Measurement of transverse Emittance

The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as: $\varepsilon_x = \frac{1}{\pi}\int_A dxdx'$

The measurement is based on determination of:

either profile width $\boldsymbol{\sigma_x}$ and angular width $\boldsymbol{\sigma_x'}$ at one location
or $\boldsymbol{\sigma_x}$ at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies E_{kin} < 100 MeV/u: slit-grid device, pepper-pot (suited in case of non-linear forces).
- All beams: Quadrupole variation, 'three grid' method using linear transformations (**not** well suited in the presence of non-linear forces)

Synchrotron: lattice functions results in stability criterion

$\Rightarrow$ beam width delivers emittance: $\varepsilon_x = \frac{1}{\beta_x(s)}\left[\sigma_x^2 - \left(D(s)\frac{\Delta p}{p}\right)\right]$ and $\varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)}$

The emittance characterizes the whole beam quality: $\varepsilon_x = \frac{1}{\pi}\int_A dxdx'$

Ansatz:

Beam matrix at one location: $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ with $\vec{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$

It describes a 2-dim probability distr.

The value of emittance is:

$$\varepsilon_x = \sqrt{\det \boldsymbol{\sigma}} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

For the profile and angular measurement:

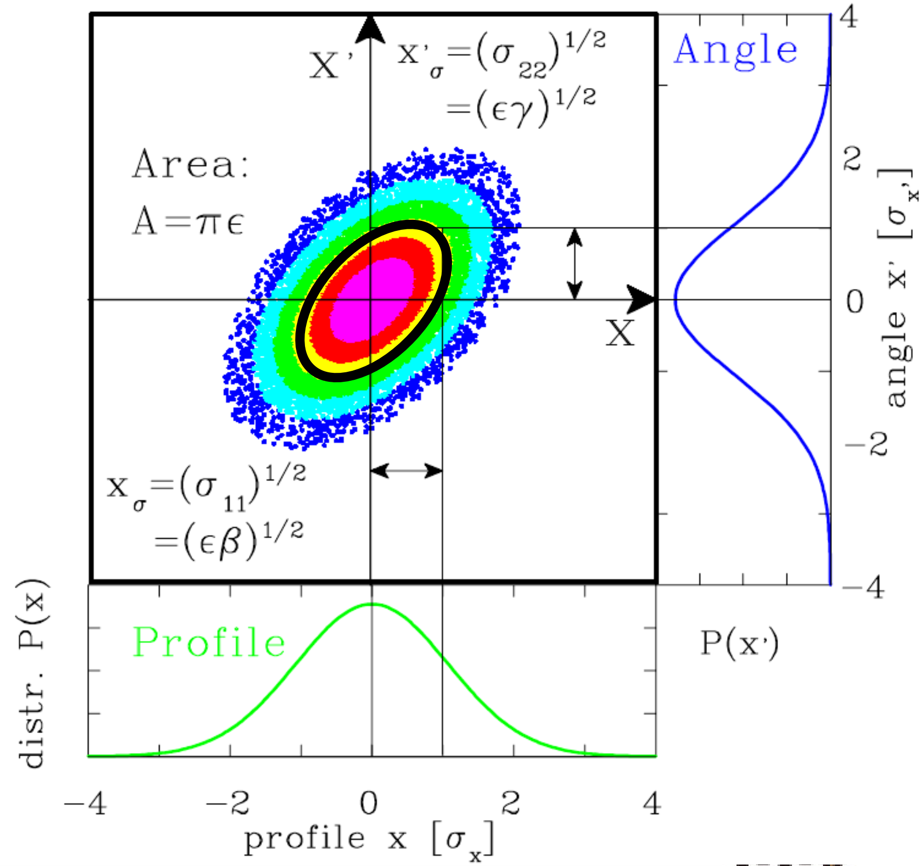
$$x_\sigma = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta} \quad \text{and}$$

$$x'_\sigma = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$$

Geometrical interpretation:

All points $\boldsymbol{x}$ fulfilling $\boldsymbol{x}^t \cdot \boldsymbol{\sigma}^{-1} \cdot \boldsymbol{x} = 1$ are located on a **ellipse**

$$\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2 = \det \sigma = \varepsilon_x^2$$



The Emittance for Gaussian Beams

The density function for a 2-dim Gaussian distribution is:

$$\rho(x,x') = \frac{1}{2\pi\epsilon}\exp\left[-\frac{1}{2}\,\vec{x}^T\,\sigma^{-1}\,\vec{x}\,\right]$$

$$= \frac{1}{2\pi\epsilon}\exp\left[\frac{-1}{2\det\sigma}\left(\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2\right)\right]$$

It describes an ellipse

with the characteristics

profile and angle Gaussian distribution

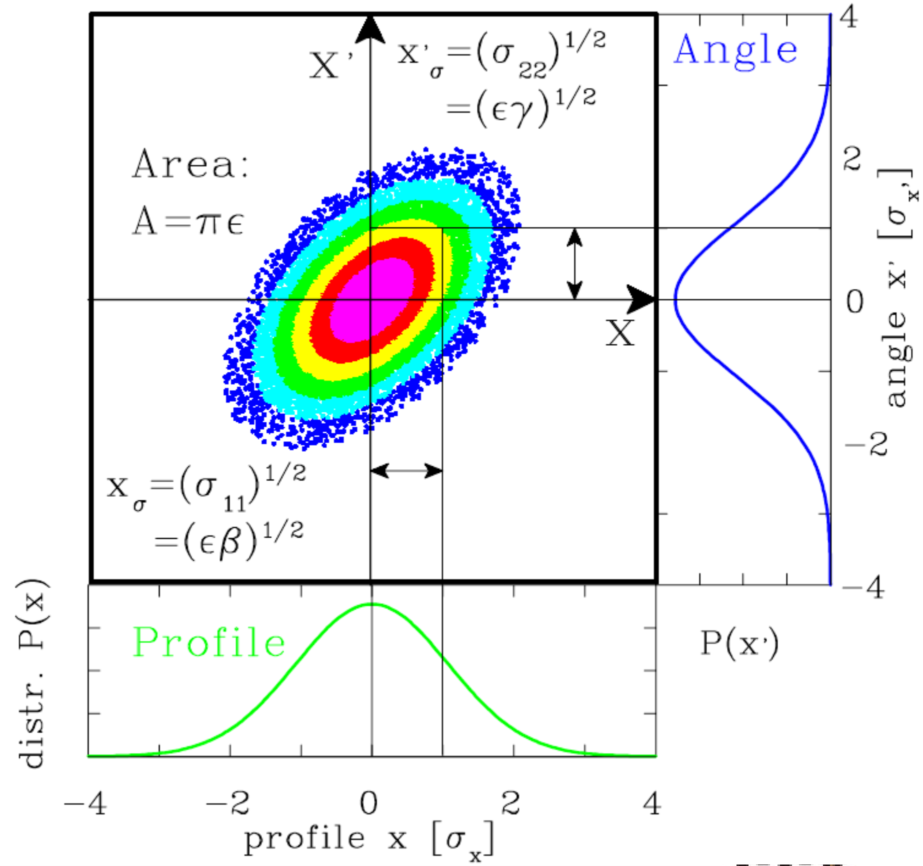
of width

$$x_\sigma = \sqrt{\sigma_{11}} \quad \text{and}$$

$$x'_\sigma = \sqrt{\sigma_{22}}$$

and the correlation or covariance

$$\text{cov} = \sqrt{\sigma_{12}}$$



The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance

using terms of 2-dim distribution:

Variances

Covariance i.e. correlation

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

It describes the value for 1 stand. derivation

The n^{th} central moment of a density distribution $\rho(x,x')$ calculated via

$$\mu \equiv \langle x \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \rho(x,x') dx' dx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x,x') dx dx'} \quad \text{and} \quad \langle x^n \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu)^n \cdot \rho(x,x') dx' dx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x,x') dx dx'}$$

Covariance: $$\langle xx' \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu) \cdot (x'-\mu') \cdot \rho(x,x') dx dx'}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x,x') dx dx'}$$

The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance using terms of 2-dim distribution:

Variances

Covariance i.e. correlation

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

It describes the value for 1 stand. derivation

For Gaussian beams only:

