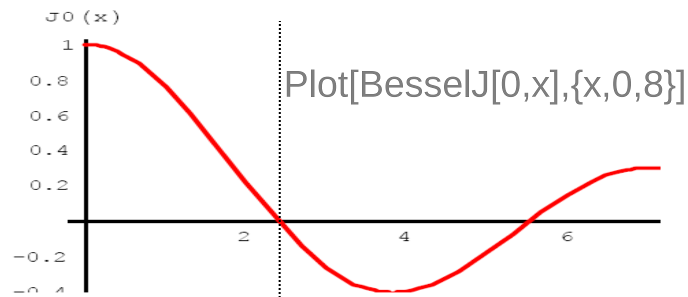
Pill box resonator: Field distribution 1/2

$$\Rightarrow E = \left(1 - \frac{1}{2^2}\left(\frac{\omega r}{c}\right)^2 + \frac{1}{2^2 \cdot 4^2}\left(\frac{\omega r}{c}\right)^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(\frac{\omega r}{c}\right)^6\right) \cdot E_0 e^{i\omega t} =$$

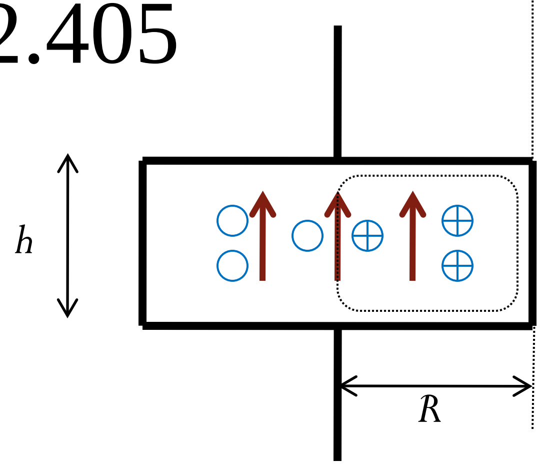
$$= \left(1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega r}{2c}\right)^6 + \right) \cdot E_0 e^{i\omega t} =$$

