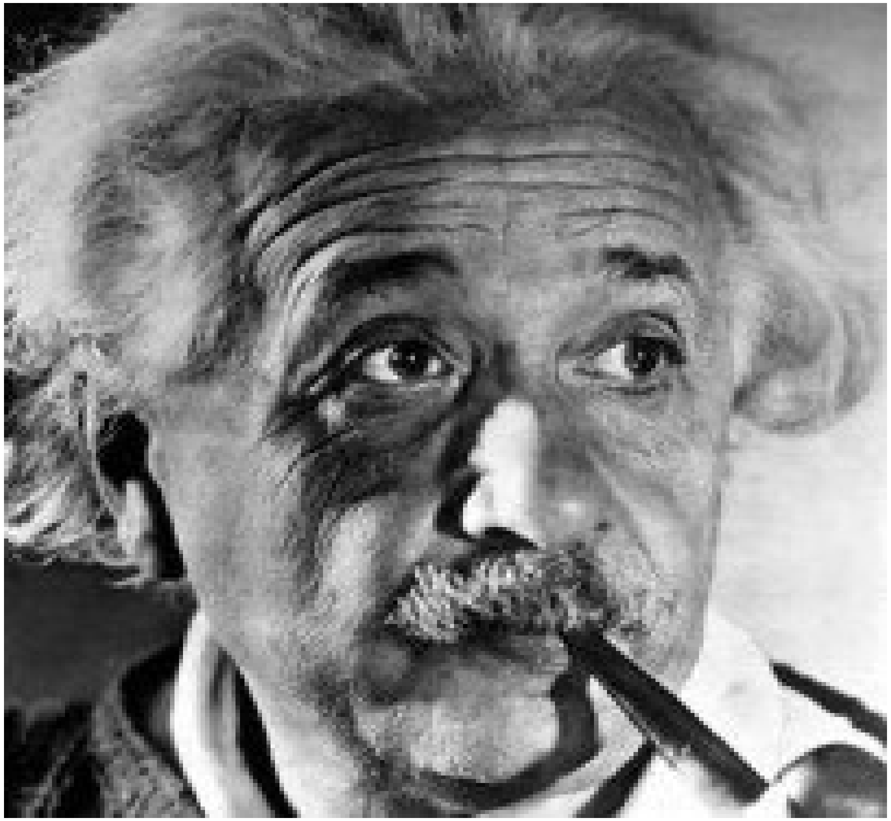
Review of Special Relativity



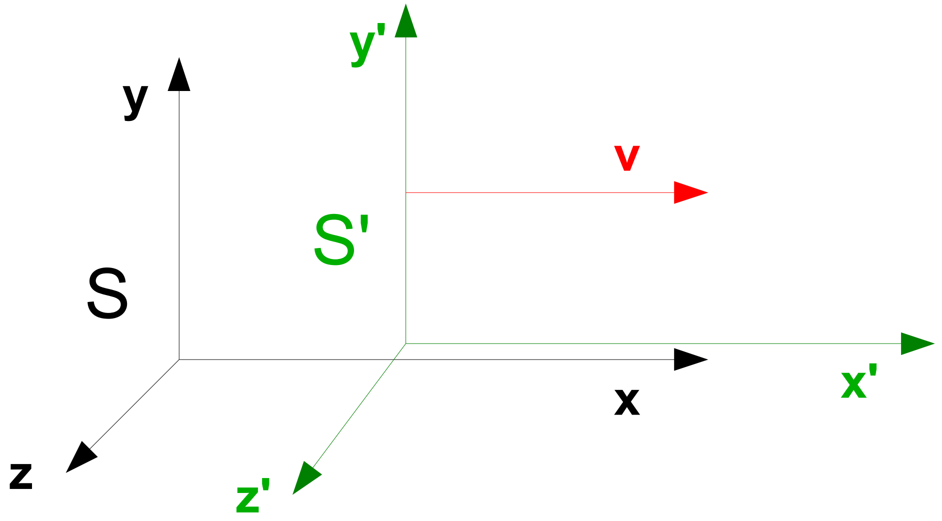
This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Why was „Special Relativity“ needed?

Mechanical laws (Newton's laws) are the same for all inertial systems.

They are invariant under a Galilean transformation (G-T):

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$



El.-mag. laws are not invariant under a G-T.

Take the wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

it transforms to

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2} - 2\frac{v}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right]\Phi = 0$$

moreover, it contains the speed of light as a constant, independent of the reference system, contradicting the deep belief in a supporting media (ether) for the waves.

Many experiments tried to prove el.-mag. theory wrong.

They all failed!

Michelson-Morley experiment showed that c is a constant and that there exists no „ether".

→ The Newton-Galileo concept of space and time had to be modified

Relativistic Kinematics

Einstein based his theory on two postulates:

1. All inertial frames are equivalent wrt. all laws of physics.
2. The speed of light is equal in all reference frames.

1st postulate:

Space is isotropic (all directions are equivalent) and homogeneous (all points are equivalent)

Homogeneity and form-invariance under transformation require a linear transformation.

$$ct' = a_{00}\,ct + a_{01}\,x + a_{02}\,y + a_{03}\,z$$
$$x' = a_{10}\,ct + a_{11}\,x + a_{12}\,y + a_{13}\,z$$
$$y' = a_{20}\,ct + a_{21}\,x + a_{22}\,y + a_{23}\,z$$
$$z' = a_{30}\,ct + a_{31}\,x + a_{32}\,y + a_{33}\,z$$

Successive use of homogeneity, isotropy and the speed of light determines the constants, such that the Lorentz Transform (L-T) is obtained:

$$ct' = \gamma\,(ct - \beta\,x) \qquad y' = y$$
$$x' = \gamma\,(x - \beta\,ct) \qquad z' = z$$

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

The inverse transform is obtained by replacing the primed variables by the unprimed, the unprimed by the primed and by replacing β by -β:

$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta ct')$$
$$y = y', \qquad z = z'$$

We write the L-T as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

L is an affine transformation. It preserves the rectilinearity and parallelisme of straight lines.

$$\underline{\underline{L}}=\begin{bmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \underline{\underline{L}}^{-1}=\begin{bmatrix}\gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \underline{\underline{L}}\,\underline{\underline{L}}^{-1}=\underline{1}$$

Time dilatation:

Two events in S' at t_1', t_2' and at location $x_1'=x_2'$

$$c(t_2-t_1)=\gamma(ct_2'-ct_1')+\beta(x_2'-x_1')$$

$$\Delta t=\gamma\Delta t'$$

Two events in S at t_1, t_2 and at location $x_1=x_2$

$$c(t_2'-t_1')=\gamma(ct_2-ct_1)-\beta(x_2-x_1)$$

$$\Delta t'=\gamma\Delta t$$

Length contraction:

A meter in S' is extending from x_1' to x_2'.

It is measured in S at the time $t_1 = t_2$

$$x_2' - x_1' = \gamma(x_2 - x_1) - \beta\gamma c(t_2 - t_1)$$

$$\Delta x = \frac{1}{\gamma}\Delta x'$$

Perpendicular dimensions remain: $\Delta y = \Delta y'$

Time intervals and distances depend on the motion of the observer.

$$\Delta t = \gamma \Delta t' \text{ and } \Delta x = \frac{1}{\gamma}\Delta x'$$

are not standard equations!!

A space-time interval

$$ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2$$

is invariant under L-T.

We write the space-time interval as

$$ds = cdt\sqrt{1 - \frac{1}{c^2}\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)} = c\frac{dt}{\gamma} = c\,d\tau$$

and identify $d\tau$ as the time interval a particle moving with v would measure. τ is called proper time and is Lorentz invariant.