ε_{rms} ↔ interpreted as area containing a fraction ***f*** of ions: $\varepsilon(f) = -2\pi\varepsilon_{rms} \cdot \ln(1-f)$

factor to ϵ_{rms}	$1 \cdot \epsilon_{rms}$	$\pi \cdot \epsilon_{rms}$	$2\pi \cdot \epsilon_{rms}$	$4\pi \cdot \epsilon_{rms}$	$6\pi \cdot \epsilon_{rms}$	$8\pi \cdot \epsilon_{rms}$
faction of beam f [%]	15	39	63	86	95	98

Care: no common definition of emittance concerning the fraction ***f***

Outline:

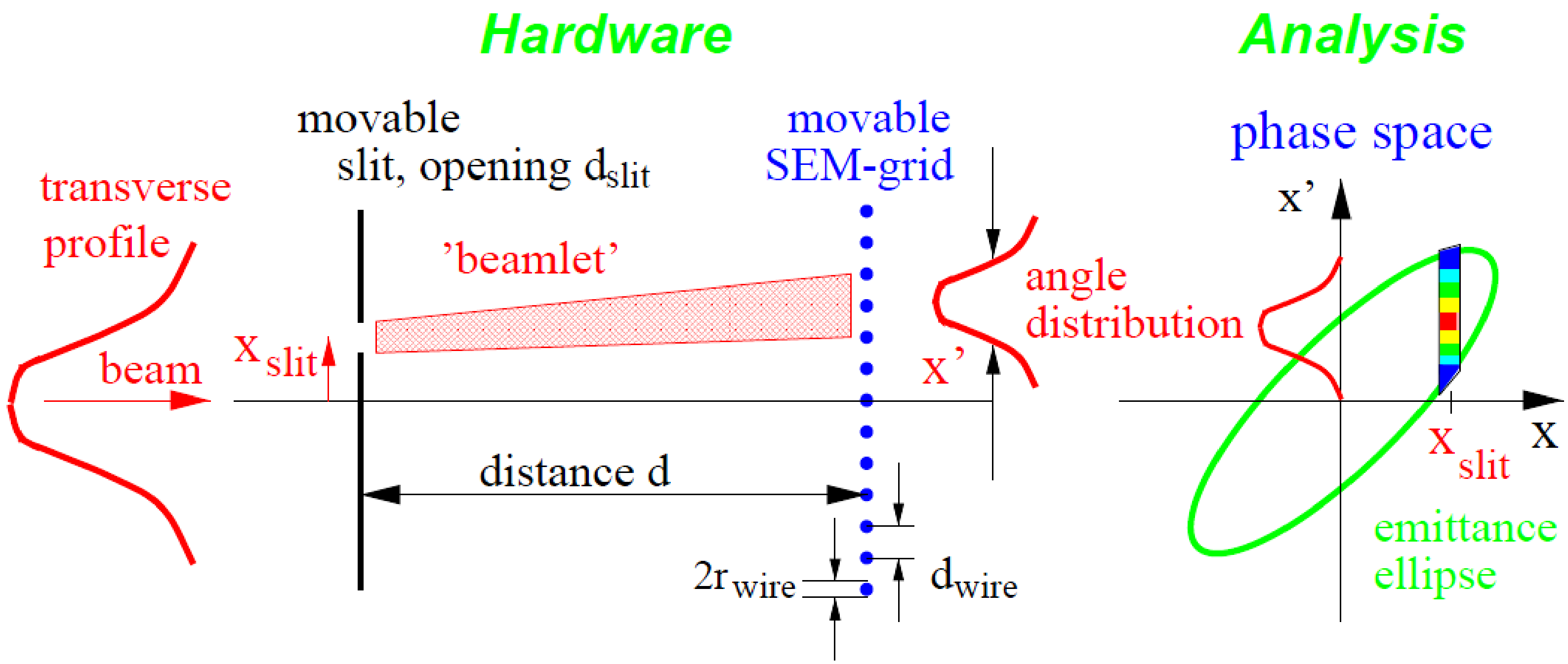
- Definition and some properties of transverse emittance
- **Slit-Grid device: scanning method**

  scanning slit → beam position & grid → angular distribution
- **Pepper-pot device: single shot device**
- **Quadrupole strength variation and position measurement**
- **Summary**

The Slit-Grid Measurement Device

Slit-Grid: Direct determination of position and angle distribution.

Used for protons/heavy ions with $E_{kin} < 100$ MeV/u $\Rightarrow$ range $R < 1$ cm.



Slit: position ***P(x)*** with typical width: 0.1 to 0.5 mm

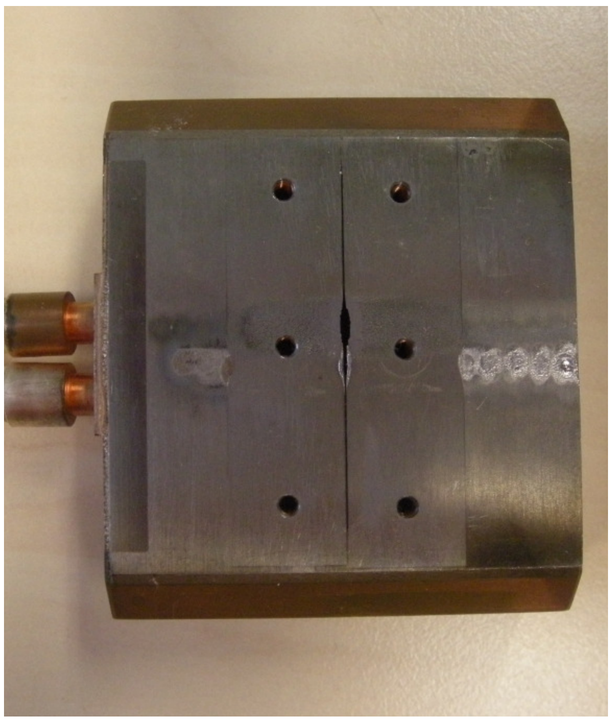
Distance: 10 cm to 1 m (depending on beam velocity)

SEM-Grid: angle distribution ***P(x')***

Slit with e.g. 0.1 mm thickness

→ Transmission only from Δx.

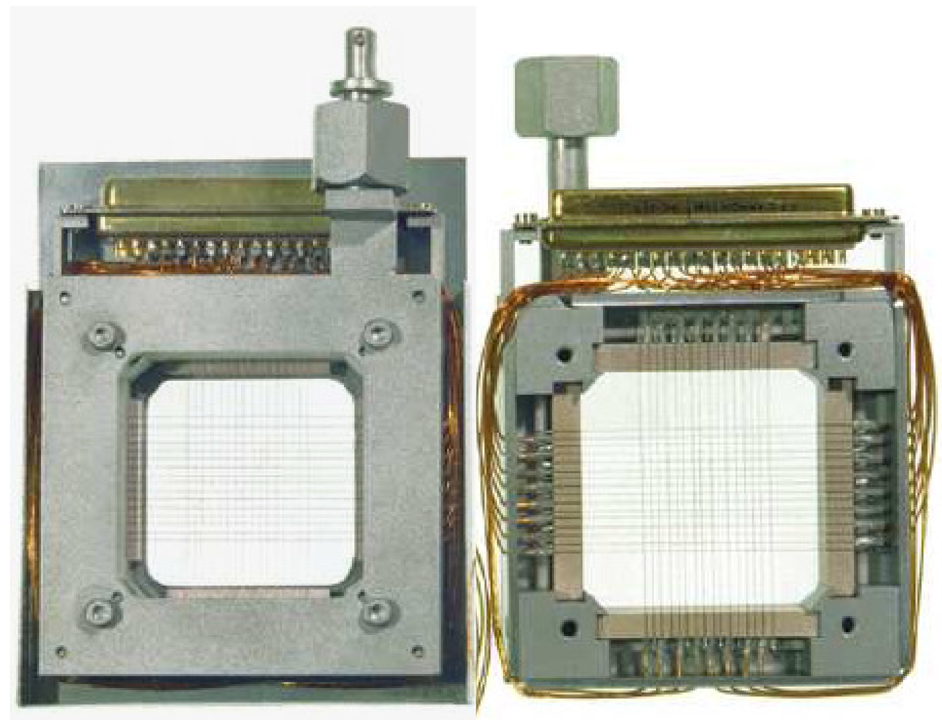
Example: Slit of width 0.1 mm (defect)
Moved by stepping motor, 0.1 mm resolution



