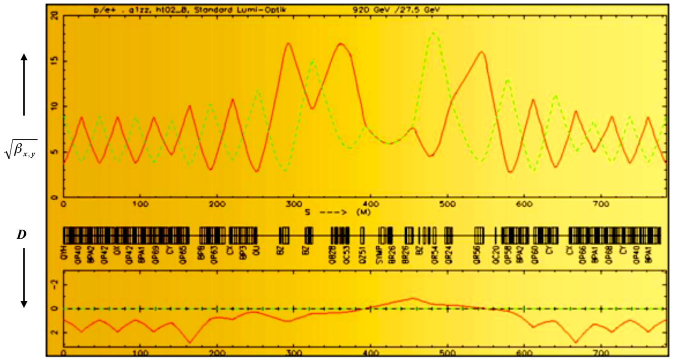
III.) The „not so ideal“ World

Lattice Design in Particle Accelerators

Bernhard Holzer, CERN-LHC



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

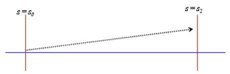
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\; \dots \textit{ hor. plane:}$$

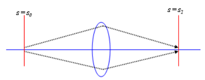
$$K = k \qquad \dots \textit{ vert. Plane:}$$

Solution of Trajectory Equations

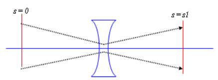
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l) & \cos(\sqrt{|K|}\,l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}\,l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}\,l) & \cosh(\sqrt{|K|}\,l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \;\dots$$

Recapitulation: ...and for the complete particle ensemble the betas and epsilons !!!

general solution of Hill equation

$$(1)\quad x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

$$(2)\quad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \{\alpha(s) * \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\}$$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

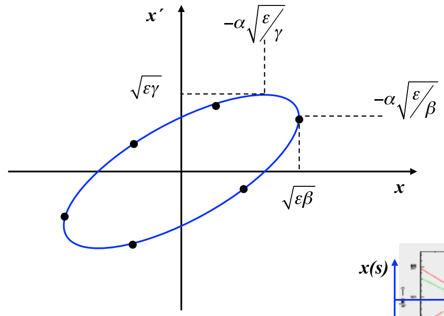
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

***** ***ε is a constant of the motion … it is independent of „s“***
***** ***parametric representation of an ellipse in the x x‘ space***
***** ***shape and orientation of ellipse are given by α, β, γ***

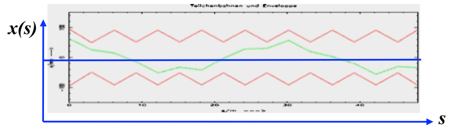
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



Liouville: ***in reasonable storage rings area in phase space is constant.***

A = π*ε=const



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

Transfer Matrix M

Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

using matrix notation in magnet parameters:

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots \qquad M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

using matrix notation in Twiss form:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}(\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$

Transformation of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

III. Lattice Design in Particle Accelerators

„… how to build a storage ring"

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

12.) Geometry of the ring:

centrifugal force = Lorentz force

Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p/\rho$$

$$\rightarrow B * \rho = p/e$$

p = momentum of the particle,
ρ = curvature radius

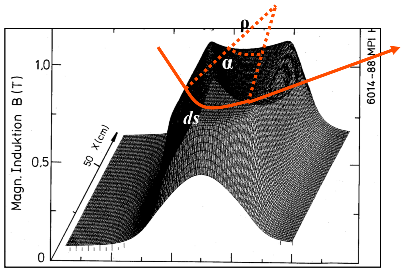
Bρ= beam rigidity

13.) Lattice Design:

$$B\,\rho = p\,/\,q$$

Circular Orbit: **dipole magnets to define the geometry**

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



field map of a storage ring dipole magnet

The angle run out in one revolution must be 2π, so

… for a full circle $$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi\,\frac{p}{q}$$ *… defines the integrated dipole field around the machine.*

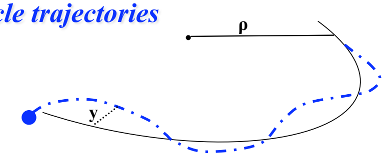
Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\ p\,/\,e$$

$$B \approx \frac{2\pi\ 7000\ 10^9\,eV}{1232\ 15\,m\ 3\ 10^8\,\frac{m}{s}\ e} = 8.3\,Tesla$$

