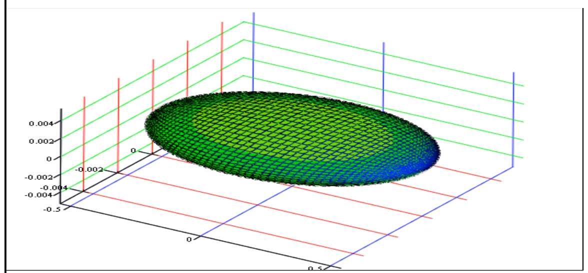
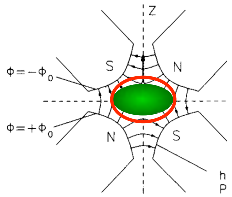
Introduction to Transverse Beam Optics

Bernhard Holzer,
CERN-LHC

II.) The Ideal World:

Particle Trajectories & Beams





Bunch in a storage ring

Reminder of Part I

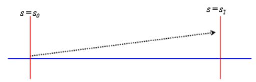
Equation of Motion:

$$x'' + K\, x = 0 \qquad K = 1/\rho^2 - k \quad \text{... hor. plane:}$$

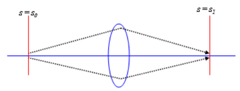
$$K = k \qquad \text{... vert. Plane:}$$

Solution of Trajectory Equations

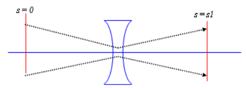
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



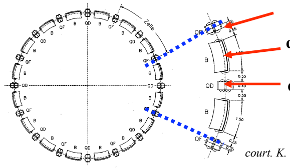
$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l \\ \sqrt{|K|}\sinh(\sqrt{|K|}l & \cosh(\sqrt{|K|}l \end{pmatrix}$$

Transformation through a system of lattice elements

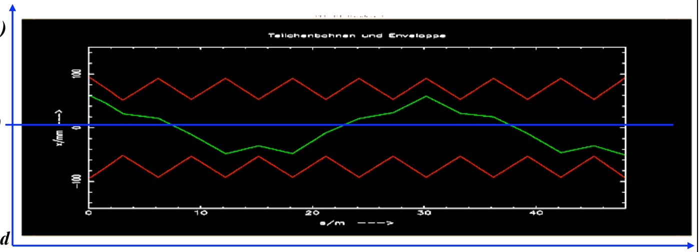
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*}.....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



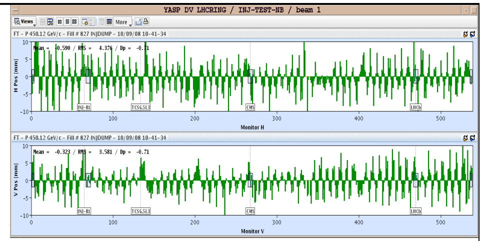
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:

$x \approx mm, x' \leq mrad$

5.) Orbit & Tune:

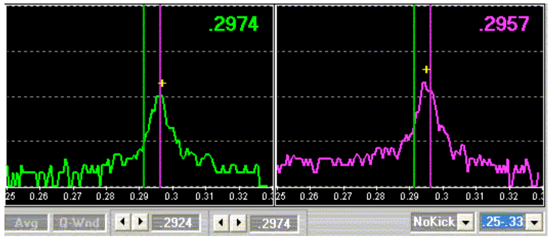


Tune: number of oscillations per turn

64.31
59.32

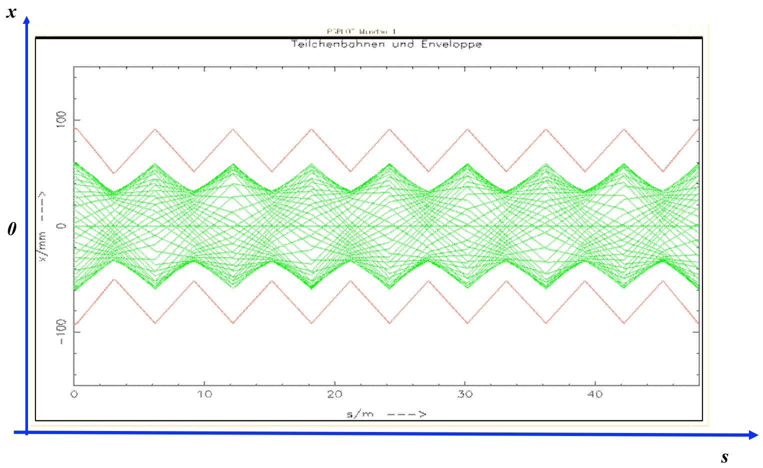
Relevant for beam stability:
non integer part