$$= J_0\left(\frac{\omega r}{c}\right) \cdot E_0 e^{i\omega t}$$



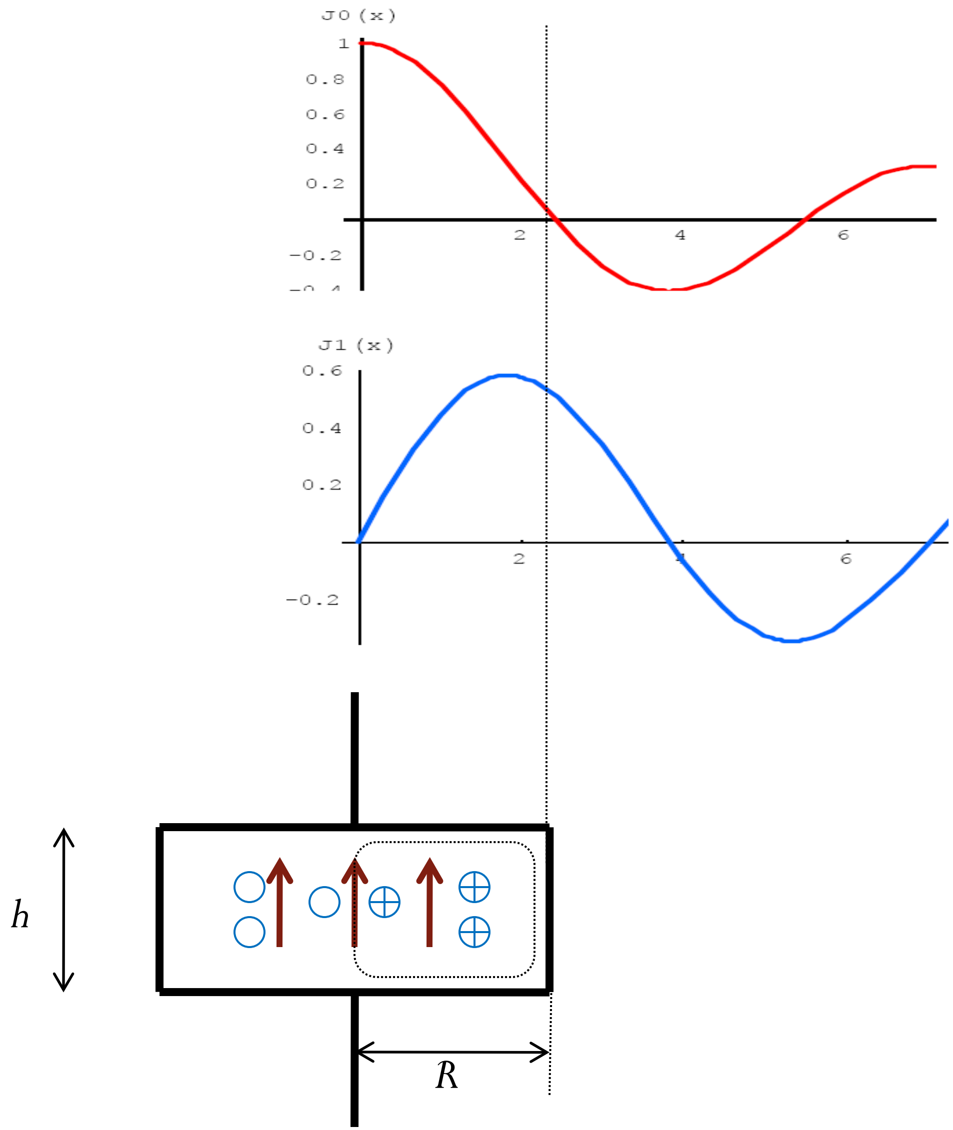
Resonance-condition

$$\frac{\omega_0 \cdot R}{c} = 2.405$$



TM$_{010}$ mode

Pill box resonator: Field distribution 2/2



$$E = J_0\left(\frac{2.405\cdot r}{R}\right)\cdot E_0 e^{i\omega_0 t}$$

$$c^2 \oint_{\Gamma} \vec{B}\cdot d\vec{s} = \frac{\partial}{\partial t} \int_{\text{inside }\Gamma} \vec{E}\cdot\vec{n}\,da$$

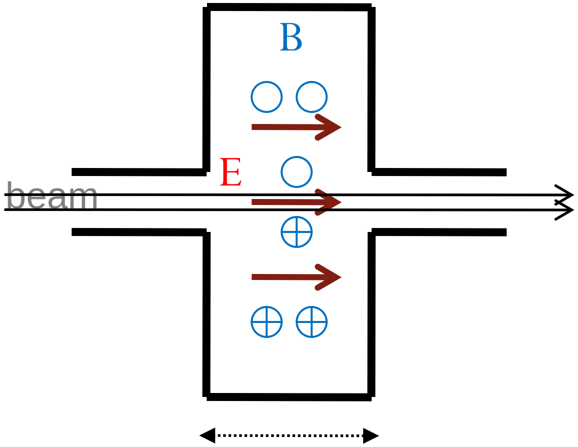
$$c^2 B_\varphi(r)\cdot 2\pi r = i\omega \int_0^{2\pi} d\varphi \int_{\text{inside }\Gamma} E(r)\cdot r\,dr$$

$$\Rightarrow B_\varphi(r) = \frac{i}{c} J_1\left(\frac{2.405\cdot r}{R}\right)\cdot E_0 e^{i\omega_0 t}$$

Pill box resonator as accelerating cavity

How to accelerate a particle beam with a pillbox resonator?

