{\sqrt{2\pi}\,T} \frac{\sin\left(\omega - \omega_w\right) T}{2\left(\omega - \omega_w\right)} \end{aligned} \tag{7.56}$$

With $\Delta\omega = \omega - \omega_w$ and (7.54) we get

$$A(\omega) = \frac{a}{\sqrt{2\pi}} \frac{\sin\left(\pi N_u \dfrac{\Delta\omega}{\omega_w}\right)}{\pi N_u \dfrac{\Delta\omega}{\omega_w}} \tag{7.57}$$

The intensity is proportional to the square of amplitude

$$I(\Delta\omega) \propto \left[\frac{\sin\left(\pi N_u \dfrac{\Delta\omega}{\omega_w}\right)}{\pi N_u \dfrac{\Delta\omega}{\omega_w}} \right]^2 \tag{7.58}$$



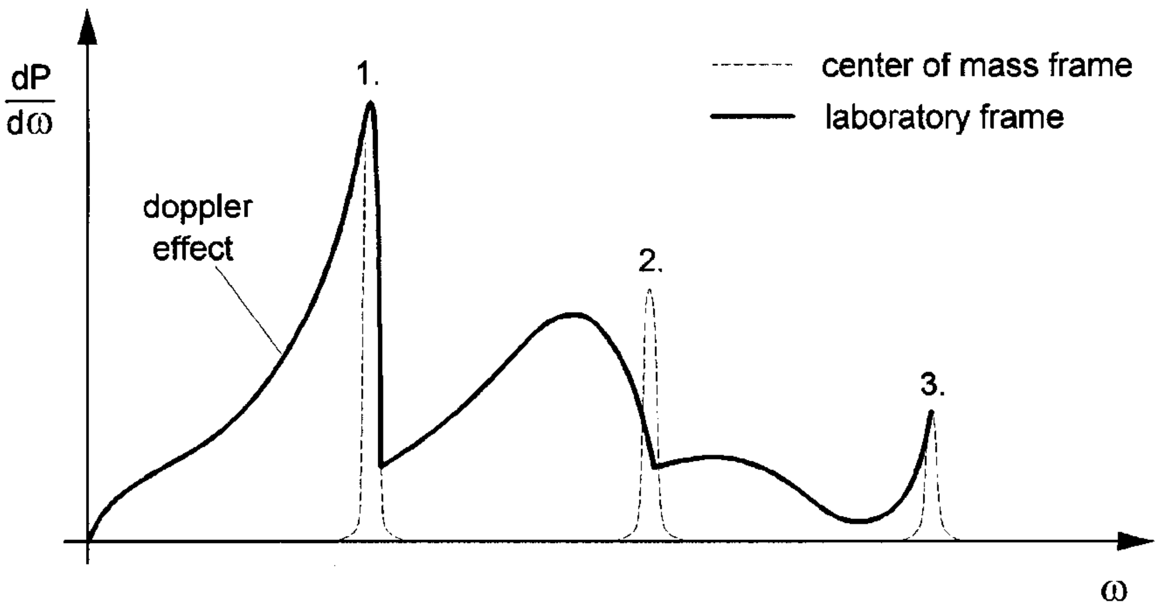
We get the half width of maximum from

$$\left(\frac{\sin x}{x}\right)^2 = \frac{1}{2} \qquad \text{with} \qquad x = \pi N_u \frac{\Delta\omega}{\omega_w} = 1.392 \tag{7.59}$$

and find

$$\frac{2\Delta\omega}{\omega_w} = \frac{2x}{\pi N_u} = \frac{0.886}{N_u} \approx \frac{1}{N_u} \tag{7.60}$$

The spectrum of an undulator is



8 Appendix B: Longitudinal Phase Space

8.1 Particle distribution in longitudinal phase space

In chapter 4 we have discussed the radiation damping. This effect alone would reduce the amplitude of the synchrotron and betatron oscillations to zero. The photons, however, are emitted randomly and we have sudden emission of many single photons. Every emission of a photon excites synchrotron and betatron oscillations. We will now evaluate the energy distribution in a bunch due to quantum effects caused by synchrotron radiation.

The power of the photons with the energy ε emitted from an energy interval $\{\varepsilon, \varepsilon + d\varepsilon\}$ can be derived from the equation (3.58)

$$dP_\gamma = \varepsilon\, \dot{n}(\varepsilon)\, d\varepsilon \tag{8.1}$$

with

$$\dot{n}(\varepsilon) = \frac{P_0}{\varepsilon_c}\frac{1}{\varepsilon} S_s\left(\frac{\varepsilon}{\varepsilon_c}\right) \tag{8.2}$$