Let us define contra- and covariant 4-vectors

$$\textit{contravariant} \quad X^{\mu}=(X^0,X^1,X^2,X^3)=(ct,x,y,z)$$

$$\textit{covariant} \quad X_{\mu}=(X_0,X_1,X_2,X_3)=(ct,-x,-y,-z)$$

then a scalar product is L-T invariant

$$X^{\mu}X_{\mu}=X^0X_0+X^1X_1+X^2X_2+X^3X_3=$$
$$=(ct)^2-x^2-y^2-z^2=X'^{\mu}X'_{\mu}$$

In general: The scalar product of any two 4-vectors (whose components transform like (ct,x,y,z)) is Lorentz invariant

$$A^{\mu}B_{\mu}=A'^{\mu}B'_{\mu} \quad \textit{in particular} \quad A^{\mu}A_{\mu}=A'^{\mu}A'_{\mu}$$

Transformation of velocity

A particle moving with velocity u' in S' has velocity u in S

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \gamma\left(\frac{dx'}{dt'} + \beta c\right)\frac{dt'}{dt} = \gamma(u'_x + v)\frac{dt'}{dt}$$

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{\beta}{c}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{v}{c^2}u'_x\right)$$

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \quad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}$$

Example 1:

$$v = u'_x = 30\, km/s = 10^{-4}\, c$$

$$u_x = 2\, 10^{-4}\, c / (1 + 10^{-8}) = 60\, (1 - 10^{-8})\, km/s$$

Example 2:

$$v = u'_x = 0.9\, c$$

$$u_x = 1.8\, c / (1 + 0.81) = 0.995\, c$$

Example 3:

$$Motion\ in\ two\ directions: \quad \vec{u}' = (u'_x, u'_y, 0)$$

$$\tan(\vartheta') = \frac{u'_y}{u'_x} \quad \rightarrow \quad \tan(\vartheta) = \frac{u_y}{u_x} = \frac{1}{\gamma} \frac{1}{1 + v/u'_x} \tan(\vartheta')$$

Transformation of acceleration

A particle moving with u' in S' and experiencing an acceleration a' has a in S.

$$a_x = \frac{du_x}{dt} = \frac{du_x}{dt'}\frac{dt'}{dt} = \frac{d}{dt'}\frac{u'_x + v}{1 + vu'_x/c^2}\frac{dt'}{dt} =$$

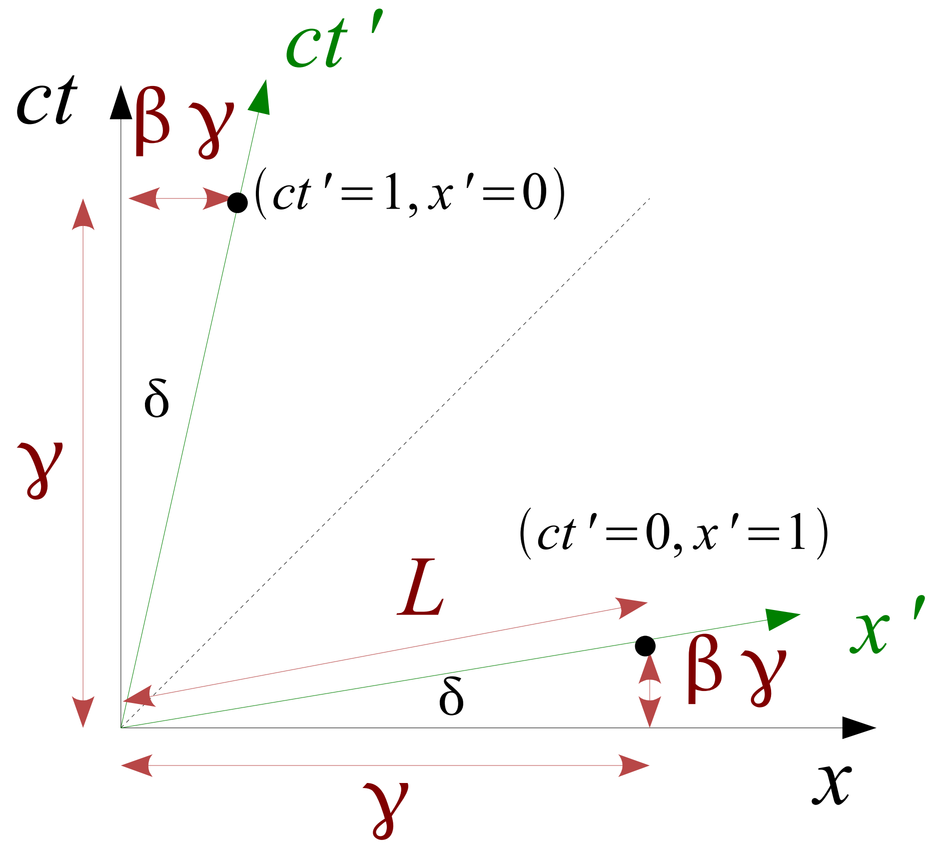
$$= \frac{a'_x}{\gamma^3 (1 + vu'_x/c^2)^3}$$

$$a_y = \frac{a'_y}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_y/c^2)a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

$$a_z = \frac{a'_z}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_z/c^2)a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

Acceleration in an inertial system is possible !!