Beam surface interaction: e^- emission

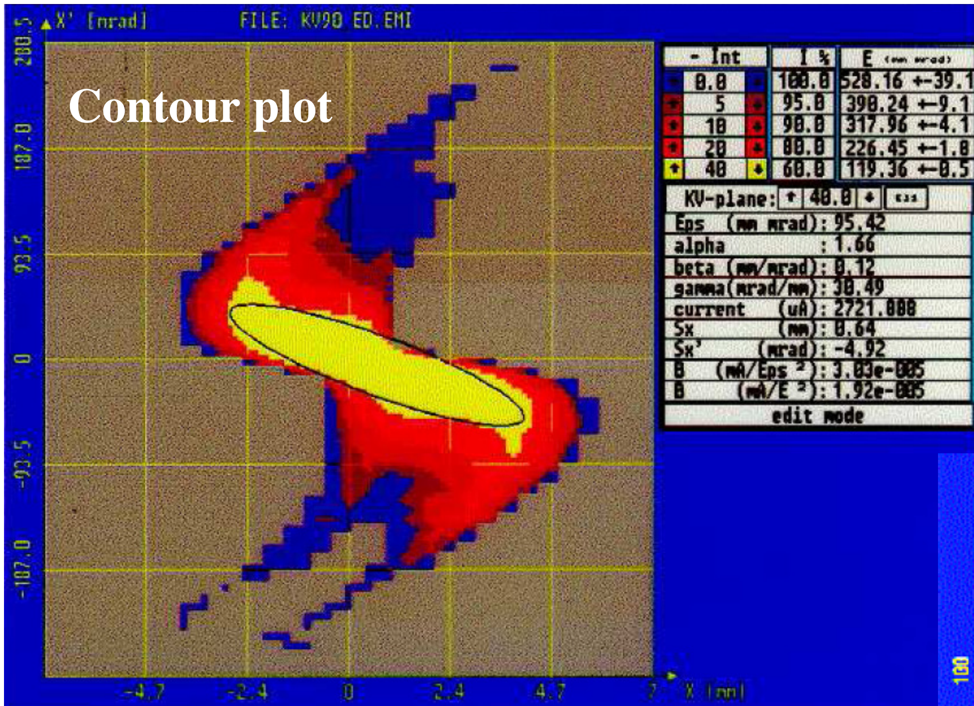
→ measurement of current.

Example: 15 wire spaced by 1.5 mm:



Each wire is equipped with one I/U converter
different ranges settings by $\boldsymbol{R_i}$
→ very large dynamic range up to 10^6.

Result for a beam behind ion source: ➢ here aberration in quadrupoles due to large beam size



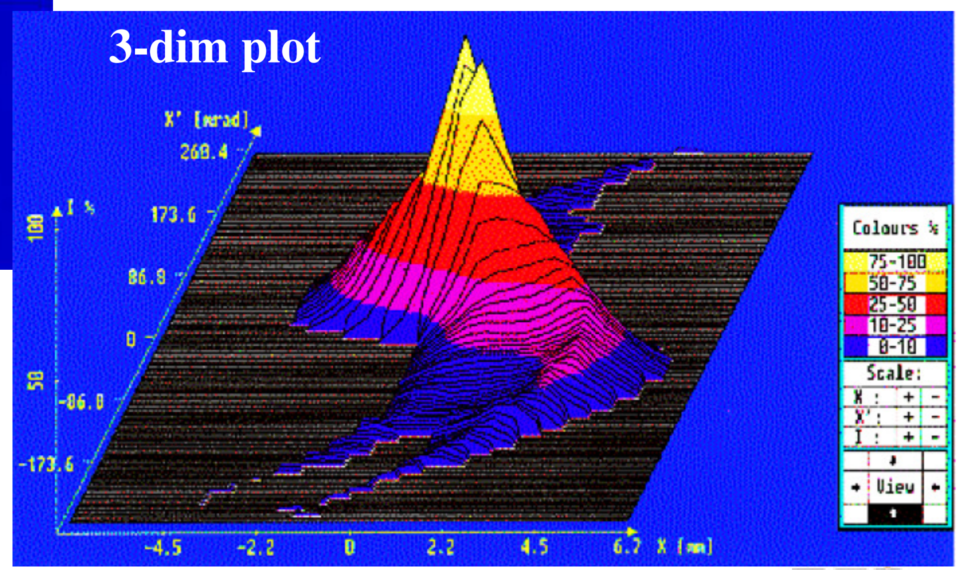
➢ different evaluation and plots possible

➢ can monitor any distribution

Calculation of moments

$\langle x^2 \rangle$, $\langle x'^2 \rangle$ and $\langle xx' \rangle$ and

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$



Low energy ion beam:

→ well suited for emittance showing space-charge effects or aberrations.

The Resolution of a Slit-Grid Device

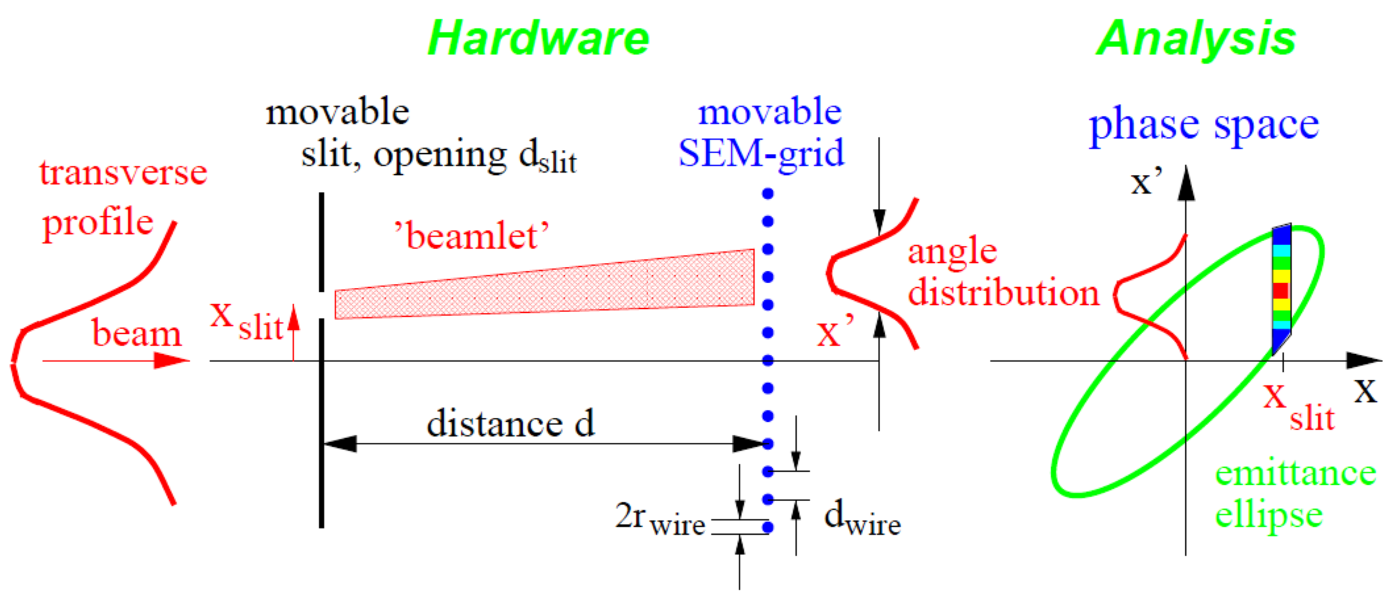
The width of the slit d_{slit} gives the resolution in space $\Delta x = d_{slit}$.

The angle resolution is $\Delta x' = (d_{slit} + 2r_{wire})/d$

$\Rightarrow$ discretization element $\Delta x \cdot \Delta x'$.

By scanning the SEM-grid the angle resolution can be improved.

Problems for small beam sizes or parallel beams.