The unavoidable beam tube opening is considered to be small compared to *l*



$$h = \beta \cdot \frac{\lambda}{2}$$

$$E(r,t) = J_0\left(\frac{2.405 \cdot r}{R}\right) \cdot E_0 \sin(\omega_0 \cdot t)$$

$$t(z) = z/(\beta \cdot c)$$

$$W\big|_{r=0} = e\int_0^h E(r=0, t(z))dz = eE_0 \cdot \int_0^h \sin(\omega_0 z/(\beta c))dz =$$

$$= -eE_0 \cdot \frac{\beta c}{\omega_0} \cdot \cos(\omega_0 z/(\beta c))\Big|_{z=0}^{z=h} = eE_0 \cdot \frac{\beta c}{\omega_0} \cdot (1 - \cos(\omega_0 h/(\beta c))) =$$

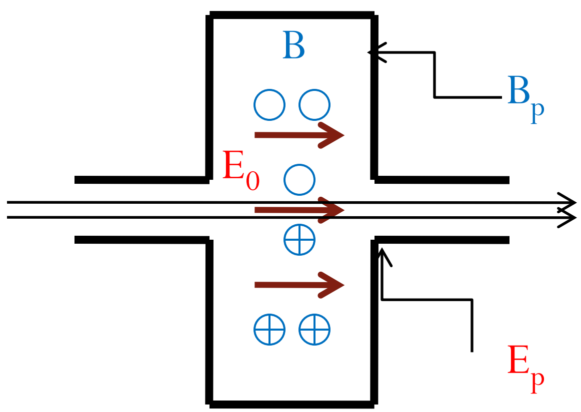
$$= eE_0 \cdot h \cdot \frac{\sin^2(\omega_0 h/(2\beta c))}{\frac{\omega_0 h}{2\beta c}}$$

$$\frac{\omega_0 h}{2\beta c} \xrightarrow{h=\frac{\beta\lambda}{2}} \frac{\omega_0 \beta\lambda}{2 \cdot 2 \cdot \beta c} = \frac{2\pi f \lambda}{2 \cdot 2 \cdot f\lambda} = \frac{\pi}{2}$$

$$V_a = \frac{W}{e} = E_a \cdot h = E_0 \cdot h \cdot \frac{\sin^2(\omega_0 h/(2\beta c))}{\frac{\omega_0 h}{2\beta c}} \xrightarrow{h=\frac{\beta\lambda}{2}} \frac{2E_0 h}{\pi} \Rightarrow E_a = \frac{2E_0}{\pi}$$

Pill box resonator: Cavity characteristics 1/6

The **peak surface electric and magnetic fields** constitute the ultimate limit for the accelerating gradient => minimize the ratio E_p/E_a and B_p/E_a.



Remember:

$$E = J_0\left(\frac{2.405\cdot r}{R}\right)\cdot E_0 e^{i\omega_0 t}$$

$$B_\varphi(r) = \frac{i}{c} J_1\left(\frac{2.405\cdot r}{R}\right)\cdot E_0 e^{i\omega_0 t}$$

$$E_a = \frac{2E_0}{\pi}$$

$$\frac{B_p}{E_a} = \underset{r\in[0,R]}{Max}\frac{\frac{1}{c}J_1\left(\frac{2.405\cdot r}{R}\right)\cdot E_0}{\frac{2E_0}{\pi}} = \frac{0.582\cdot\pi}{2\cdot c} \approx 3.07\left[\frac{\mathrm{mT}}{\mathrm{MV/m}}\right]$$

$$\frac{E_p}{E_a} = \underset{r\in[0,R]}{Max}\frac{J_0\left(\frac{2.405\cdot r}{R}\right)\cdot E_0}{\frac{2E_0}{\pi}} = \frac{\pi}{2} \approx 1.57$$