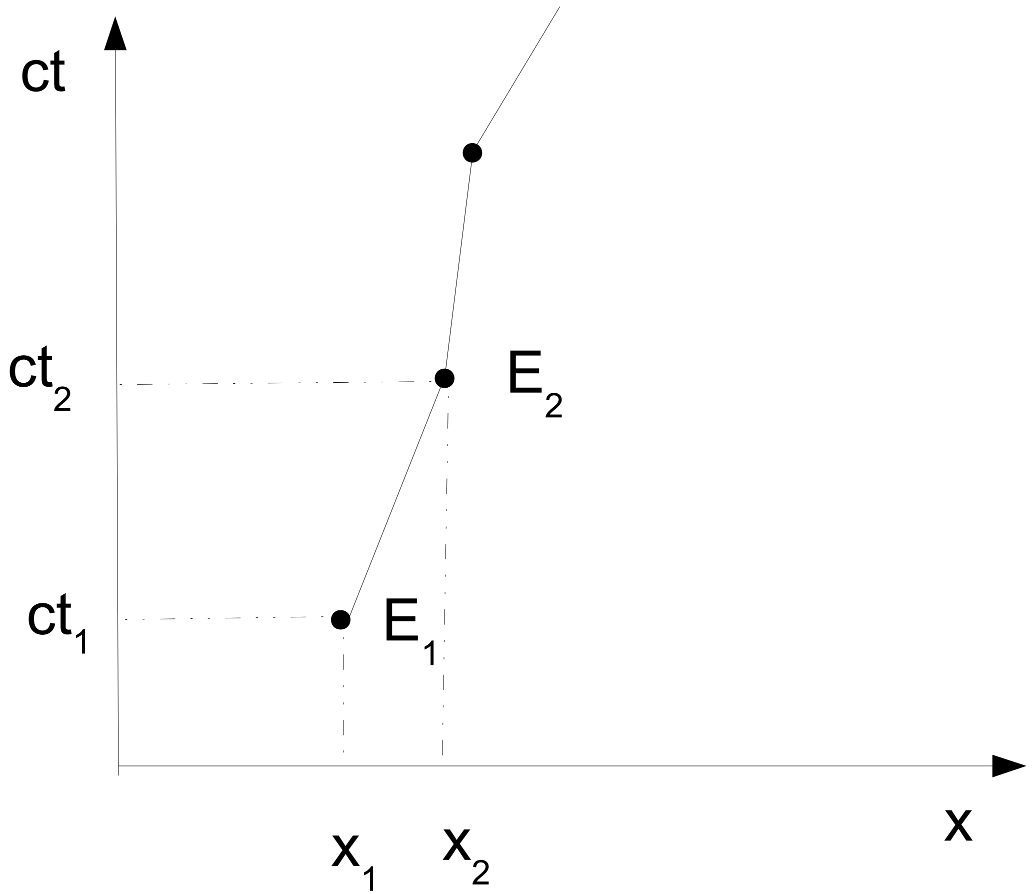
Minkowski diagram



$$\tan(\delta)=\beta$$

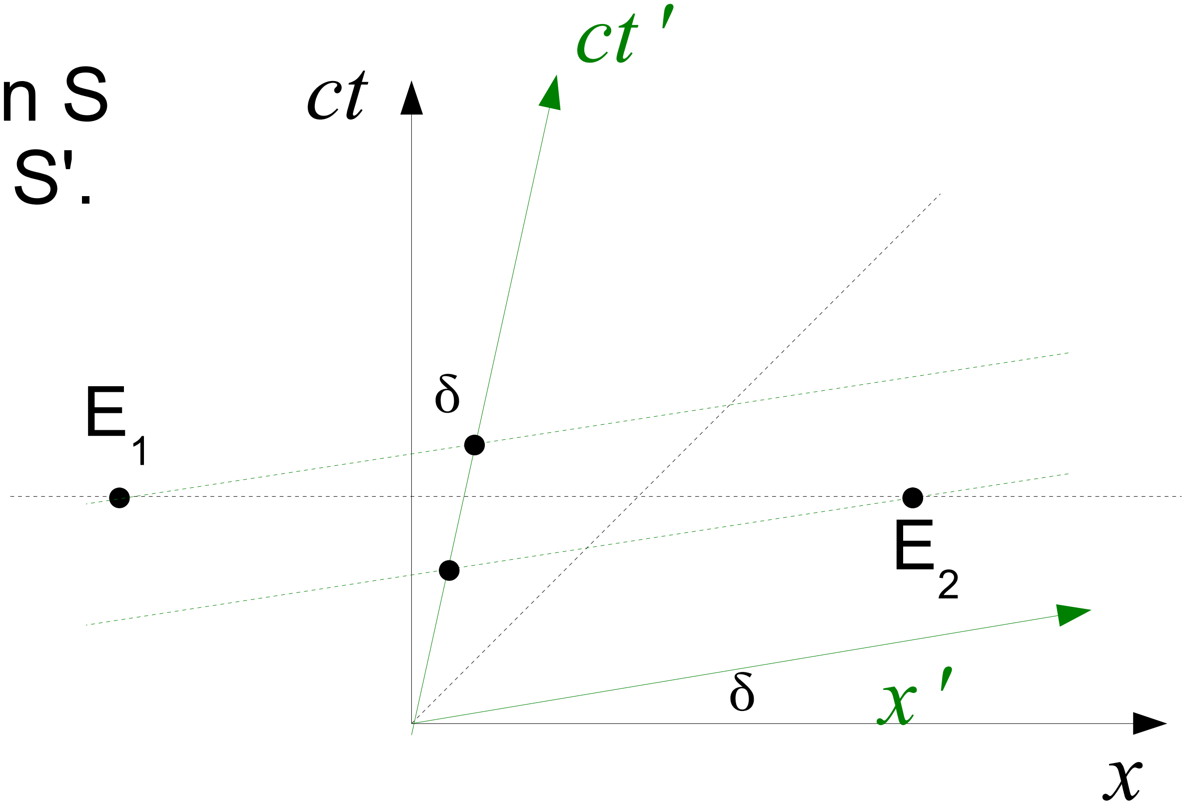
Scale in S': $$L=\sqrt{\gamma^2+\beta^2\gamma^2}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

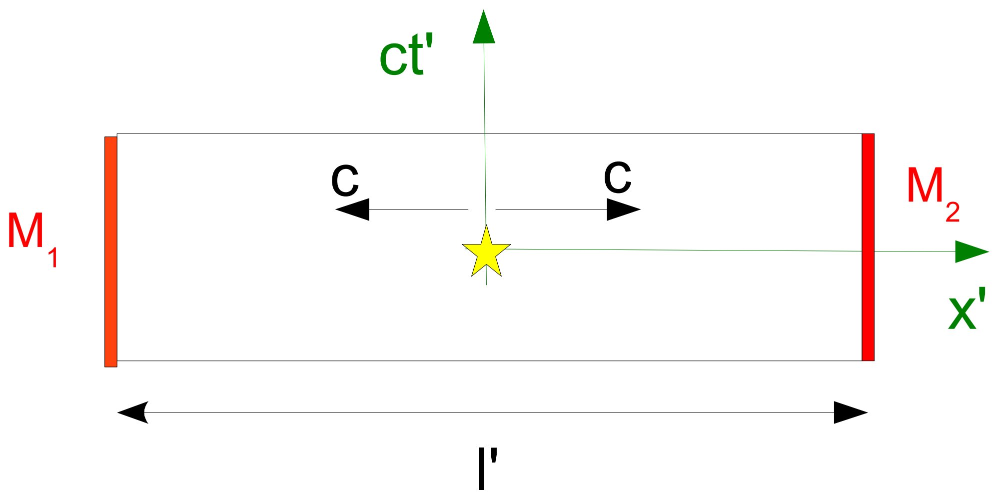
World-line (path-time diagram)



v=+c/-c is straight line with slope +1/- 1

2 events simultaneous in S are not simultaneous in S'.