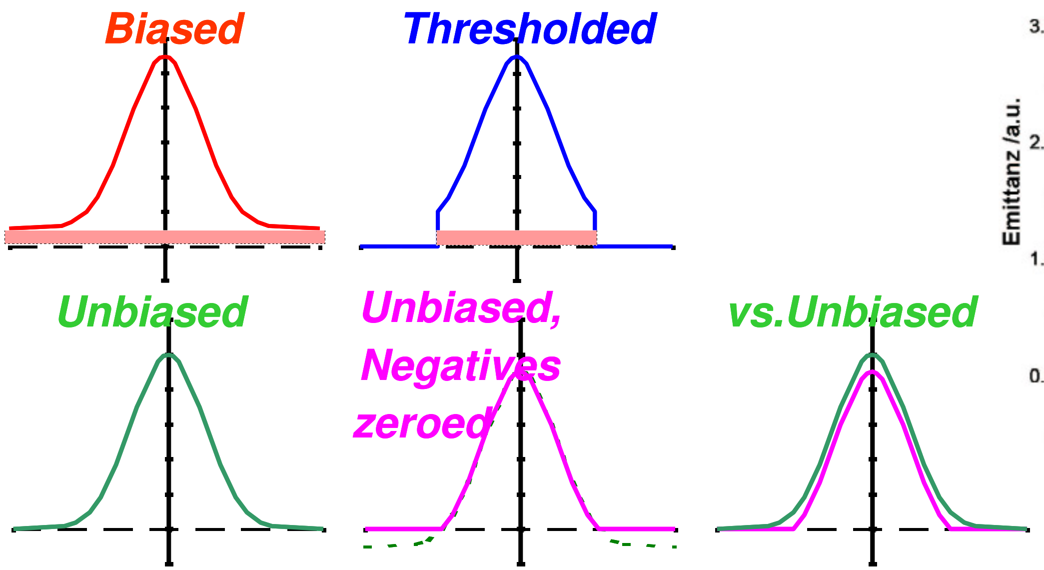


For pulsed LINACs: Only one measurement each pulse → long measuring time required.

The Noise Influence for Emittance Determination

A real measurement of beamlets contains:

- **Noise i.e. fluctuation of the output**
- **Bias i.e. electrical offset from amplifier**

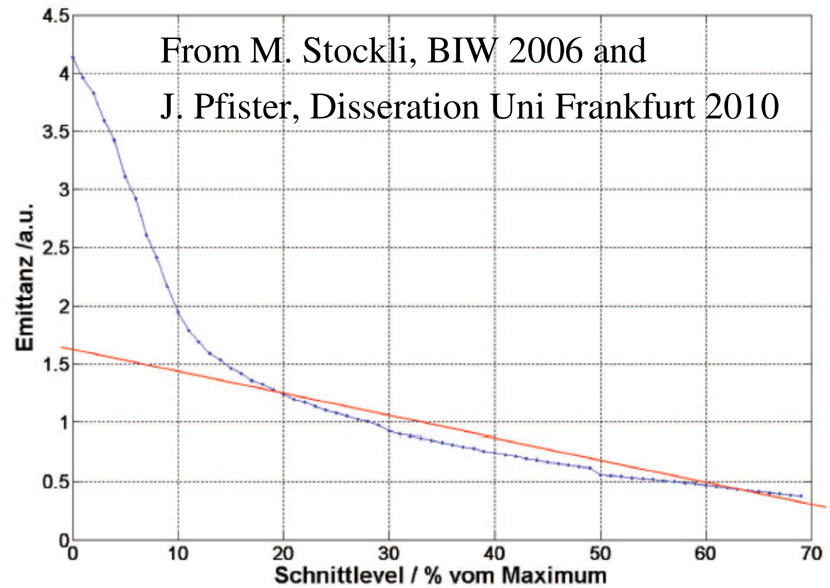


→ Strong influence of noise reduction to numerical values of ***<x>, <x'²>*** and ***<xx'>*** and on ε_{rms}

⇒Algorithm & cut-level must be given for evaluation

General: Typical error ***Δε/ε > 10 %***

Example: Dependence of ε_{rms} on threshold value



$$\langle x'^2 \rangle = \frac{\int x'^2 \cdot \rho(x,x')\, dx\, dx'}{\int \rho(x,x')\, dx\, dx'}$$ for continous values

$$= \frac{\sum_{i,j} x'^{2}_{ij} \cdot P(x_{ij}, x'_{ij})}{\sum_{i,j} P(x_{ij}, x'_{ij})}$$ for discrete values

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

Outline:

- Definition and some properties of transverse emittance
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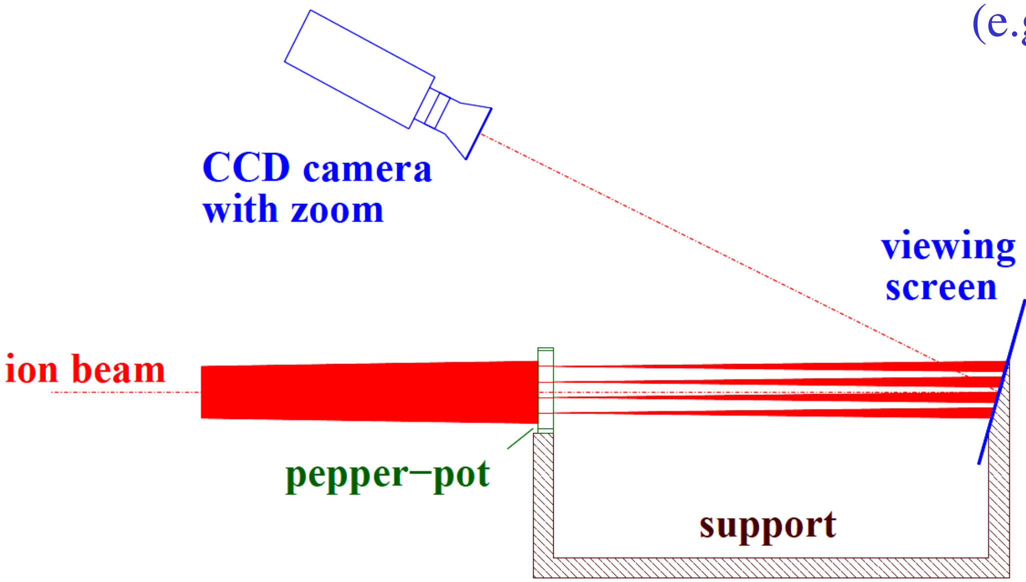
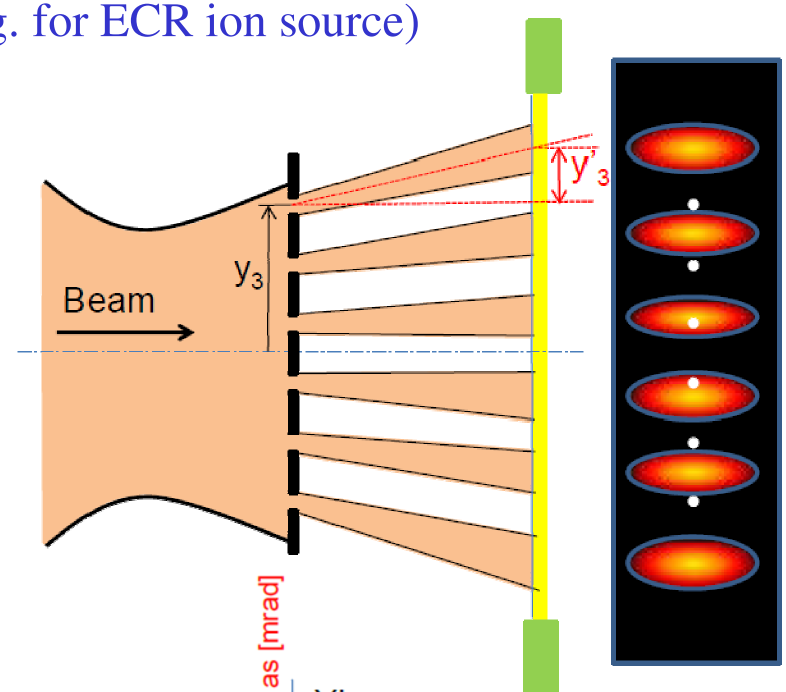
  scanning slit → beam position & grid → angular distribution
- **Pepper-pot device: single shot device**

  hole-plate → beam position & screen → angular distribution
- **Quadrupole strength variation and position measurement**
- **Summary**