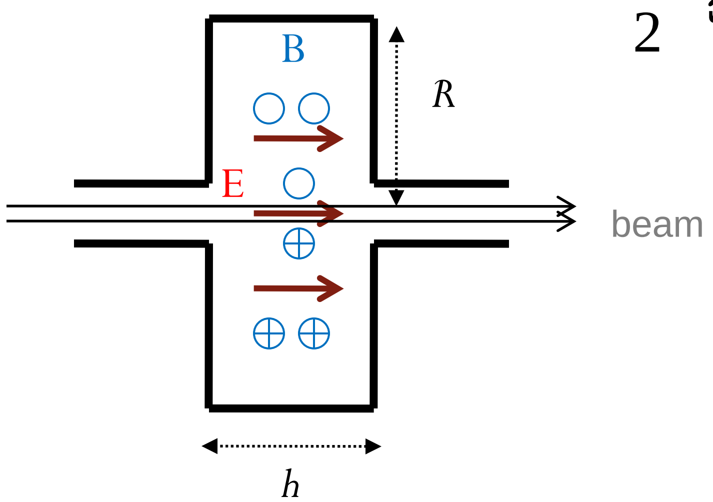
 Cavity characteristics

Stored energy U

$$U = \frac{\varepsilon_0}{2}\int_0^{2\pi} d\varphi \int_0^h dz \int_0^R r dr \cdot |E(r)|^2 =$$

$$= \frac{\varepsilon_0}{2}\int_0^{2\pi} d\varphi \int_0^h dz \int_0^R r dr \left| J_0\left(\frac{2.405 \cdot r}{R}\right)\right|^2 \cdot |E_0|^2 =$$

$$= \frac{\varepsilon_0}{2} \cdot \underbrace{\pi R^2 h}_{V} \cdot \left| \underbrace{J_1(2.405)}_{0.519} \right|^2 |E_0|^2$$



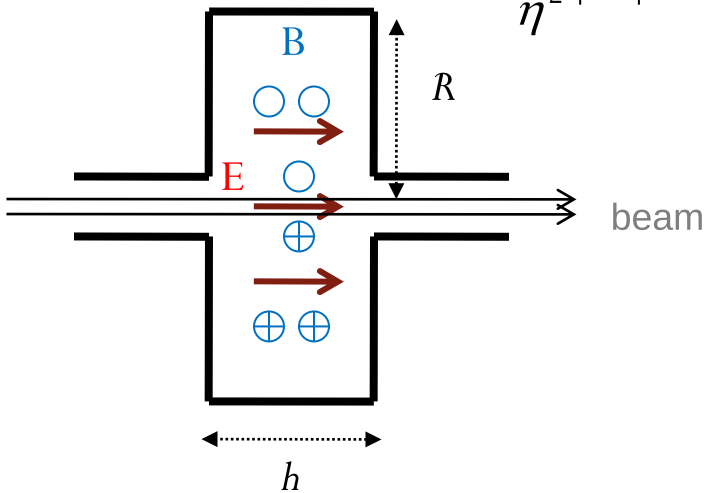
Cavity characteristics 3/6

Power loss

$$P = \frac{R_s}{2\mu_0^2}\int_0^{2\pi} d\varphi \cdot \left(\int_0^h dz \left|B_\varphi(R)\right|^2 + 2\cdot\int_0^R r dr \left|B_\varphi(r)\right|^2 \right) =$$

$$= \frac{R_s}{2\mu_0^2}\frac{1}{c^2}\left|E_0\right|^2 \int_0^{2\pi} d\varphi \cdot \left(R\int_0^h dz \left|J_1(2.405)\right|^2 + 2\cdot \underbrace{\int_0^R r dr \left|J_1\left(\frac{2.405\cdot r}{R}\right)\right|^2}_{\frac{1}{2}R^2\left|J_1(2.405)\right|^2} \right) =$$

$$= R_s \frac{1}{\eta^2}\left|E_0\right|^2 \cdot \pi R\cdot(h+R)\cdot \left|\underbrace{J_1(2.405)}_{0.519}\right|^2 ; \qquad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$$