14.) Focusing forces … single particle trajectories



$$x'' + K * x = 0$$

$K = -k + 1/\rho^2$ **hor. plane**

$K = k$ **vert. plane**

dipole magnet $\frac{1}{\rho} = \frac{B}{p/q}$

quadrupole magnet $k = \frac{g}{p/q}$

Example: LHC Ring:
Bending radius: ρ = 2.8 km
Quadrupol Gradient: g = 220 T/m

k = $9.4 * 10^{-3}$ /m²
1/ρ² = $1.3 * 10^{-7}$ /m²

For estimates in large accelerators the weak focusing term 1/ρ² can in general be neglected

Solution for a focusing magnet

$$x(s) = x_0 * \cos(\sqrt{K} * s) + \frac{x_0'}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$x'(s) = -x_0 \sqrt{K} * \sin(\sqrt{K} * s) + x_0' * \cos(\sqrt{K} * s)$$

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

Question: „ What does that mean ???? "

Most simple example: drift space

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \begin{array}{l} x(l) = x_0 + l * x_0' \\ x'(l) = x_0' \end{array}$$

transformation of twiss parameters:

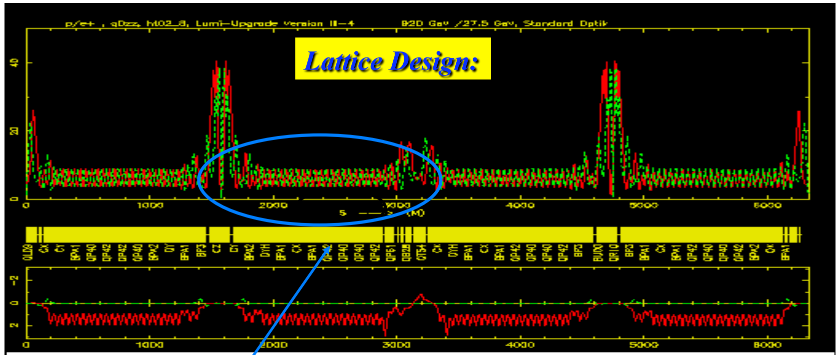
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.



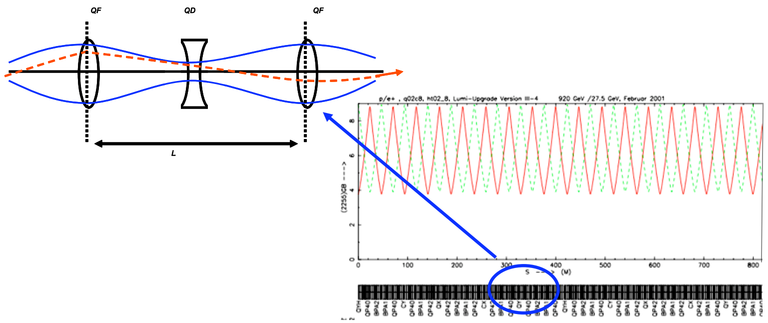
Arc: ***regular (periodic) magnet structure:***

bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: ***drift spaces for injection, dispersion suppressors,***
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

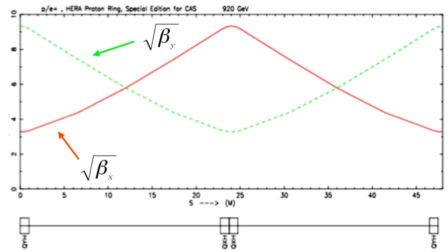
3.) The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$0.125 * 2\pi = 45^0$

Can we understand, what the optics code is doing?

matrices $M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l_q) & \cos(\sqrt{|K|}l_q) \end{pmatrix}$ $M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$

strength and length of the FoDo elements

K = +/- 0.54102 m^-2
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

1.) is the motion stable?

$trace(M_{FoDo}) = 1.415 \rightarrow$ < 2

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}Trace(M) = 0.707$$

$$\psi = arc\cos\left(\frac{1}{2}Trace(M)\right) = 45°$$

3.) hor β-function

$$\beta = \frac{M_{1,2}}{\sin\psi} = 11.611\ m$$

4.) hor α-function ♪

$$\alpha = \frac{M_{1,1} - \cos\psi}{\sin\psi} = 0$$

Can we do it a little bit easier ?
We can: … the „thin lens approximation“

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the focal length *f* is much larger than the length of the quadrupole magnet,