P_0 is the total power of all photons as given in (3.59) and $\varepsilon_c = \hbar\,\omega_c$ the critical energy derived from equation (3.57). The function $S_s(\xi)$ is the spectral function (3.61). The total rate of quantum emission is then

$$\dot{N}_{\text{tot}} = \int_0^\infty \dot{n}(\varepsilon)\, d\varepsilon = \frac{P_0}{\varepsilon_c}\int_0^\infty \frac{1}{\varepsilon} S_s\left(\frac{\varepsilon}{\varepsilon_c}\right) d\varepsilon = \frac{15\sqrt{3}}{8}\frac{P_0}{\varepsilon_c} \tag{8.3}$$

With this relations the mean quantum energy is

$$\langle\varepsilon\rangle = \frac{P_0}{\dot{N}_{\text{tot}}} = \frac{1}{\dot{N}_{\text{tot}}}\int_0^\infty \varepsilon\, \dot{n}(\varepsilon)\, d\varepsilon = \frac{8}{15\sqrt{3}}\varepsilon_c \tag{8.4}$$

More important is the mean of the square energy

$$\langle\varepsilon^2\rangle = \frac{1}{\dot{N}_{\text{tot}}}\int_0^\infty \varepsilon^2\, \dot{n}(\varepsilon)\, d\varepsilon \qquad \Rightarrow \qquad \dot{N}_{\text{tot}}\langle\varepsilon^2\rangle = \int_0^\infty \varepsilon^2\, \dot{n}(\varepsilon)\, d\varepsilon \tag{8.5}$$

The synchrotron oscillation is an energy oscillation with the frequency Ω, as shown in chapter 4.1.1. Without damping we have

$$\Delta E(t) = \Delta E_0 \exp i\Omega(t - t_0) \tag{8.6}$$

After emission of a photon at the time t_i with the energy ε the amplitude is reduced according to

$$\begin{aligned} \Delta E(t) &= \Delta E_0 \exp i\Omega(t - t_0) - \varepsilon \exp i\Omega(t - t_i) \\ &= \Delta E_1 \exp i\Omega(t - t_1) \end{aligned} \tag{8.7}$$

with

$$\Delta E_1^2 = \Delta E_0^2 + \varepsilon^2 - 2\,\Delta E_0\,\varepsilon \cos\Omega(t_i - t_0) \tag{8.8}$$

The phase is completely random and the expectation value of $\cos\Omega(t_i - t_0)$ vanishes. The probable amplitude change is then

$$\langle\delta\Delta E^2\rangle = \langle\Delta E_1^2 - \Delta E_0^2\rangle = \varepsilon^2 \tag{8.9}$$

One can see that the square of the amplitude change is proportional to the square of the energy of the emitted photons.

We take now all photons emitted from an interval $\{\varepsilon, \varepsilon + \Delta\varepsilon\}$ of the radiation spectrum. Since the number of photons per second is $\dot{n}(\varepsilon)\,\Delta\varepsilon$ the contribution to the rate of amplitude change is

$$\Delta\left\{\frac{d\left\langle\Delta E^2\right\rangle}{dt}\right\} = \varepsilon^2\,\dot{n}(\varepsilon)\,\Delta\varepsilon \tag{8.10}$$

Integration over all energies of the spectrum gives with (8.5)

$$\frac{d\left\langle\Delta E^2\right\rangle}{dt} = \int_0^\infty \varepsilon^2 \dot{n}(\varepsilon)\,d\varepsilon = \dot{N}_{\text{tot}}\left\langle\varepsilon^2\right\rangle \tag{8.11}$$

On the other hand we have the radiation damping of the energy oscillations

$$\Delta E(t) = \Delta E_0 \exp\left(-a_s t\right) \qquad \Rightarrow \qquad \Delta E^2(t) = \Delta E_0^2 \exp\left(-2a_s t\right) \tag{8.12}$$

with the time derivative

$$\frac{d\Delta E^2(t)}{dt} = -2a_s\,\Delta E_0^2 \exp\left(-2a_s t\right) = -2a_s\,\Delta E^2(t) \tag{8.13}$$

or after averaging

$$\frac{d\left\langle\Delta E^2\right\rangle}{dt} = -2a_s\left\langle\Delta E^2\right\rangle \tag{8.14}$$

The two effects the quantum excitation (8.11) and the damping (8.14) compensate each other and we get

$$\dot{N}_{\text{tot}}\left\langle\varepsilon^2\right\rangle - 2a_s\left\langle\Delta E^2\right\rangle = 0 \tag{8.15}$$

The energy oscillations are sinusoidal and the probable amplitude square is just ½ of the peak amplitude. Therefore, we get from (8.15)

$$\boxed{\sigma_{\text{E}}^2 = \frac{\left\langle\Delta E^2\right\rangle}{2} = \frac{1}{4a_s}\dot{N}_{\text{tot}}\left\langle\varepsilon^2\right\rangle} \tag{8.16}$$