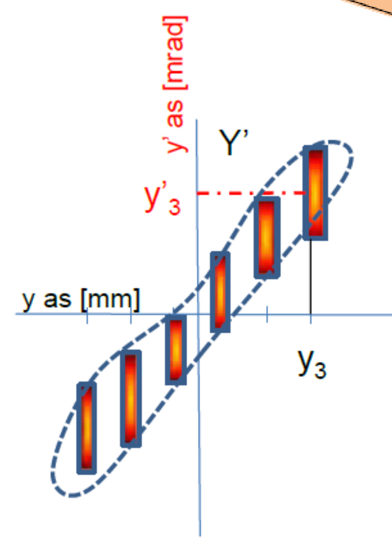
- For pulsed LINAC: Measurement within one pulse is an advantage
- If horizontal and vertical direction coupled → 2-dim evaluation **required** (e.g. for ECR ion source)





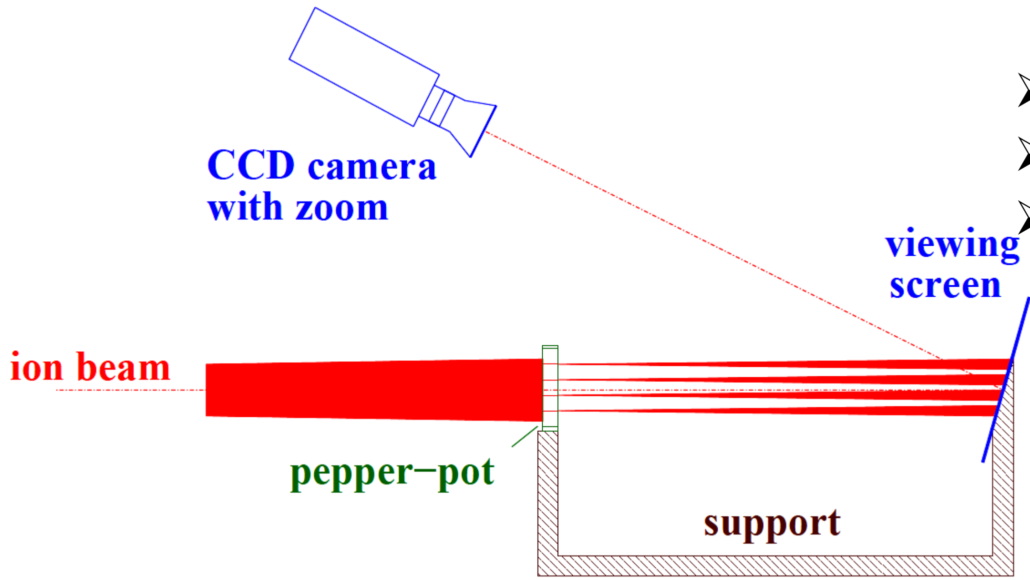
Good **spatial** resolution if many holes are illuminated.

Good **angle** resolution ***only*** if spots do not overlap.



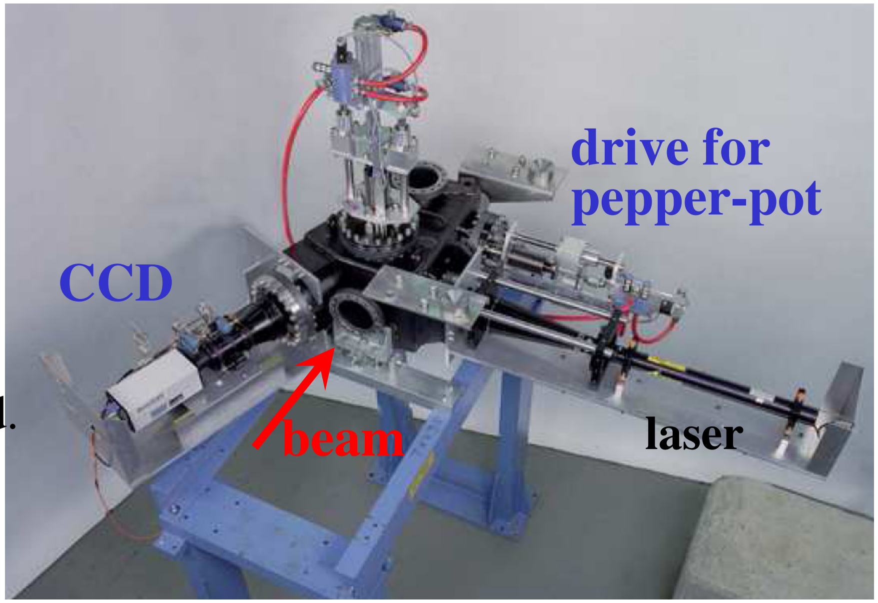
Partly from H.R. Kremers et al., ECRIS 2010

The Pepperpot Emittance Device at GSI UNILAC



Example GSI-LINAC 0.12 to 11 MeV/u:

- ***Pepper-pot***: 15 × 15 holes with Ø 0.1mm on a 50 × 50 mm^2 copper plate
- ***Distance***: pepper-pot-screen: 25 cm
- ***Screen:*** Al_2O_3, Ø 50 mm
- ***Data acquisition***: high resolution CCD

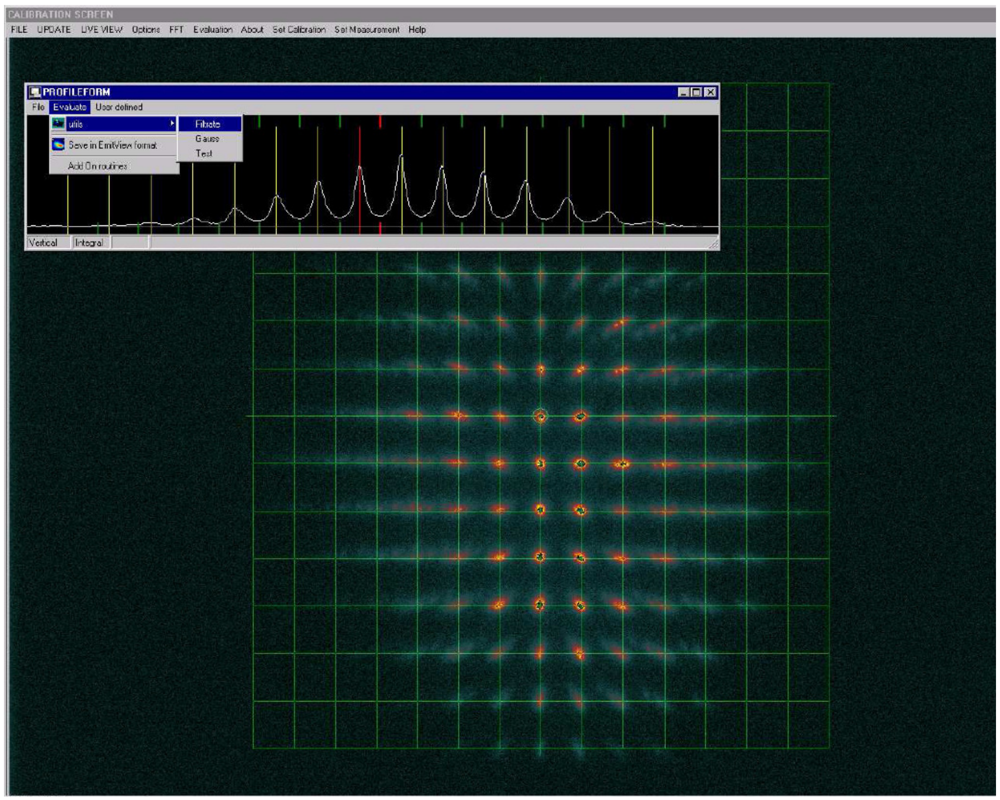


Good **spatial** resolution if many holes are illuminated.