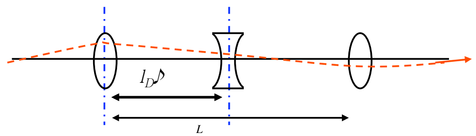
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be aproximated using ♪

$$kl_q = const, \; l_q \to 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

FoDo in thin lens approximation



$$l_D = L/2, \quad \tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$\boldsymbol{M}_{half\ cell} = \boldsymbol{M}_{qdh} * \boldsymbol{M}_{ld} * \boldsymbol{M}_{qf\ h}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f → -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell:

$$M = \begin{pmatrix} 1+\frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1-\frac{l_D}{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1-\frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1+\frac{l_D}{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi = \frac{1}{2}\,\boldsymbol{trace}(\boldsymbol{M}) = \frac{1}{2}\left(2-\frac{4\boldsymbol{l}_D^2}{\tilde{\boldsymbol{f}}^2}\right) = 1-\frac{2\boldsymbol{l}_D^2}{\tilde{\boldsymbol{f}}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos\psi_{cell} = 1 - 2\sin^2(\frac{\psi_{cell}}{2}) = 1 - \frac{2l_D^2}{\tilde{f}^2}$$

$$\sin\frac{\psi_{cell}}{2} = \frac{l_D}{\tilde{f}} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin\frac{\psi_{cell}}{2} = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$

$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$

$$\sin\frac{\psi_{cell}}{2} \approx \frac{L_{cell}}{4f} = 0.405$$

$\rightarrow\ \psi_{cell} \approx 47.8°$

$\rightarrow\ \beta \approx 11.4\ m$

Remember:
Exact calculation yields:

$\rightarrow\ \psi_{cell} \approx 45°$

$\rightarrow\ \beta \approx 11.6\ m$

III.) The „ not so ideal world "

Acceleration and Momentum Spread

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $x''(s) - k(s)\,x(s) = 0$

general solution of Hills equation: $x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\varphi)$

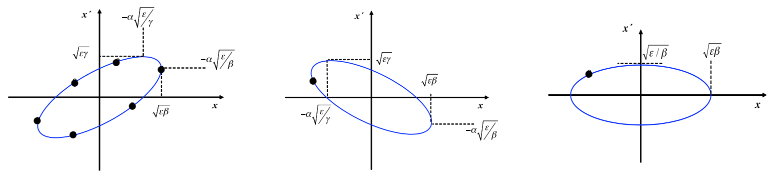
beam size: $\sigma = \sqrt{\varepsilon\beta} \approx \text{"mm"}$

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

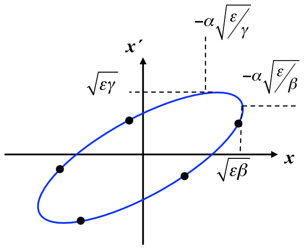
$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

* ε *is a constant of the motion … it is independent of „s"*
* *parametric representation of an ellipse in the x x' space*
* *shape and orientation of ellipse are given by α, β, γ*



16.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq const$!

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = $\gamma m v = m c \gamma \beta_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} \qquad \text{where } \beta_x = v_x / c$$

$$\int p\,dq = mc\int \gamma\beta_x\,dx$$

$$\int p\,dq = mc\gamma\beta\underbrace{\int x'dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

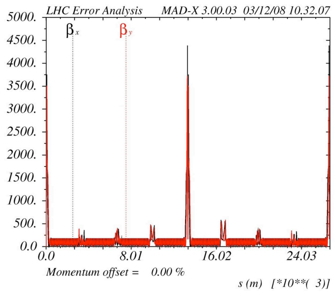
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

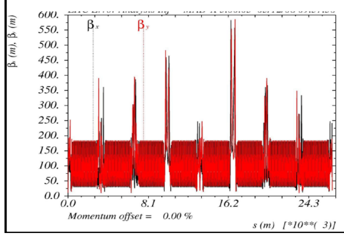
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

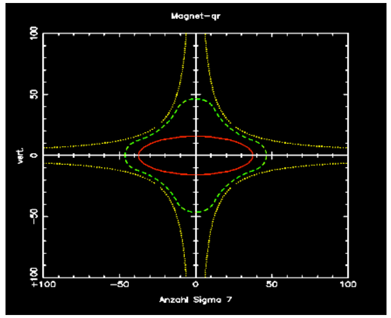


LHC injection optics at 450 GeV

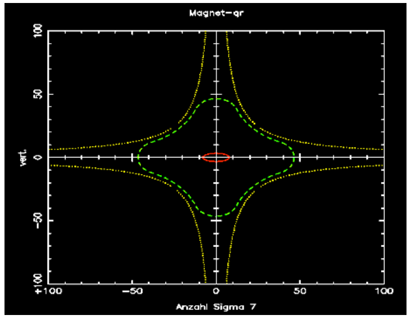
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε (40GeV) $= 1.2 * 10^{-7}$
ε (920GeV) $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