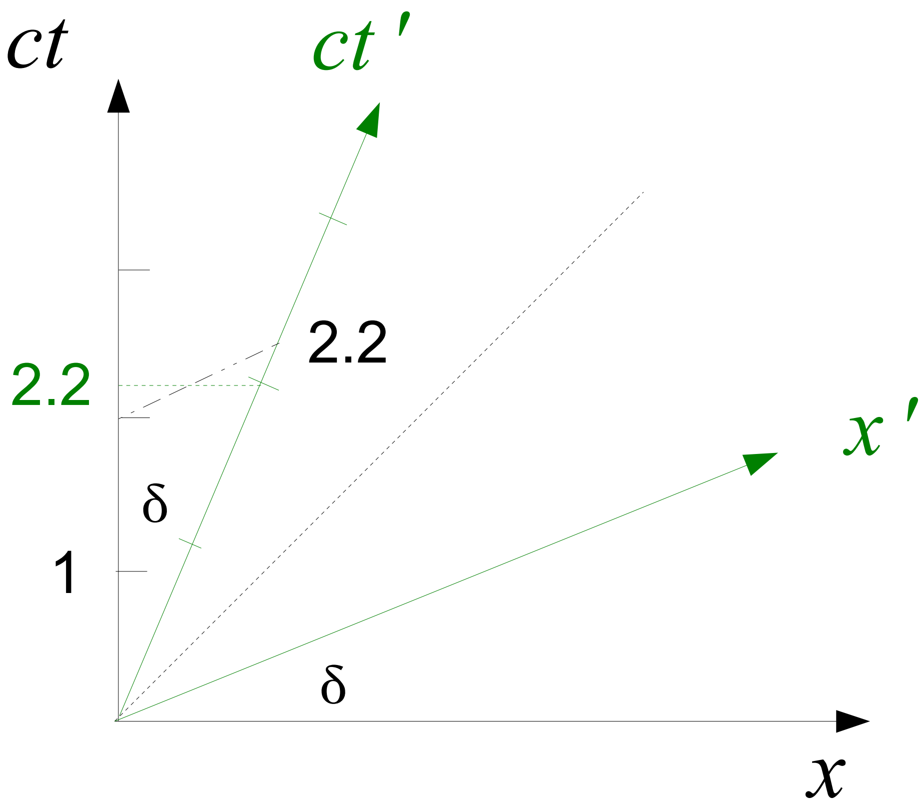




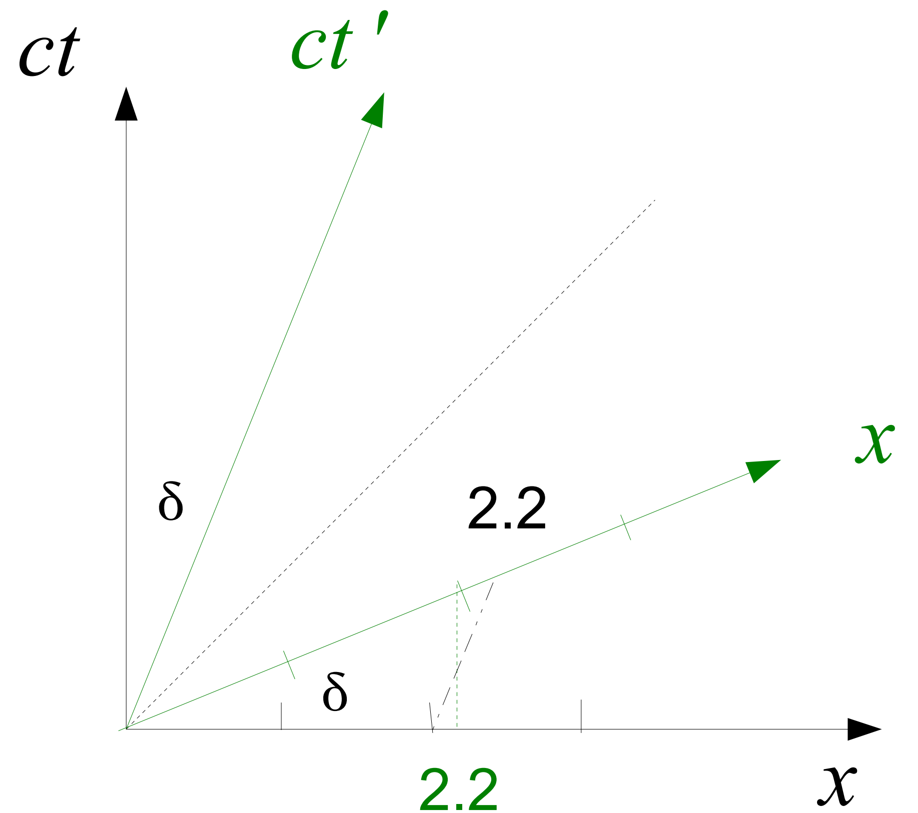
$$ct_1 = \frac{l}{2} - vt_1, \quad ct_2 = \frac{l}{2} + vt_2$$

$$t_2 - t_1 = \gamma \frac{\beta}{c} l'$$

Time dilatation:
B=0.42, δ=22.8°, γ=1.1, L=1.2



Length contraction:
β=0.42, δ=22.8°, γ=1.1, L=1.2



Relativistic Dynamics

Two principles:

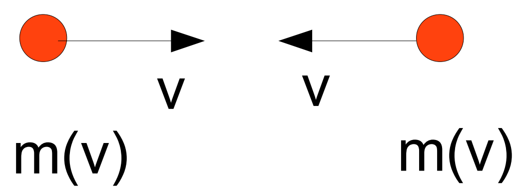
1. Conservation of linear momentum
2. Conservation of energy

Derivation of moving mass

Because of $E=mc^2$ we choose as ansatz $m=m(v)$

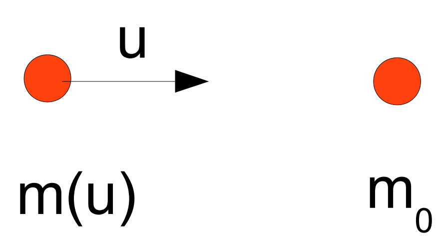
Inelastic collision between 2 identical particles:

Laboratory frame

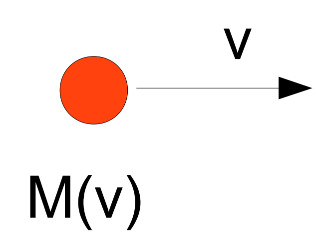


Composite particle at rest after collision

Rest frame of right particle:



Composite particle moving after collision (center of mass frame $\Sigma p=0$)



Conservation of momentum $$m(u)u = M(v)v \qquad (1)$$

$$u = \frac{2v}{1+(v/c)^2} \qquad (2)$$

Conservation of energy $$m(u)c^2 + m_0 c^2 = M(v)c^2 \qquad (3)$$

From (1), (2), (3) after eliminating M

$$m(u) = \frac{m_0}{\sqrt{1-(u/c)^2}} = \gamma_u m_0 \qquad (4)$$

From (1), (2), (4) $$M(v) = \frac{2m_0}{1-(v/c)^2} = \gamma M_0, \quad M_0 = 2\gamma m_0$$

$$M_0 - 2m_0 = 2m_0(\gamma - 1) > 0$$ *rest mass is not conserved*

Mass $$m(v)=\gamma_v m_0$$

Momentum $$\vec{p}(v)=m(v)\vec{v}$$

Force $$\vec{f}=\frac{d\vec{p}}{dt}=m_0\frac{d\gamma_v}{dt}\vec{v}+m_0\gamma_v\frac{d\vec{v}}{dt}=$$

$$=\gamma_v^3\frac{m_0}{c^2}(\vec{v}\cdot\vec{a})\vec{v}+\gamma_v m_0\vec{a}$$

$\vec{f}$ and $\vec{a}$ *are not* parallel. *With* $\vec{f}\cdot\vec{v}=\gamma_v^3 m_0(\vec{v}\cdot\vec{a})$ *we can solve for* $\vec{a}$

$$\vec{a}=\frac{1}{\gamma m_0}\left(\vec{f}-\frac{1}{c^2}(\vec{f}\cdot\vec{v})\vec{v}\right)$$

$$\text{for } \vec{v}=(v,0,0): \qquad \vec{F}=(\gamma^3 m_0 a_x, \gamma m_0 a_y, \gamma m_0 a_z)$$

$$\text{longitudinal mass} \quad \gamma^3 m_0, \quad \parallel \vec{v}$$

$$\text{transverse mass} \quad \gamma m_0, \quad \perp \vec{v}$$

Example: A charge q is at rest in S. At t=0 an electric field E_x is turned on. Calculate the velocity.

$$f_x=\frac{dp_x}{dt}=qE_x \quad \rightarrow \quad p_x=qE_x t=\frac{m_0 u_x}{\sqrt{1-(u_x/c)^2}}$$

$$u_x=\frac{qE_x t/m_0 c}{\sqrt{1+(qE_x t/m_0 c)^2}} \qquad u_x=\frac{qE_x t}{m_0 c}$$

Energy

A particle moves with $\vec{u}=(u,0,0)$ and *experiences a force* f_x. *Work done at path dx is*

$$dE_{kin}=f_x\,dx=\gamma^3 m_0 a_x\,dx=\gamma^3 m_0 u\,du$$

$$E_{kin}=m_0c^2\int_o^{\beta}\frac{\beta_u\,d\beta_u}{(1-\beta_u^2)^{3/2}}=\gamma_u m_0c^2-m_0c^2=E-E_0$$

E_{kin} *is difference of two energies. The total energy* $E=\gamma_u m_0 c^2$ *minus rest energy* $E_0=m_0c^2$.

1mg of mass corresponds to the energy to lift 1000t to a height of 1000m.

$$\frac{dE_{kin}}{dt} = \frac{dm}{dt} c^2$$

$$\vec{f} = \frac{d\vec{p}}{dt} = \frac{dm}{dt}\vec{u} + m\frac{d\vec{u}}{dt} = \frac{1}{c^2}\frac{dE_{kin}}{dt}\vec{u} + \gamma_u m_0 \vec{a}$$

$$\vec{a} = \frac{1}{\gamma_u m_0}\left(\vec{f} - \frac{1}{c^2}(\vec{f}\cdot\vec{u})\vec{u}\right)$$

$$\rightarrow \quad P = \frac{dE_{kin}}{dt} = \vec{f}\cdot\vec{u}$$

The temporal change of E_{kin} of a body, or the power it absorbs, is the scalar product of $\vec{f}$ and $\vec{u}$.