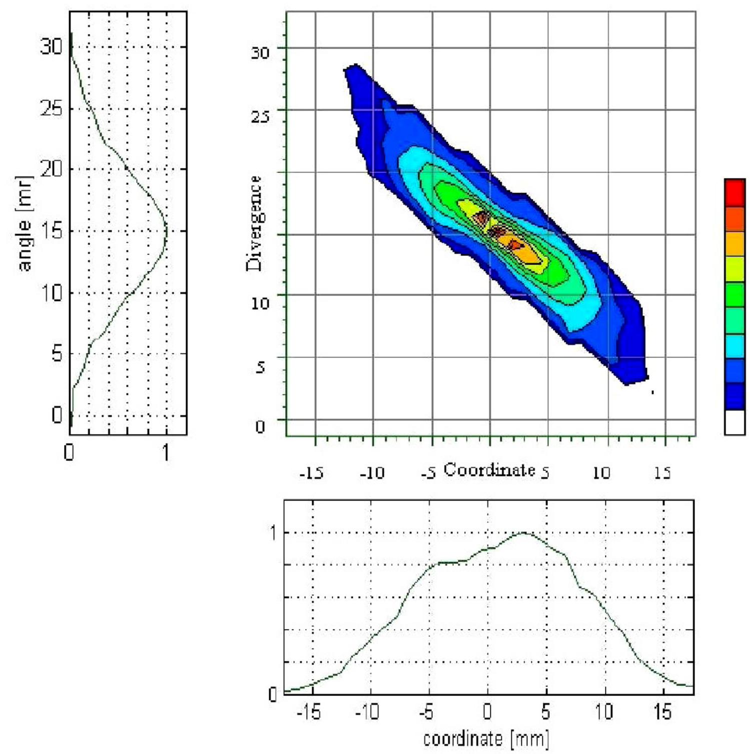
Good **angle** resolution ***only*** if spots do not overlap.

Result of a Pepperpot Emittance Measurement

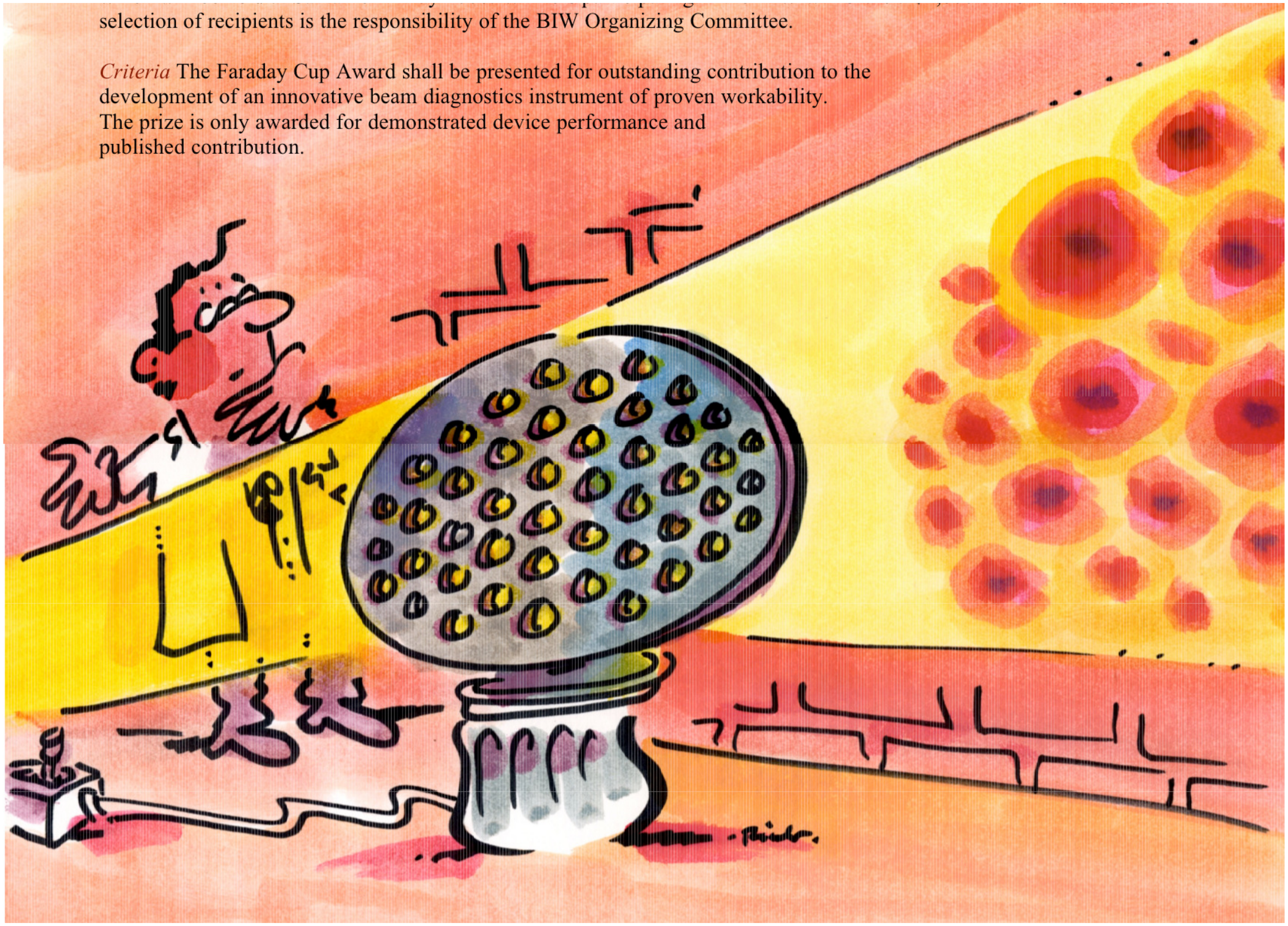
***Example*:** Ar $^{1+}$ ion beam at 1.4 MeV/u, screen image from single shot at GSI:



Data analysis:
Projection on horizontal and vertical plane
→ analog to slit-grid.



The Artist View of a Pepperpot Emittance Device



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  scanning slit → beam position & grid → angular distribution
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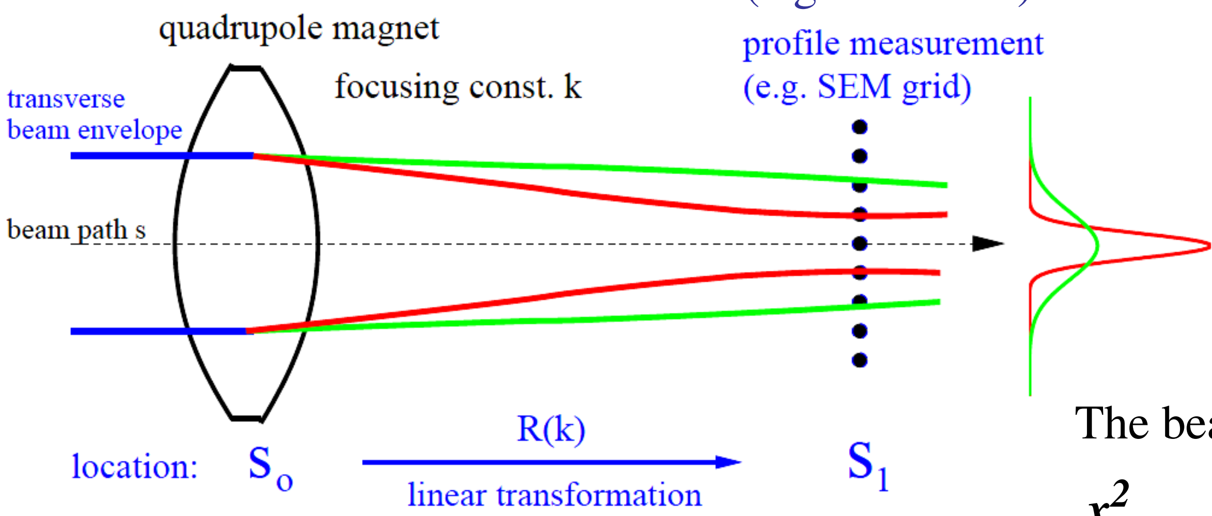
  hole-plate → beam position & screen → angular distribution
- **Quadrupole strength variation and position measurement**

  emittance from several profile measurement and beam optical calculation
- **Summary**

Emittance Measurement by Quadrupole Variation

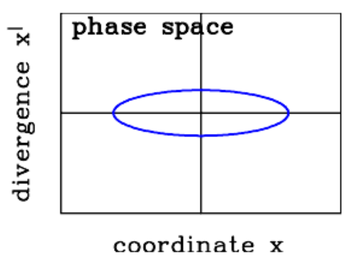
From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.



The beam width x_{max} and

$x^2_{max} = \sigma_{11}(1, k)$ is measured,

matrix **R(k)** describes the focusing.