LHC revolution frequency: 11.3 kHz $0.31 * 11.3 = 3.5 kHz$



***Question:* what will happen, if the particle performs a second turn ?**

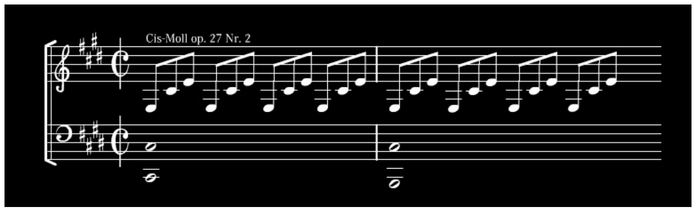
... or a third one or ... 10^{10} turns



19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II, 1801)



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force ≠ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

General solution of Hill´s equation:

(i) $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance" of the oscillation between point „0" and „s" in the lattice.
For one complete revolution: number of oscillations per turn „Tune"

$$Q_y = \frac{1}{2\pi} \cdot \oint \frac{ds}{\beta(s)}$$

8.) Beam Emittance and Phase Space Ellipse:

general solution of Hill equation

(1) $x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$

(2) $x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \left\{ \alpha(s) * \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

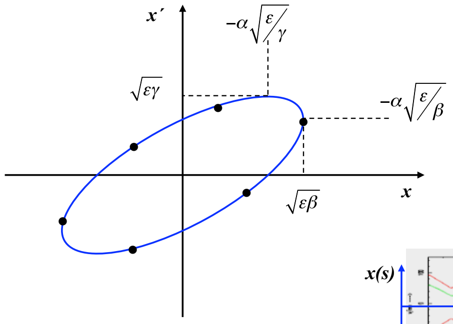
***** *ε is a constant of the motion … it is independent of „s“***

***** *parametric representation of an ellipse in the x x‘ space***

***** *shape and orientation of ellipse are given by α, β, γ***

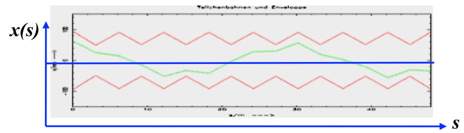
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

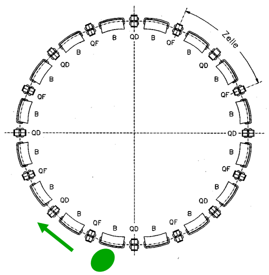


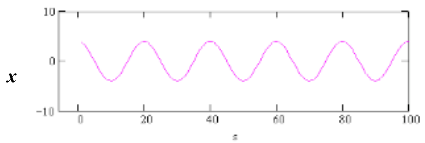
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

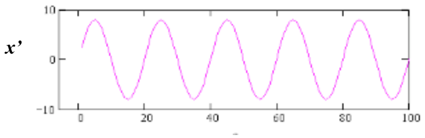
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

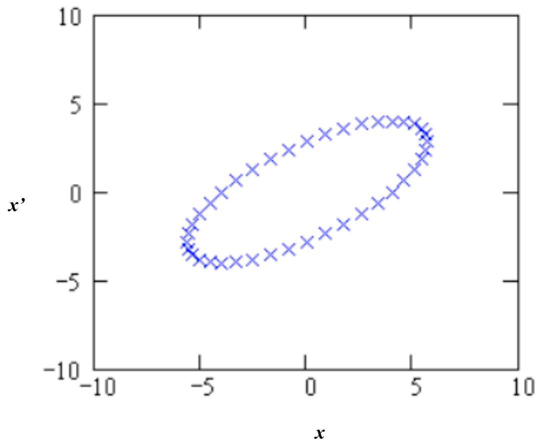






… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



Emittance & The Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ ⟶ ***x´ at that position …?***

… put $\hat{x}(s)$ ***into*** $\varepsilon = \gamma(s)*x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$ ***and solve for x´***

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$$

- ***A high β-function means a large beam size and a small beam divergence.*** ***!***
  … et vice versa !!!