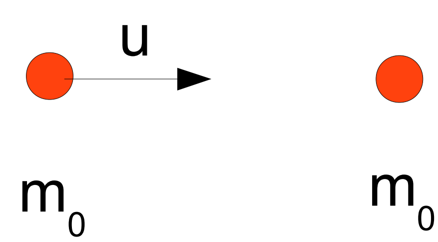
Examples: Collider

1) 3.5 TeV head-on p-p collider

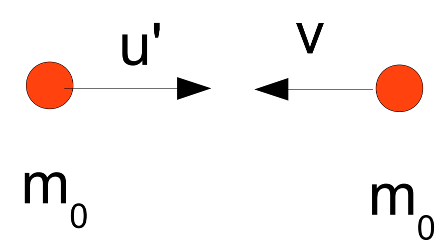
$$M_0 = 2\gamma m_0 \quad \rightarrow \quad E_{CM} = M_0 c^2 = 2E$$

$$E_{p0} = 938\,MeV, \quad \gamma = 3.7\,10^3, \quad E_{CM} = 7\,TeV$$

2) 3.5 TeV fixed target p-machine



Center of mass frame (Σp=0)
moves with -v



$$u' = \frac{u - v}{1 - uv/c^2} \quad \rightarrow \quad \gamma_{u'} = \gamma_u \gamma_v (1 - \beta_u \beta_v)$$

$$\Sigma p = \gamma_{u'} m_0 u' - \gamma_v m_0 v = 0 \quad \rightarrow \quad \beta_v = \frac{\beta_u}{1 + \sqrt{1 - \beta_u^2}}$$

$$E_{CM} = \gamma_{u'} m_0 c^2 + \gamma_v m_0 c^2 = \sqrt{2(1 + \gamma_u)} E_0 =$$

$$= \sqrt{2(E + E_0) E_0} = 81\, GeV$$

Energy-momentum equation and diagram

$$E^2 = (mc^2)^2 = (m_0 c^2)^2 \frac{(1-\beta^2)+\beta^2}{1-\beta^2} = E_0^2 + (pc)^2$$

$$\frac{E}{c} = \sqrt{(m_0 c)^2 + p^2}, \qquad mass-less\,particle: \quad E = |p|c$$

Since conserved quantities are plotted, arrows can be added like vectors.

All interactions are allowed in which energy-momentum vectors c, d after interaction add to vector s.

Photon absorption by a particle at rest

Absorption only for massive particles with excited states.

Photon emission by a massive particle at rest

$\Delta E > E_b$

Difference in recoil of particle

L-T of energy and momentum

A particle with m_0 moves in S with $\vec{u}=(u,0,0)$.
In S' it's velocity is $\vec{u}\,'=(u',0,0)$.

$$p_x=\gamma_u m_0 u=E\frac{u}{c^2}, \qquad E=\gamma_u m_0 c^2$$

$$u'=\frac{u-v}{1-uv/c^2} \quad \rightarrow \quad \gamma_{u'}=\gamma_u\gamma_v\left(1-\frac{uv}{c^2}\right)$$

$$\frac{E'}{c}=\gamma_{u'}m_0 c=\gamma_v\left(\frac{E}{c}-\beta_v p_x\right)$$

$$p_x'=\gamma_{u'}m_0 u'=\gamma_v\left(p_x-\beta_v\frac{E}{c}\right), \quad p_y'=p_y, \quad p_z'=p_z$$

E/c transforms like ct and $\vec{p}$ transforms like $\vec{r}$

Derivation of Planck's hypothesis E=hν

A photon with energy E' in S' travels in -x' direction

$$p_x' = -\frac{E'}{c}$$

$$E = \gamma_v (E' + v\, p_x') = \sqrt{\frac{1-\beta_v}{1+\beta_v}} E'$$

Frequency Doppler shift

$$\nu = \sqrt{\frac{1-\beta_v}{1+\beta_v}} \nu' \quad \rightarrow \quad \frac{E}{\nu} = \frac{E'}{\nu'} = h$$

4-vectors

Position vector $$X^{\mu}=(ct,x,y,z)$$

Energy-momentum vector $$P^{\mu}=(\frac{E}{c},p_x,p_y,p_z)$$

1. Derivation of energy momentum equation:

In *a frame where momentum does not vanish*:

$$P^{\mu}P_{\mu}=(\frac{E}{c})^2-p_x^2-p_y^2-p_z^2=(\frac{E}{c})^2-p^2$$

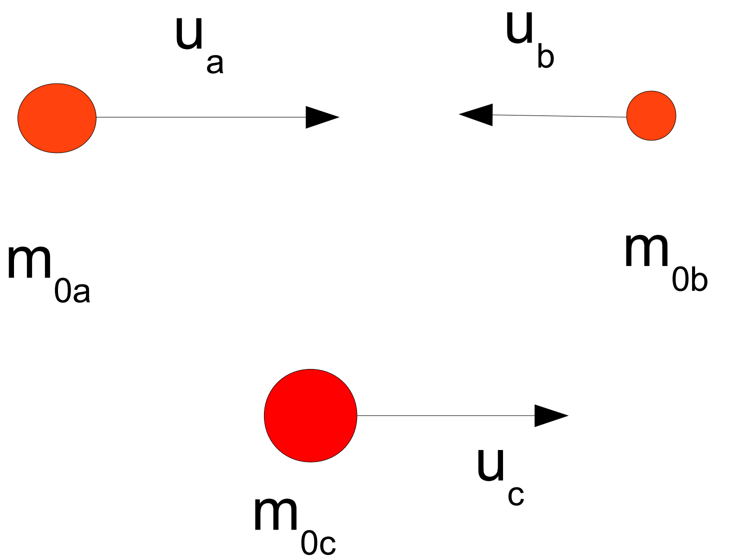
If momentum vanishes: $$P'^{\mu}P'_{\mu}=(\frac{E_0}{c})^2$$

$$P^{\mu}P_{\mu}=P'^{\mu}P'_{\mu} \quad \rightarrow \quad E^2=E_0^2+(pc)^2$$

2. Inelastic collision:

before collision

after collision



$$E_a + E_b = E_c\,, \quad \vec{p}_a + \vec{p}_b = \vec{p}_c \quad \rightarrow$$

$$P_a^\mu + P_b^\mu = P_c^\mu \qquad \cdot\left(P_{a\mu} + P_{b\mu} = P_{c\mu}\right)$$

$$P_a^\mu P_{a\mu} + 2\mathrm{P}_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \qquad (i)$$

rest frames for a, b, c

$$P_a^\mu P_{a\mu} = (m_{0a} c)^2, \quad P_b^\mu P_{b\mu} = (m_{0b} c)^2, \quad P_c^\mu P_{c\mu} = (m_{0c} c)^2$$

laboratory frame

$$P_a^\mu = (\gamma_a m_{0a} c, \gamma_a m_{0a} u_a, 0, 0)$$
$$P_b^\mu = (\gamma_b m_{0b} c, \gamma_b m_{0b} u_b, 0, 0)$$
$$2P_a^\mu P_{b\mu} = 2\gamma_a \gamma_b m_{0a} m_{0b} (c^2 - u_a u_b)$$