At first we use (8.11) and evaluate

$$\dot{N}_{\text{tot}}\left\langle\varepsilon^2\right\rangle = \int_0^\infty \varepsilon^2 \dot{n}(\varepsilon)\,d\varepsilon = \frac{P_0}{\varepsilon_c}\int_0^\infty \varepsilon\, S\left(\frac{\varepsilon}{\varepsilon_c}\right)d\varepsilon = \frac{55}{24\sqrt{3}}\varepsilon_c P_0 \tag{8.17}$$

Using the formula (3.57) we get the critical energy

$$\varepsilon_c = \hbar\omega_c = \frac{3}{2}\frac{\hbar\, c\,\gamma^3}{\rho} \tag{8.18}$$

The emitted photon power is given in (3.59) in the form

$$P_0 = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2} \tag{8.19}$$

The average is

$$\langle P_0\rangle = \frac{e^2 c\,\gamma^4}{6\pi\varepsilon_0}\left\langle\frac{1}{\rho^2}\right\rangle \qquad \Rightarrow \qquad P_0 = \frac{\langle P_0\rangle}{\langle 1/\rho^2\rangle}\frac{1}{\rho^2} \tag{8.20}$$

Insertion of (8.18) and (8.20) into (8.17) gives

$$\dot N_{\text{tot}}\langle\varepsilon^2\rangle = \frac{55}{24\sqrt{3}}\varepsilon_c\,P_0 = \frac{55}{16\sqrt{3}}\hbar\,c\,\gamma^3\langle P_0\rangle\frac{1}{\langle 1/\rho^2\rangle}\frac{1}{\rho^3} \tag{8.21}$$

With $W_0 = P_0\,T_0$ the damping constant in (4.65) becomes

$$a_s = \frac{W_0\,J_0}{2\,T_0\,E} = \frac{\langle P_0\rangle J_s}{2\,E} = \frac{\langle P_0\rangle J_s}{2\,\gamma\,m_0 c^2} \tag{8.22}$$

Replacing this expression in (8.16) the probable amplitude square is then

$$\sigma_{\text{E}}^2 = \frac{\dot N_{\text{tot}}\langle\varepsilon^2\rangle}{4\,a_s} = \frac{55}{32\sqrt{3}}\frac{\hbar\,c\,m_0c^2\,\gamma^4}{J_s}\frac{\left\langle\frac{1}{\rho^3}\right\rangle}{\left\langle\frac{1}{\rho^2}\right\rangle} \tag{8.23}$$

Usually the relative energy spread is more interesting and with $E = \gamma\,m_0c^2$ we get

$$\boxed{\frac{\sigma_{\text{E}}^2}{E^2} = \frac{55}{32\sqrt{3}}\frac{\hbar c\gamma^2}{J_s m_0 c^2}\frac{\left\langle\frac{1}{\rho^3}\right\rangle}{\left\langle\frac{1}{\rho^2}\right\rangle}} \tag{8.24}$$

8.2 Bunch length

The synchrotron oscillation causes a periodic phase and an energy shift. In equation (4.11) it was shown that these two physical quantities have the relation

$$\Delta\dot\Psi = \frac{2\pi\,q\,\alpha}{T_0}\frac{\Delta E}{E} = \omega_u\,q\,\alpha\frac{\Delta E}{E} \tag{8.25}$$

and we find

$$\frac{\Delta E}{E} = \frac{\Delta\dot\Psi}{\omega_u q\,\alpha} \tag{8.26}$$

The phase is a real number and the phase oscillation has the form

$$\begin{aligned}\Delta\Psi(t) &= \hat\Psi\sin(\Omega\,t+\varphi)\\ \Delta\dot\Psi(t) &= \Omega\,\hat\Psi\cos(\Omega\,t+\varphi)\end{aligned} \tag{8.27}$$

We are in the following only interested in the amplitude of the phase. Then we get from (8.26)

$$\frac{\Delta E}{E} = \frac{\Omega\,\hat\Psi}{\omega_u q\,\alpha} \tag{8.28}$$

The phase amplitude can then be expressed in the form

$$\hat{\Psi} = \frac{\omega_u q \alpha}{\Omega} \frac{\Delta E}{E} \tag{8.29}$$

The bunch length σ_s is strongly correlated with the phase amplitude namely

$$\sigma_s = \frac{\lambda_{rf}}{2\pi} \hat{\Psi} = \frac{c}{q\,\omega_u} \hat{\Psi} = \frac{c\,\alpha}{\Omega} \frac{\Delta E}{E} \tag{8.30}$$

We replace the synchrotron frequency by the expression (4.18) and set the energy deviation to the natural energy fluctuation as calculated in (8.24) we get the bunch length in the form

$$\boxed{\sigma_s = \frac{c}{\omega_u} \sqrt{-\frac{2\pi\alpha E}{qeU\cos\Psi_s}} \frac{\sigma_E}{E}} \tag{8.31}$$

It is important to mention that the bunch length decreases with decreasing momentum compaction factor α and increasing rf-voltage U as

$$\sigma_s \propto \sqrt{\frac{\alpha}{U}} \tag{8.32}$$