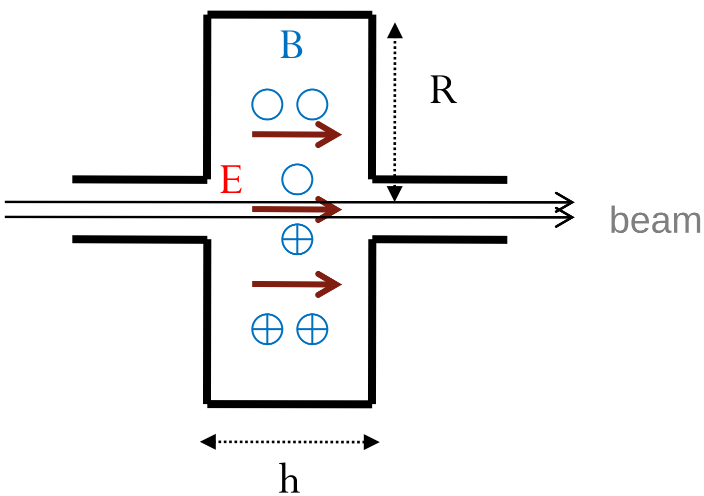


Pill box resonator: Cavity characteristics 4/6

The **Q-factor** measures the dissipation of the stored energy to the cavity wall consequent to the unavoidable surface currents associated with that stored energy.

$$Q=\frac{\text{Stored energy}\,U}{\text{Energy lost during 1 RF period}}=\frac{\text{Stored energy}\,U}{\Delta \mathrm{U}/(2\pi)}=$$

$$=2\pi\cdot\frac{\text{Stored energy}\,U}{\text{Dissipated power}\,P\cdot\underbrace{T}_{f^{-1}}}=\underbrace{2\pi f}_{\omega}\cdot\frac{\text{Stored energy}\,U}{P}=\omega\cdot\frac{U}{P}$$



$$Q=\omega\cdot\frac{\varepsilon_0\cdot\eta^2}{2R_s\cdot\left(\frac{1}{R}+\frac{1}{h}\right)}=\qquad \eta=\sqrt{\frac{\mu_0}{\varepsilon_0}}=377\,\Omega$$

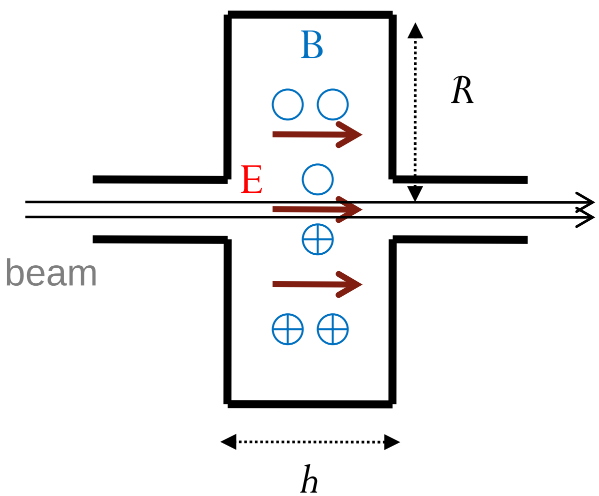
$$=\frac{1}{R_s}\cdot\eta\cdot\underbrace{\frac{2.405}{2\left(1+\frac{R}{h}\right)}}_{=G\text{ (Geometry-factor)}}=\frac{G}{R_s}\xrightarrow{h=\frac{\lambda}{2}}\frac{257\,\Omega}{R_s}\qquad \frac{\omega_0\cdot R}{c}=2.405$$

$$G\approx 257\ \Omega$$

$$\frac{R}{h}\xrightarrow{h=\frac{\lambda}{2}}\frac{2R}{\lambda}=\frac{2R}{\lambda}=\frac{2\cdot 2.405c}{\lambda\omega_0}=\frac{2\cdot 2.405\cdot\lambda f}{\lambda\cdot 2\pi f}=\frac{2.405}{\pi}$$

Cavity characteristics 5/6

The **shunt impedance R_{shunt}** measures the acceleration action of the beam of charged particles in terms of the unavoidable dissipation of energy in the cavity wall.