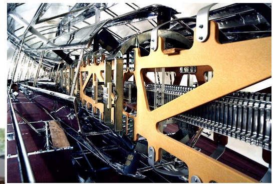
The „ not so ideal world “

17.) The „ Δp / p ≠ 0" Problem

ideal accelerator: all particles will see the same accelerating voltage.
→ $\Delta p / p = 0$

„nearly ideal“ accelerator: Cockroft Walton or van de Graaf

$\Delta p / p \approx 10^{-5}$



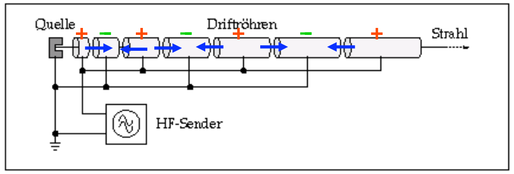
Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

1928, Wideroe

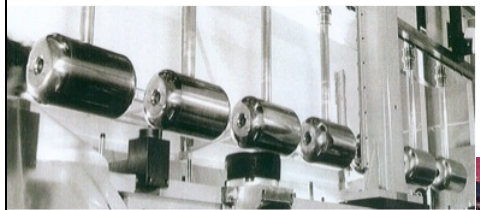
schematic Layout:



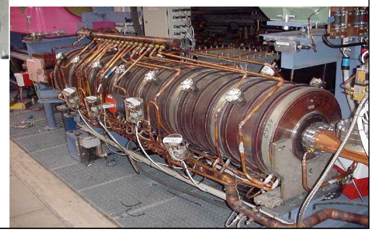
Energy Gain per „Gap“:

$$W = q\, U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac



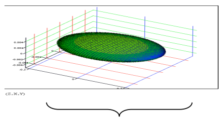
500 MHz cavities in an electron storage ring



* RF Acceleration: multiple application of the same acceleration voltage; brillant idea to gain higher energies

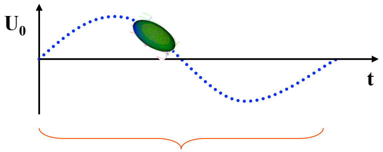
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Example: HERA RF:

Bunch length of Electrons ≈ 1cm



$$\left.\begin{array}{l}\nu = 500\,MHz \\ c = \lambda\,\nu\end{array}\right\}\ \lambda = 60\ cm$$

$\lambda = 60\ cm$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

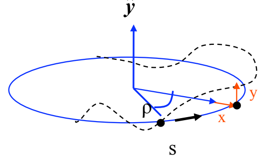
typical momentum spread of an electron bunch: $\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$

18.) Dispersion: trajectories for $\Delta p / p \neq 0$

***Question:* do you remember last session, page 12 ? … sure you do**

***Force* acting on the particle**

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$



remember:** $x \approx mm$, $\rho \approx m$ … → **develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \to s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\,B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \longrightarrow x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

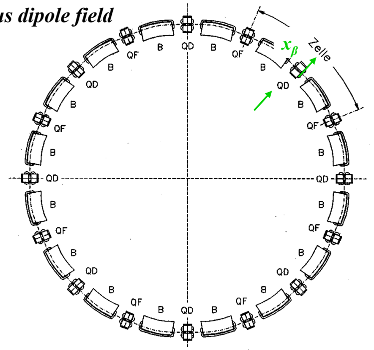
$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

* *is that special orbit, an ideal particle would have for Δp/p = 1*
* *the orbit of any particle is the sum of the well known x_β and the dispersion*
* *as D(s) is just another orbit it will be subject to the focusing properties of the lattice*

Dispersion

Example: homogeneous dipole field



ıit for Δp/p > 0

$: D(s) \cdot \frac{\Delta p}{p}$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Resume´:

beam emittance	$\varepsilon \propto \frac{1}{\beta\gamma}$
beta function in a drift	$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$
… and for α = 0	$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$
particle trajectory for Δp/p ≠ 0 inhomogenious equation	$x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0} \frac{1}{\rho}$
… and its solution	$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$