- ***In the middle of a quadrupole β is maximum,***
  α = zero } $x' = 0$

  … and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

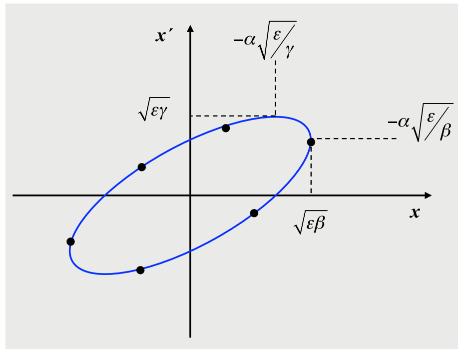
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha\cdot xx' + \beta\cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha\cdot x \pm\sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

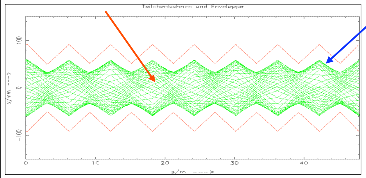
$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

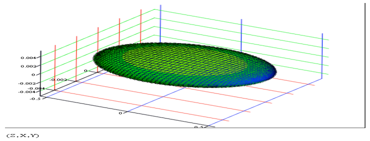
Emittance of the Particle Ensemble:

$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$ $\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$



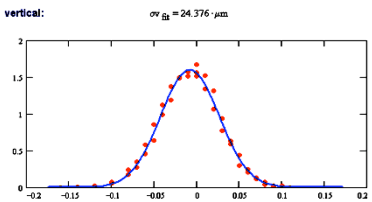
single particle trajectories, $N \approx 10^{11}$ per bunch

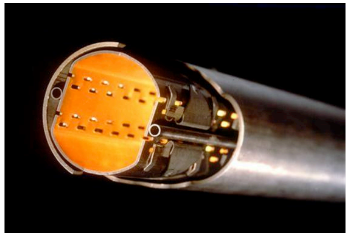
particle bunch



Gauß Particle Distribution: $\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles

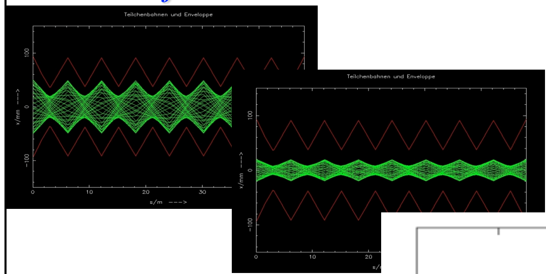




LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10} m * 180 m} = 0.3\, mm$

aperture requirements: $r_0 \geq 10 * \sigma$

Emittance of the Particle Ensemble:



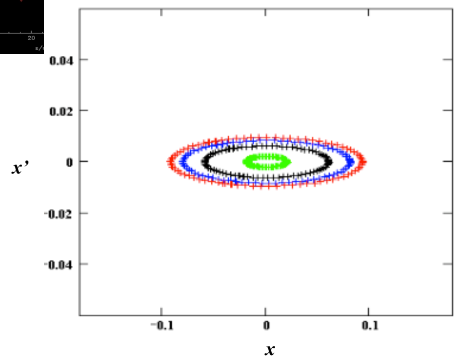
Example: LHC

beam parameters in the arc

$\beta(x) \approx 180\, m$

$\varepsilon \approx 5*10^{-10}\, rad \cdot m \qquad (\leftrightarrow 1\sigma)$

$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\, mm$



9.) Transfer Matrix M

... yes we had the topic already

general solution of Hill's equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\cos\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad ,$$

$$\sin\phi = -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0\sin\psi_s\right\}x_0 + \left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