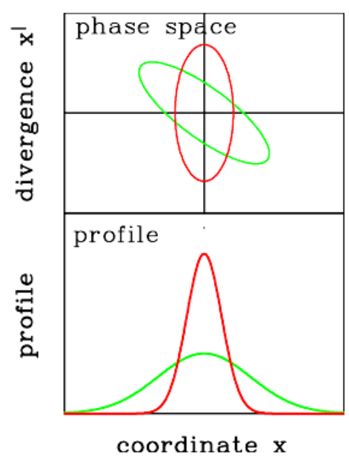


beam matrix:

(Twiss parameters)

$\sigma_{11}(0), \sigma_{12}(0), \sigma_{22}(0)$

to be determined



measurement:

$x^2(\mathbf{k}) = \sigma_{11}(1,\mathbf{k})$

Some Examples for linear Transformations

Without dispersion one can use the 2-dim sub-space (x, x').

- Drift with length L: $\mathbf{R}_{\mathbf{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
- Horizontal *focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R}_{\mathbf{focus}} = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix}$$

- Horizontal *de-focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R}_{\mathbf{defocus}} = \begin{pmatrix} \cosh\sqrt{k}L & \frac{1}{\sqrt{k}}\sinh\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sinh\sqrt{k}L & \cosh\sqrt{k}L \end{pmatrix}$$

For a (ideal) quadrupole with field gradient $g = B_{pole}/a$, B_{pole} is the field at the pole and a the aperture, the quadrupole constant $k = |g|/\left(B\rho\right)_0$ for a magnetic rigidity $(B\rho)_0$.

Measurement of transverse Emittance

- The beam width x_{max} at s_1 is measured, and therefore $\sigma_{11}(1,k_i) = x^2_{max}(k_i)$.
- Different focusing of the quadrupole $k_1, k_2...k_n$ is used: $\Rightarrow \mathbf{R}_{\mathbf{focus}}(k_i)$, including the drift, the transfer matrix is changed $\mathbf{R}(k_i) = \mathbf{R}_{\mathbf{drift}} \cdot \mathbf{R}_{\mathbf{focus}}(k_i)$.
- Task: Calculation of *beam* matrix $\sigma(0)$ at entrance s_0 (size and orientation of ellipse)
- The transformations of the beam matrix are: $\sigma(1,k) = \mathbf{R}(k) \cdot \sigma(0) \cdot \mathbf{R}^{\mathbf{T}}(k)$.

  $\Longrightarrow$ Resulting in a redundant system of linear equations for $\sigma_{ij}(0)$:

$$
\begin{array}{rcll}
\sigma_{11}(1,k_1) & = & R^2_{11}(k_1)\cdot\sigma_{11}(0) + 2R_{11}(k_1)R_{12}(k_1)\cdot\sigma_{12}(0) + R^2_{12}(k_1)\cdot\sigma_{22}(0) & \text{focusing } k_1 \\
 & \vdots & & \\
\sigma_{11}(1,k_n) & = & R^2_{11}(k_n)\cdot\sigma_{11}(0) + 2R_{11}(k_n)R_{12}(k_n)\cdot\sigma_{12}(0) + R^2_{12}(k_n)\cdot\sigma_{22}(0) & \text{focusing } k_n
\end{array}
$$

- To learn something on possible errors, $n > 3$ settings have to be performed. A setting with a focus close to the SEM-grid should be included to do a good fit.
- *Assumptions:*
  - Only elliptical shaped emittance can be obtained.
  - No broadening of the emittance e.g. due to space-charge forces.
  - If *not* valid: A self-consistent algorithm has to be used.

Measurement of transverse Emittance

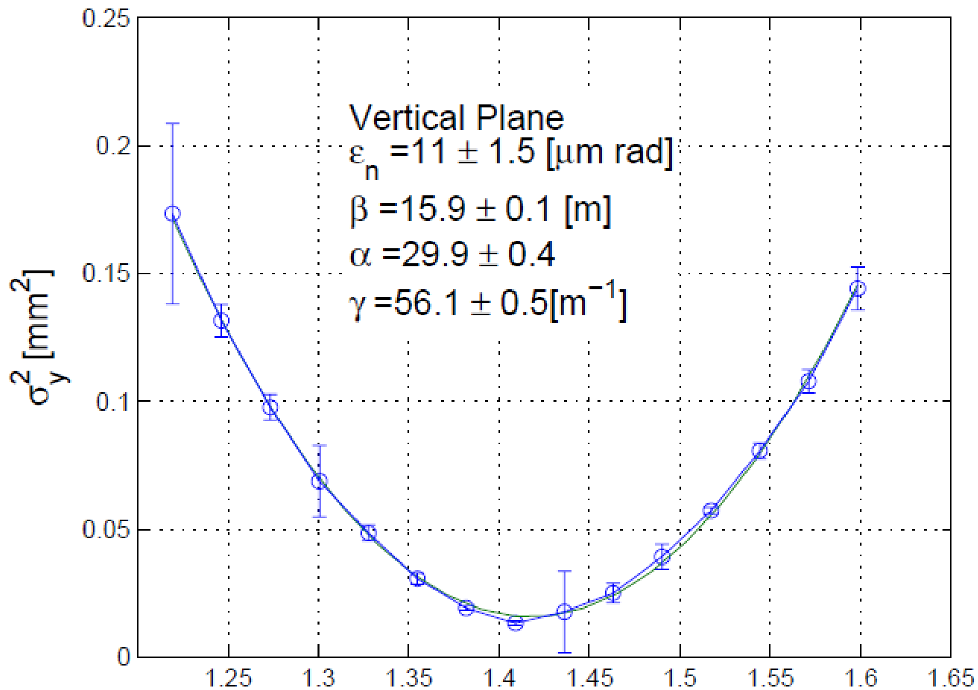
Using the 'thin lens approximation' i.e. the quadrupole has a focal length of ***f***:

$$R_{focus}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \Rightarrow R(L,K) = R_{drift}(L) \cdot R_{focus}(K) = \begin{pmatrix} 1+LK & L \\ K & 1 \end{pmatrix}$$

Example: Square of the beam width at ELETTRA 100 MeV e⁻ Linac, YAG:Ce:

Measurement of $\boldsymbol{\sigma(1,K) = R(K) \cdot \sigma(0) \cdot R^T(K)}$

$$\sigma_{11}(1,K) = L^2\sigma_{11}(0) \cdot K^2$$
$$+ 2 \cdot (\ L\sigma_{11}(0) + L^2\sigma_{12}(0)\ ) \cdot K$$
$$+ L^2\sigma_{22}(0) + \sigma_{11}(0)$$
$$\equiv a \cdot K^2 \ - \ 2ab \cdot K \ + \ ab^2 + c$$



Focusing strength K [m^{-1}]

G. Penco et al., EPAC'08

The σ-matrix at quadrupole is:

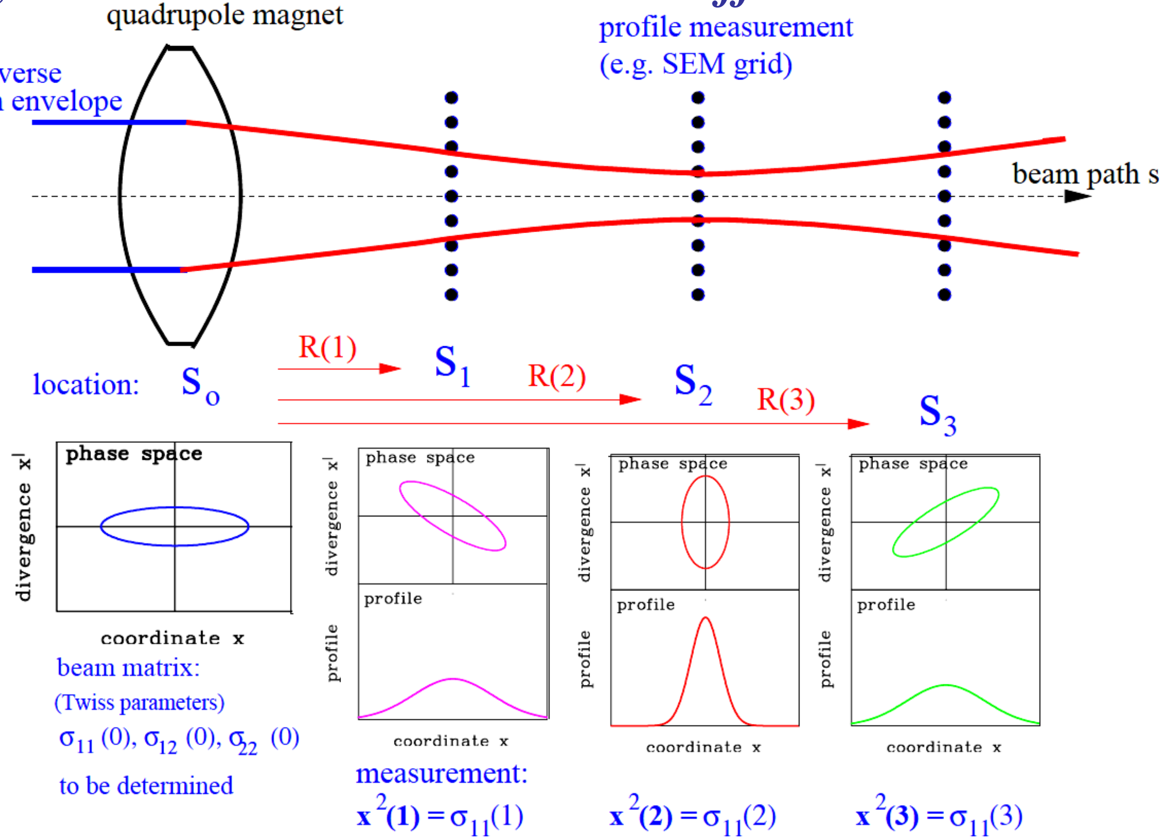
$$\sigma_{11}(0) = \frac{a}{L^2}$$
$$\sigma_{12}(0) = -\frac{a}{L^2}\left(\frac{1}{L} + b\right)$$
$$\sigma_{22}(0) = \frac{1}{L^2}\left(ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2}\right)$$

$$\epsilon = \sqrt{\det \sigma(0)} = \sqrt{\sigma_{11}(0)\sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac}/L^2$$

Instead of quadrupole variation, the beam width is measured at ***different*** locations:

The procedure is:

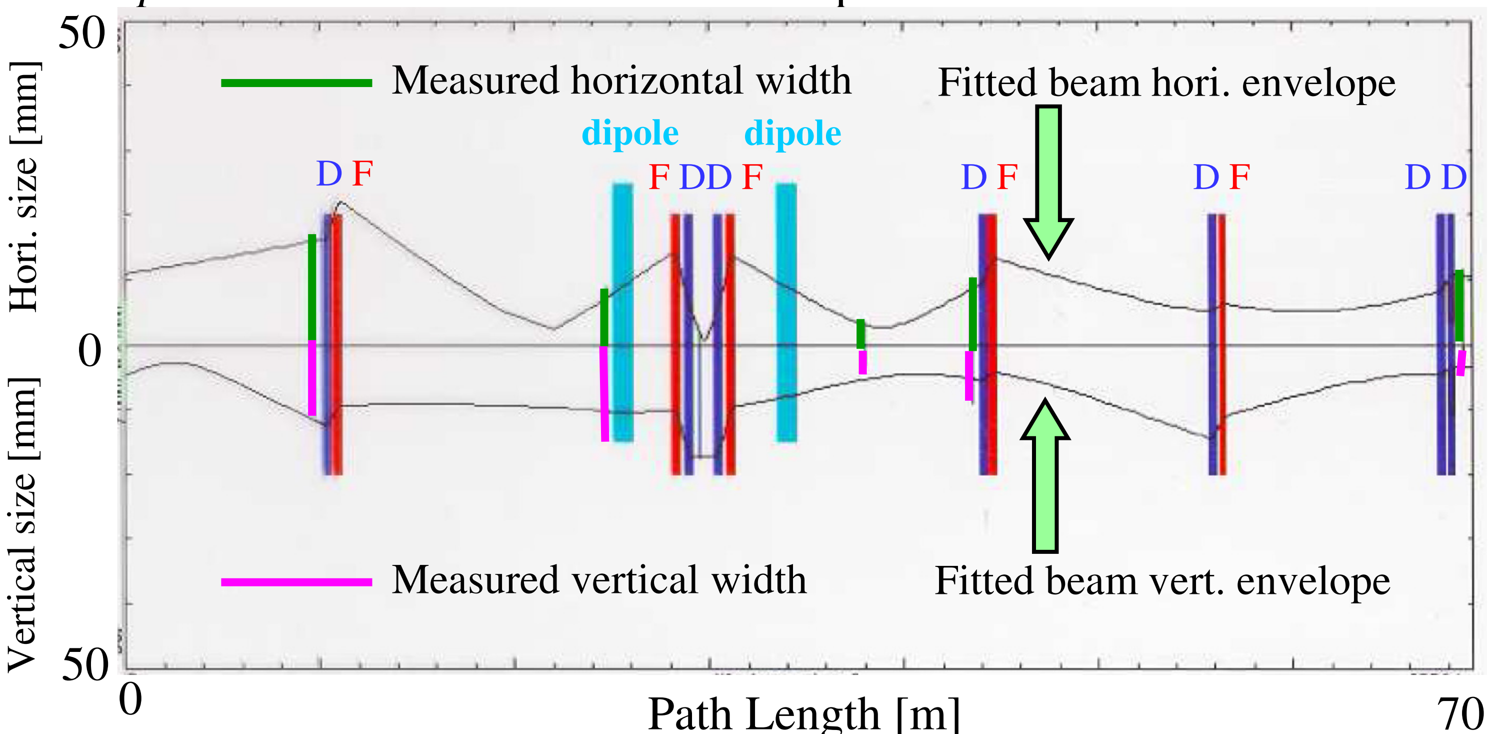
- Beam width $x(i)$ measured at the locations s_i
  $\Rightarrow$ beam matrix element
  $x^2(i) = \sigma_{11}(i)$.
- The transfer matrix $\mathbf{R}(i)$ is known. (without dipole a 3 × 3 matrix.)
- The transformations are:
  $\sigma(i) = \mathbf{R}(i)\sigma(0)\mathbf{R}^T(i)$
  $\Rightarrow$ redundant equations:



$$
\begin{aligned}
\sigma_{11}(1) &= R_{11}^2(1)\cdot\sigma_{11}(0) + 2R_{11}(1)R_{12}(1)\cdot\sigma_{12}(0) + R_{12}^2(1)\cdot\sigma_{22}(0) \qquad \mathbf{R}(1): s_0 \rightarrow s_1\\
\sigma_{11}(2) &= R_{11}^2(2)\cdot\sigma_{11}(0) + 2R_{11}(2)R_{12}(2)\cdot\sigma_{12}(0) + R_{12}^2(2)\cdot\sigma_{22}(0) \qquad \mathbf{R}(2): s_0 \rightarrow s_2\\
&\vdots\\
\sigma_{11}(n) &= R_{11}^2(n)\cdot\sigma_{11}(0) + 2R_{11}(n)R_{12}(n)\cdot\sigma_{12}(0) + R_{12}^2(n)\cdot\sigma_{22}(0) \qquad \mathbf{R}(n): s_0 \rightarrow s_n
\end{aligned}
$$

Solution: Solving the linear equations like for quadrupole variation or fitting the profiles with linear optics code (e.g. TRANSPORT, WinAgile, MadX).

Example: The hor. and vert. beam envelope and the beam width at a transfer line:



Assumptions:
- constant emittance, in particular no space-charge broadening
- 100 % transmission i.e. no loss due to vacuum pipe scraping
- no misalignment, i.e. beam center equals center of the quadrupoles.

Summary for transverse Emittance Measurement

Emittance measurements are very important for comparison to theory.

It includes size (value of ε) and orientation in phase space (σ_{ij} or α, β and γ)

(three independent values)

Techniques for transfer lines (synchrotron: width measurement sufficient):

Low energy beams → direct measurement of x- and x'-distribution

- ***Slit-grid:*** movable slit → ***x***-profile, grid → ***x'***-profile
- ***Pepper-pot***: holes → ***x***-profile, scintillation screen → ***x'***-profile

All beams → profile measurement + linear transformation:

- ***Quadrupole variation:*** one location, different setting of a quadrupole
- ***'Three grid method':*** different locations
- ***Assumptions:***
  - well aligned beam, no steering
  - no emittance blow-up due to space charge.