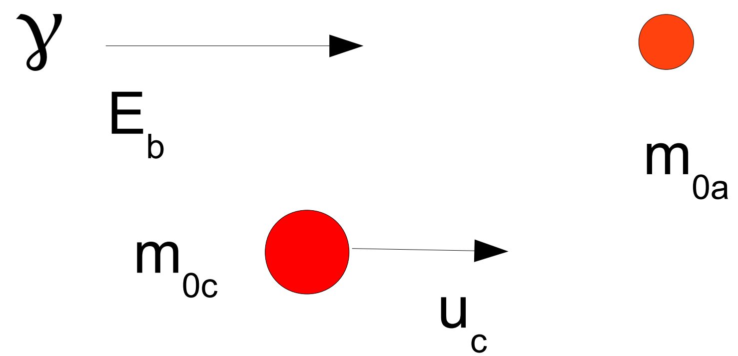
substituted in (i)

$$m_{0c} = \sqrt{m_{0a}^2 + m_{0b}^2 + 2 m_{0a} m_{0b} \gamma_a \gamma_b \left(1 - \frac{u_a u_b}{c^2}\right)} \geq m_{0a} + m_{0b}$$

3. Absorption of a photon by an atom at rest

before absorption

after absorption



$$P_a^{\mu} P_{a\mu} + 2P_a^{\mu} P_{b\mu} + P_b^{\mu} P_{b\mu} = P_c^{\mu} P_{c\mu} \qquad (i)$$

rest frame of a before absorption

$$P_a^{\mu} = (m_{0a} c, 0, 0, 0), \quad P_b^{\mu} = \left(\frac{E_b}{c}, p_{bx}, 0, 0\right)$$

rest frame of c before absorption $\quad P_c^{\mu} = (m_{0c} c, 0, 0, 0)$

scalar product for photons

$$P_b^{\mu} P_{b\mu} = \left(\frac{E}{c}\right)^2 - p_{bx}^2 = \left(\frac{h\nu}{c}\right)^2 - \left(\frac{h\nu}{c}\right)^2 = 0$$

substituted in (i)

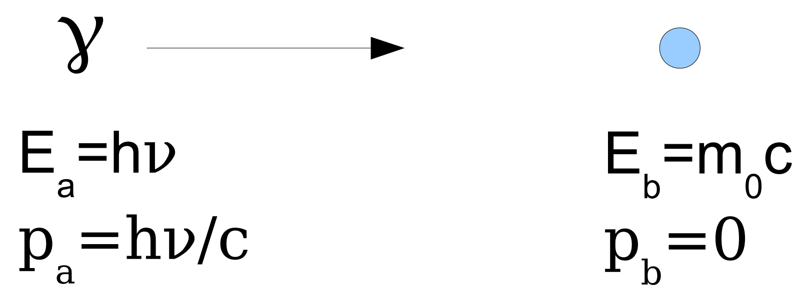
$$(m_{0a} c)^2 + 2 \mathrm{m}_{0a} c \frac{h\nu}{c} + 0 = (m_{0c} c)^2$$

$$m_{oc} = \sqrt{m_{0a}^2 + 2 \mathrm{m}_{0a} \frac{h\nu}{c^2}} = m_{0a} \sqrt{1 + 2 \frac{h\nu}{m_{0a} c^2}}$$

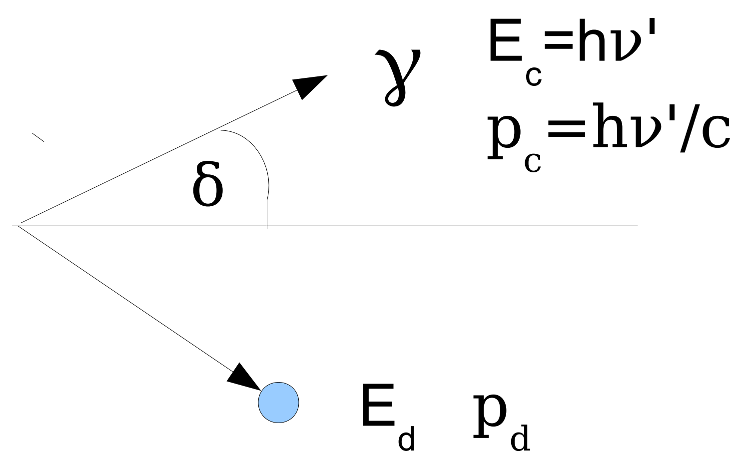
$$\rightarrow \quad m_{oc} c^2 \approx m_{oa} c^2 + h\nu$$

4. Compton effect (photon scattered at electron)

before collision



after collision



scalar product of $P_a^\mu + P_b^\mu = P_c^\mu + P_d^\mu \quad (i)$

scalar product with itself →

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} + 2P_c^\mu P_{d\mu} + P_d^\mu P_{d\mu}$$

$$P_a^\mu P_{a\mu} = P_c^\mu P_{c\mu} = 0$$

$P_b^\mu P_{b\mu}$ in S_b equals $P_d^\mu P_{d\mu}$ in S_d

$$\rightarrow \quad P_a^\mu P_{b\mu} = P_c^\mu P_{d\mu}$$

multiplication of (i) with $P_{c\mu}$

$$P_a^\mu P_{c\mu} + P_b^\mu P_{c\mu} = P_d^\mu P_{c\mu} = P_a^\mu P_{b\mu} \qquad (ii)$$

$$P_a^\mu = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0\right)$$

$$P_b^\mu = (m_0 c, 0, 0, 0)$$

$$P_c^\mu = \left(\frac{h\nu'}{c}, \frac{h\nu'}{c}\cos(\vartheta), \frac{h\nu'}{c}\sin(\vartheta), 0\right)$$

substituted in (ii)

$$\left(\frac{h}{c}\right)^2 \nu\nu'(1-\cos(\vartheta)) + m_0 h\nu' = m_0 h\nu$$

and with $\nu = c/\lambda$

$$\lambda' - \lambda = \frac{h}{m_0 c}(1-\cos(\vartheta)) \qquad \textit{Compton equation}$$

$$\frac{h}{m_0 c} = 2{,}42.10^{-12}\, m \qquad \textit{Compton wavelength}$$

Velocity 4-vector (physically meaningless)

$$U^\mu = \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{dt}\frac{dt}{d\tau} = \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau} =$$

$$= \gamma_u (c, u_x, u_y, u_z)$$

$$U^\mu U_\mu = U'^\mu U'_\mu = c^2$$

Acceleration 4-vector

$$A^\mu = \frac{dU^\mu}{dt}\frac{dt}{d\tau} = \gamma_u \left[\frac{d\gamma_u}{dt}(c, u_x, u_y, u_z) + \gamma_u \frac{d}{dt}(c, u_x, u_y, u_z)\right]$$

$$A^\mu = \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})(c, \vec{u}) + \gamma_u^2 (0, \vec{a}) \qquad U^\mu A_\mu = 0$$

$$A^\mu A_\mu = -\frac{\gamma_u^6}{c^2}(\vec{u}\cdot\vec{a})^2 - \gamma_u^4 a^2 \qquad (i)$$

In instantaneous rest frame S' of a particle

$$u'=0, \quad \gamma_{u'}=1 \quad \rightarrow \quad A'^{\mu} A'_{\mu}=-a^2=-\alpha^2$$

α *is proper acceleration*

Linear acceleration, $\vec{u} \parallel \vec{a}$ and *use of* (i):

$$\alpha^2=\gamma_u^6 \beta_u^2 a^2+\gamma_u^4 a^2=\gamma_u^6 a^2 \quad \rightarrow \quad \alpha=\gamma_u^3 a$$

Radial acceleration, $\vec{u} \perp \vec{a}$ and *use of* (i):

$$\alpha^2=\gamma_u^4 a^2 \quad \rightarrow \quad \alpha=\gamma_u^2 a=\gamma_u^2 \frac{u^2}{r}$$

Frequency-wavenumber 4-vector

Plane wave: $\vec{E} = \vec{E}_0 \sin(\omega t - \vec{k}\cdot\vec{r}), \quad |\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

Phase at a fixed position must be the same for all reference systems.