** we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.*

** and nothing but the α β γ at these positions.*

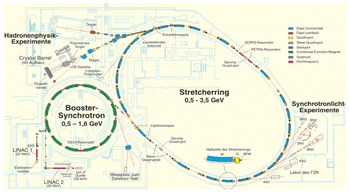
** … !*

** Äquivalenz der Matrizen*

10.) Periodic Lattices

transfer matrix for particle trajectories as a function of the lattice parameters

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



Delta Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete turn …"

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

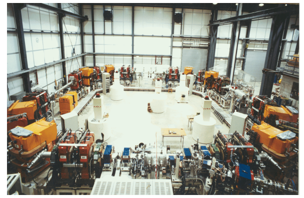
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

***Tune:* Phase advance per turn in units of 2π**

$$Q = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)}$$

Stability Criterion:



***Question:* what will happen, if we do not make too many mistakes and your particle performs one complete turn ?**

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

Matrix for N turns:

$$\boldsymbol{M}^N = \left(\boldsymbol{I}\cos\psi + \boldsymbol{J}\sin\psi\right)^N = \boldsymbol{I}\cos N\psi + \boldsymbol{J}\sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| < 1 \quad \leftrightarrow \quad |Trace(M)| < 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = (\boldsymbol{I} * \cos\psi_1 + \boldsymbol{J} * \sin\psi_1) * (\boldsymbol{I} * \cos\psi_2 + \boldsymbol{J} * \sin\psi_2)$$

$$= \boldsymbol{I}^2 * \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ} * \cos\psi_1 \sin\psi_2 + \boldsymbol{JI} * \sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\boldsymbol{I} * \boldsymbol{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{J} * \boldsymbol{I} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{I} * \boldsymbol{J} = \boldsymbol{J} * \boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I} * \cos(\psi_1 + \psi_2) + \boldsymbol{J} * \sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I} * \cos(2\psi) + \boldsymbol{J} * \sin(2\psi)$$

11.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

where …

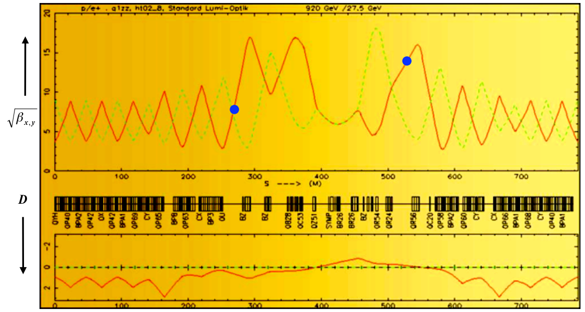
$$M = M_{QF} \cdot M_{QD} \cdot M_B \cdot M_{Drift} \cdot M_{QF} \cdot \ldots$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const (Liouville):

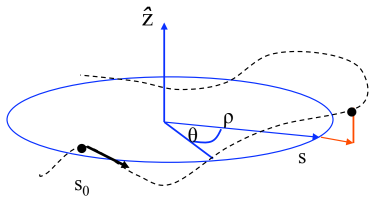
$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$



Beta function in a storage ring

express x_0, x'_0 as a function of x, x'.



$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

... remember W = CS′-SC′ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x'_0 &= -C'x + Cx' \end{aligned}$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x′and compare the coefficients to get ....

$$\beta(s) = C^2 \beta_0 - 2SC\alpha_0 + S^2 \gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2 \beta_0 - 2S'C'\alpha_0 + S'^2 \gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

Résumé:

equation of motion:	$x''(s) + K(s)\,x(s) = 0 \quad , \qquad K = 1/\rho^2 - k$
general solution of Hill´s equation:	$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$
phase advance & tune:	$\psi_{12}(s) = \int_{s1}^{s2} \frac{1}{\beta(s)} ds \quad , \qquad Q(s) = \frac{1}{2\pi} \oint \frac{1}{\beta(s)} ds$
emittance:	$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$
transfer matrix from s_1 → s_2:	$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{21} + \alpha_1 \sin\psi_{21}) & \sqrt{\beta_2 \beta_1}\sin\psi_{21} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{21} - (1 + \alpha_1\alpha_2)\sin\psi_{21}}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{21} - \alpha_2 \sin\psi_{21}) \end{pmatrix}$
matrix for 1 turn:	$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$
stability criterion:	$\lvert Trace(M) \rvert < 2$