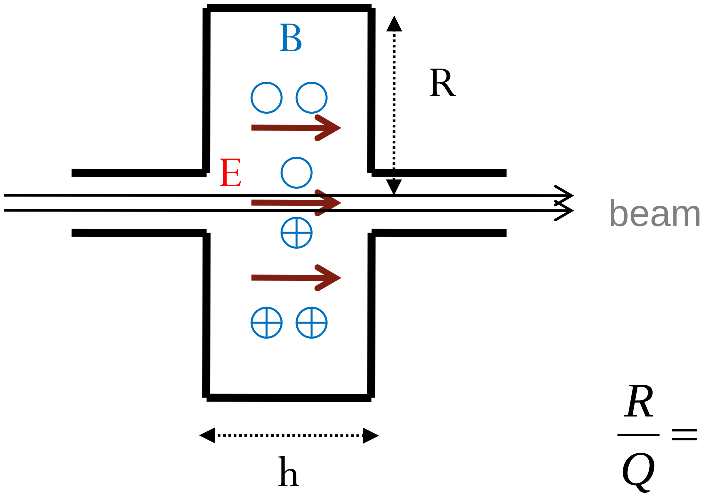


$$R_{shunt} = \frac{V_a^2}{2\cdot P} = \frac{\left(E_0\cdot h\cdot\frac{\sin^2(\omega_0 h/(2\beta c))}{\frac{\omega_0 h}{2\beta c}}\right)^2}{2\cdot R_s\frac{1}{\eta^2}|E_0|^2\cdot\pi R\cdot(h+R)\cdot\left|\underbrace{J_1(2.405)}_{0.519}\right|^2} \xrightarrow{h=\frac{\beta\lambda}{2}}$$

$$\xrightarrow{h=\frac{\beta\lambda}{2}} \frac{2\eta^2}{R_s\cdot\pi^2\cdot 2.405\cdot\left(1+\frac{2.405}{\pi}\right)\cdot\left|\underbrace{J_1(2.405)}_{0.519}\right|^2} = \frac{25182\,\Omega}{R_s[\Omega]}$$

Pill box resonator: Cavity characteristics 6/6

The quantity ***R/Q*** measures the interaction of the cavity with the beam.



$$\frac{R}{Q}=\underbrace{\frac{V_a^2}{2\cdot P}}_{R}\cdot\underbrace{\frac{P}{\omega\cdot U}}_{1/Q}=\frac{V_a^2}{2\cdot\omega\cdot U}\xrightarrow{h=\frac{\beta\lambda}{2}}\frac{\dfrac{25182\,\Omega}{R_s[\Omega]}}{\dfrac{257\,\Omega}{R_s}}=98\,\Omega$$

Cavity characteristics - Summary

Symbol	Name	Definition	Pillbox cavity [0.35 GHz, 4.2 K, Nb]	1-cell Accelerating cavity [0.35 GHz, 4.2 K, Nb]*)
E_p/E_a	Peak normalized surface electric field	n/a	1.6	2
B_p/E_a [mT/(MV/m)]	Peak normalized surface magnetic field	n/a	3.1	4.1
R_s [nΩ]	Surface resistance	E_x/H_y	40	40
h [m]	Cavity length	$h=\lambda/2$	0.43	0.43
E_a [MV/m]	Accelerating gradient	(1/e) ·Energy gain/length	10	10
V [MV]	Accelerating voltage	$V=E_a \cdot h$	4.3	4.3
G [Ω]	Geometry factor	$G=R_s \cdot Q$	257	295
Q [10^9]	Quality factor	$Q=\omega U/P$	6.4	7.4
R/Q [Ω]	(R/Q) factor	$(R/Q)=V^2/(2\omega U)$	98	57
R [MΩ]	Shunt impedance	$R=V^2/(2P)$	$0.63 \cdot 10^6$	$0.42 \cdot 10^6$
U [J]	Stored energy	$U=V^2/[2\omega(R/Q)]$	43	74
P [W]	Dissipated power	$P=\omega U/Q$	15	22
h/R	Ratio cavity length to radius	n/a	1.3	0.5

*) mainly based on O. Brunner et al., Assessment of the basic parameters of the CERN Superconducting Proton Linac, PHYS. REV. ST - AB 12, 070402 (2009), Table VII