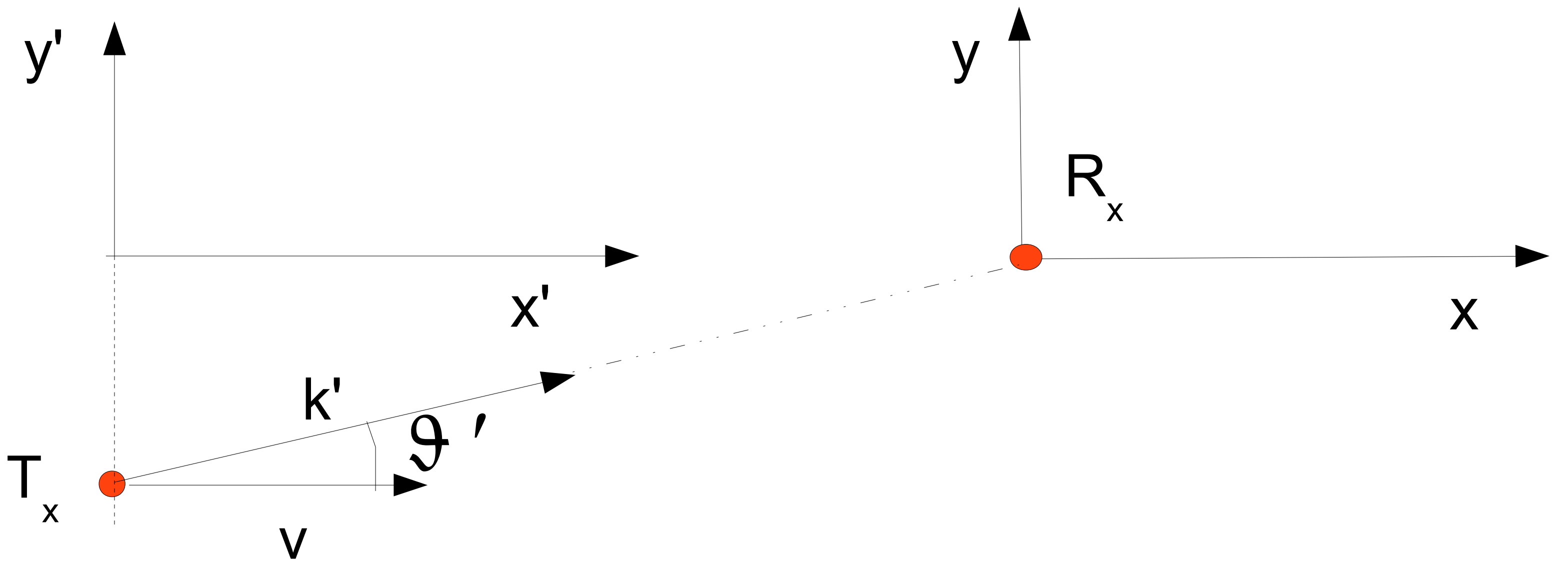
$$\Phi = \omega t - \vec{k}\cdot\vec{r} = \omega t - (k_x x + k_y y + k_z z) =$$
$$= K^\mu X_\mu = K'^\mu X'_\mu = \Phi'$$

where $$K^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$$

Since E=hν=ħω and E=pc for photons, it is p=ħω/c=ħk and

$$P^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) = \hbar K^\mu$$

Doppler effect



$$K'^{\mu} = \left(\frac{\omega'}{c}, k'_x, k'_y, k'_z\right)$$

A L-T of K'^{μ} yields the frequency shift

$$\frac{\omega}{c} = \gamma\left(\frac{\omega'}{c} + \beta k'_x\right) = \gamma\left(1 + \beta\cos(\vartheta')\right)\frac{\omega'}{c}$$

and the aberration

$$k_x = \frac{\omega}{c}\cos(\vartheta) = \gamma(\beta + \cos\vartheta')\frac{\omega'}{c}$$

$$k_y = \frac{\omega}{c}\sin(\vartheta) = \frac{\omega'}{c}\sin\vartheta', \qquad k_z = 0$$

$$\tan(\vartheta) = \frac{k_y}{k_x} = \frac{\sin(\vartheta')}{\gamma(\beta + \cos(\vartheta'))} \qquad (i)$$

with $\sin(\vartheta), \cos(\vartheta), \tan(\frac{\vartheta}{2}) = \sin(\vartheta)/(1 + \cos(\vartheta))$

we transform (i)

$$\tan(\frac{\vartheta}{2}) = \sqrt{\frac{1-\beta}{1+\beta}}\tan(\frac{\vartheta'}{2})$$

Charge-current 4-vector

charge density $\rho = \gamma_u \rho_0$

current density $\vec{j} = \rho \vec{u} = \gamma_u \rho_0 \vec{u}$

$$J^\mu = (\rho c, j_x, j_y, j_z) = \gamma_u \rho_0 (c, u_x, u_y, u_z)$$

$$\left[P^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) = \gamma_u m_0 (c, u_x, u_y, u_z) \right]$$

Power-force 4-vector (Minkowski force)

$$K^{\mu}=\frac{dP^{\mu}}{d\tau}=\frac{dP^{\mu}}{dt}\frac{dt}{d\tau}=\gamma_u\left(\frac{1}{c}\frac{dE}{dt},\frac{dp_x}{dt},\frac{dp_y}{dt},\frac{dp_z}{dt}\right)=$$
$$=\gamma_u\left(\frac{1}{c}\vec{f}\cdot\vec{u},f_x,f_y,f_z\right)$$

Relativistic Newton's 2nd law: $K^{\mu}=m_0 A^{\mu}$

$$K^0=m_0 A^0=m_0\frac{\gamma_u^4}{c}\vec{u}\cdot\vec{a}\quad ?$$

$$m_0\frac{\gamma_u^4}{c}\vec{u}\cdot\vec{a}=m_0\frac{\gamma_u^4}{c}\frac{1}{\gamma_u m_0}\vec{f}\cdot\vec{u}\left(1-\left(\frac{u}{c}\right)^2\right)=\frac{\gamma_u}{c}\vec{f}\cdot\vec{u}$$

Transformation of electromagnetic fields

Force $$\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$$

Power-force 4-vector

$$K^{\mu} = \gamma_u \left(\frac{1}{c}\vec{f}\cdot\vec{u}, f_x, f_y, f_z\right) = \gamma_u q \left(\frac{1}{c}\vec{E}\cdot\vec{u}, \vec{E} + \vec{u} \times \vec{B}\right)$$

$$\begin{bmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \end{bmatrix} = \frac{q}{c} \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \begin{bmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{bmatrix} = \frac{q}{c} \underline{F}\, U^{\mu}$$

With the Lorentz-transformation from S to S'
and the inverse transformation

$$\underline{\underline{L}}=\begin{bmatrix} \gamma_v & -\beta_v\gamma_v & 0 & 0 \\ -\beta_v\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{L}}^{-1}=\begin{bmatrix} \gamma_v & \beta_v\gamma_v & 0 & 0 \\ \beta_v\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$\underline{\underline{L}}\,\underline{\underline{L}}^{-1}=\underline{\underline{L}}^{-1}\underline{\underline{L}}=\underline{1}$$

$$K'^{\mu}=\underline{\underline{L}}K^{\mu}, \qquad K^{\mu}=\underline{\underline{L}}^{-1}K'^{\mu}$$

$$K^{\mu} = \underline{\underline{L}}^{-1} K'^{\mu} = \frac{q}{c} \underline{F} U^{\mu} = \frac{q}{c} \underline{F} \underline{\underline{L}}^{-1} U'^{\mu}$$

$$\rightarrow \quad K'^{\mu} = \frac{q}{c} \underline{\underline{L}} \underline{F} \underline{\underline{L}}^{-1} U'^{\mu} = \frac{q}{c} \underline{F}^{-1} U'^{\mu}$$

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma_v (E_y - v B_z) \qquad B'_y = \gamma_v \left(B_y + \frac{v}{c^2} E_z\right)$$

$$E'_z = \gamma_v (E_z + v B_y) \qquad B'_z = \gamma_v \left(B_z - \frac{v}{c^2} E_y\right)$$

Transformation in case of **v**=(v_x, v_y, v_z)

For **v**=(v, 0, 0) we write the transformation as

$$\vec{E}'_{\parallel}=\vec{E}_{\parallel} \qquad \vec{B}'_{\parallel}=\vec{B}_{\parallel}$$

$$\vec{E}'_{\perp}=\gamma_v(\vec{E}_{\perp}+\vec{v}\times\vec{B}) \qquad \vec{B}'_{\perp}=\gamma_v(\vec{B}_{\perp}-\frac{1}{c^2}\vec{v}\times\vec{E})$$

$\vec{E}_{\parallel}$ and $E_{\perp}$ follow through projection on $\vec{v}$

$$\vec{E}_{\parallel}=\frac{1}{v^2}(\vec{v}\cdot\vec{E})\vec{v}$$

$$\vec{E}_{\perp}=\vec{E}-\vec{E}_{\parallel}=\vec{E}-\frac{1}{v^2}(\vec{v}\cdot\vec{E})\vec{v}$$

for $\vec{B}$ correspondingly.

The general transformation is then

$$\vec{E}\,' = \gamma_v(\vec{E} + \vec{v}\times\vec{B}) - (\gamma_v - 1)\frac{(\vec{v}\cdot\vec{E})\,\vec{v}}{v^2}$$

$$\vec{B}\,' = \gamma_v(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}) - (\gamma_v - 1)\frac{(\vec{v}\cdot\vec{B})\,\vec{v}}{v^2}$$

Uniformly moving charge

Point charge at rest in origin of S'

$$\vec{E}'=\frac{1}{4\pi\epsilon_0}\frac{q}{(x'^2+y'^2+z'^2)3/2}(x',y',z'),\quad \vec{B}'=0$$

As a function of time the field at a point P=(0,a,0) in S is

$$\vec{E}'_P(t')=\frac{1}{4\pi\epsilon_0}\frac{q}{(v^2t'^2+a^2)3/2}(-vt',a,o)$$

Transformation of t'

$$t'=\gamma\left(t-\frac{v}{c^2}x\right)=\gamma t$$

Transformation of fields

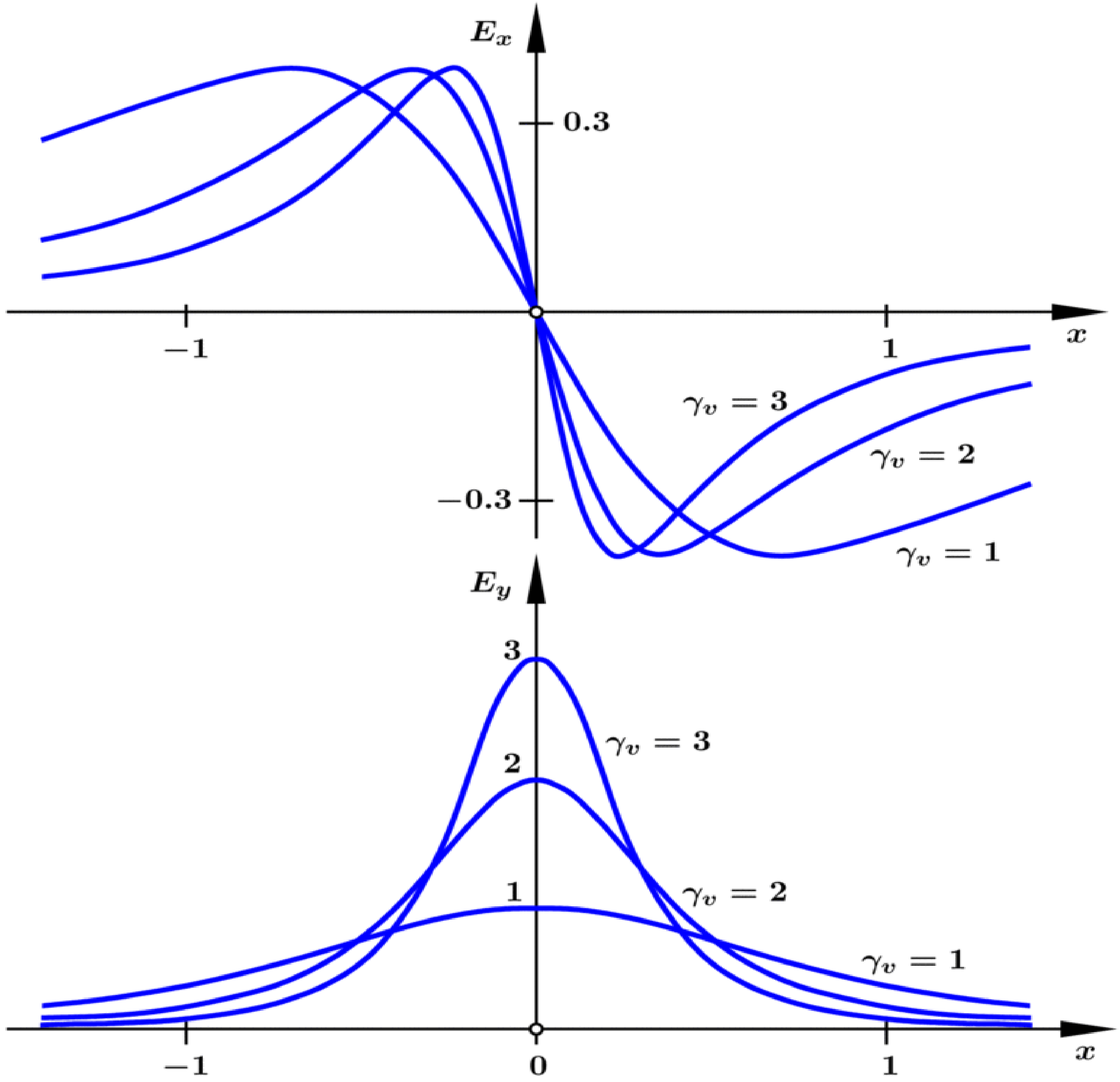
$$E_{Px} = E_{Px}', \qquad B_{Px} = 0$$

$$E_{Py} = \gamma E_{Py}', \qquad B_{Py} = -\gamma \frac{v}{c^2} E_{Pz}'$$

$$E_{Pz} = \gamma E_{Pz}', \qquad B_{Pz} = +\gamma \frac{v}{c^2} E_{Py}'$$

$$\vec{E}_P(t) = \frac{1}{4\pi\epsilon_0} \frac{q}{(\gamma^2 v^2 t^2 + a^2)^{3/2}} (-\gamma v t, \gamma a, 0)$$

$$\vec{B}_P(t) = \frac{1}{4\pi\epsilon_0} \frac{q}{(\gamma^2 v^2 t^2 + a^2)^{3/2}} (0, 0, \gamma \frac{v}{c^2} a)$$


E_x
0.3
−0.3
−1
1
x
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
E_y
3
2
1
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
−1
0
1
x

Literature:

R. P. Feynman, R. B. Leighton, M. Sands: Lectures on physics. Vol. I. Addison & Wesley, 1963

A. P. French: Special relativity. W. W. Norton & Company, 1966

J. Freund: Special relativity for beginners. World Scientific, 2008