Introduction to Transverse Beam Optics

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IV.) Errors in Field and Gradient

The „ überhaupt nicht ideal world "

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

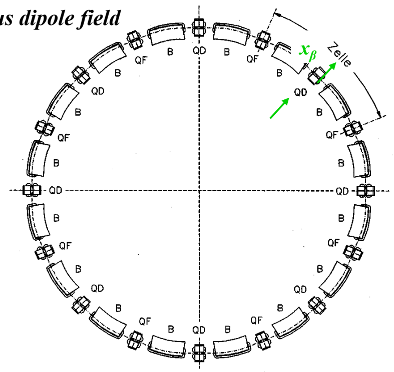
* is that special orbit, an ideal particle would have for Δp/p = 1

* the orbit of any particle is the sum of the well known x_β and the dispersion

* as D(s) is just another orbit it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field



...it for $\Delta p/p > 0$

$$\ldots D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

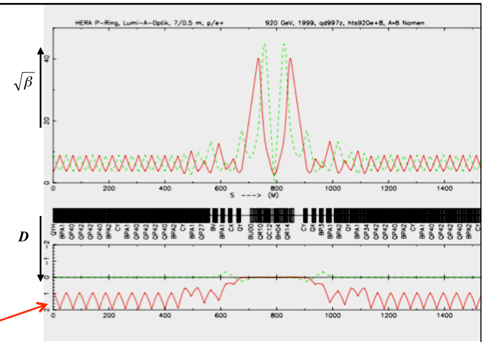
$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\ mm$$

$$D(s) \approx 1 \ldots 2\ m$$

$$\Delta p/p \approx 1\cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D′: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

Example: ***Drift***

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}}_{=0}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: ***Dispersion in a Sector Dipole Magnet***

Remember: ***Matrix of a magnetic element***

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

in general: $K = k - \frac{1}{\rho^2}$

… but in a dipole, as $k = 0$ ***…***

$$M_{foc} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

calculate the „D" elements for the marix a Sector Dipole Magnet

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D(s) = (\rho\sin\frac{l}{\rho}) * \frac{1}{\rho} * (\rho\sin\frac{l}{\rho}) - \cos\frac{l}{\rho} * \frac{1}{\rho} * \rho\cdot(-\cos\frac{l}{\rho} + 1) * \rho$$

$$D(s) = \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho} * (\cos\frac{l}{\rho} - 1)$$

$\boldsymbol{D(s) = \rho\cdot(1-\cos\frac{l}{\rho})}$, $\boldsymbol{D'(s) = \sin\frac{l}{\rho}}$ ***Dispersion elements in a sector dipole magnet***

$$M_{dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & D \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s2} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & \rho*(1-\cos\frac{l}{\rho}) \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & \sin\frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s1}$$

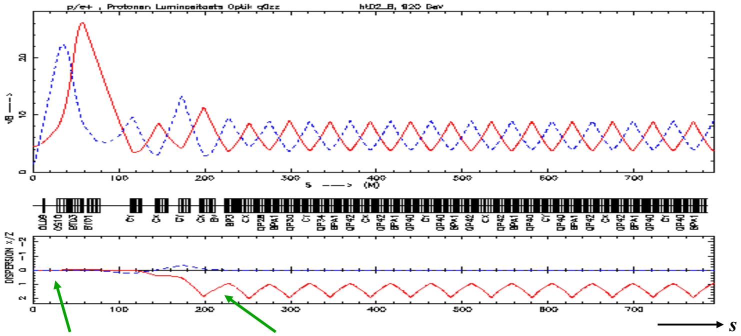
Nota bene: ***even an ideal particle with $x = x' = 0$ will start to oscillate if it passes a dipole magnet and has a momentum error $\Delta p/p$***
A dispersion trajectory will obey the same focusing forces (i.e. will be transferred by the same matrices) as a normal betatron oscillation

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) * \frac{\Delta p}{p}$$

* D(s) is created by the dipole magnets

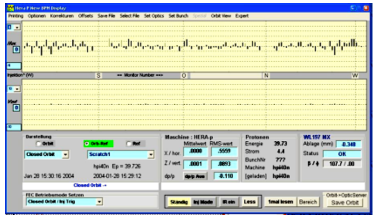
… and afterwards focused by the quadrupole fields



Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

Dispersion is visible

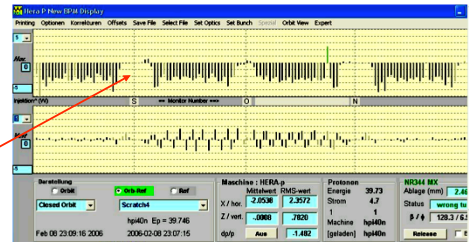


HERA Standard Orbit

dedicated energy change of the stored beam
→ closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$

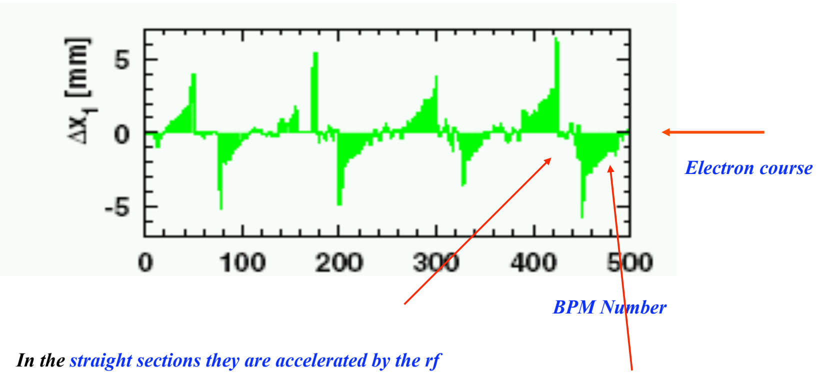
HERA Dispersion Orbit



Attention: at the Interaction Points we require D=D′= 0

Periodic Dispersion:

„Sawtooth Effect" at LEP (CERN)



In the straight sections they are accelerated by the rf cavities so much that they „overshoot" and reach nearly the outer side of the vacuum chamber.

In the arc the electron beam loses so much energy in each octant that the particle are running more and more on a dispersion trajectory.

19.) Momentum Compaction Factor:

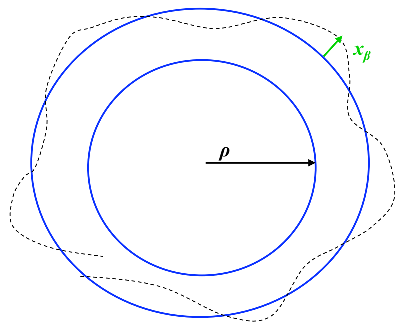
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p}$$



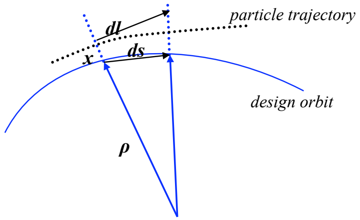
But it does much more:

it changes the length of the off - energy - orbit !!

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow\ dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p_0}$$

$$\delta l = l_{\Delta E} - l_0 = \frac{\Delta p}{p_0}\oint\left(\frac{D(s)}{\rho(s)}\right) ds$$

** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.*

Definition:

$$\frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

$$\rightarrow\ \alpha_p = \frac{1}{L_0}\oint\left(\frac{D(s)}{\rho(s)}\right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = const$$

$$\int_{dipoles} D(s)\, ds = \Sigma\,(l_{dipoles}) * \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L_0} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L_0} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L_0}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

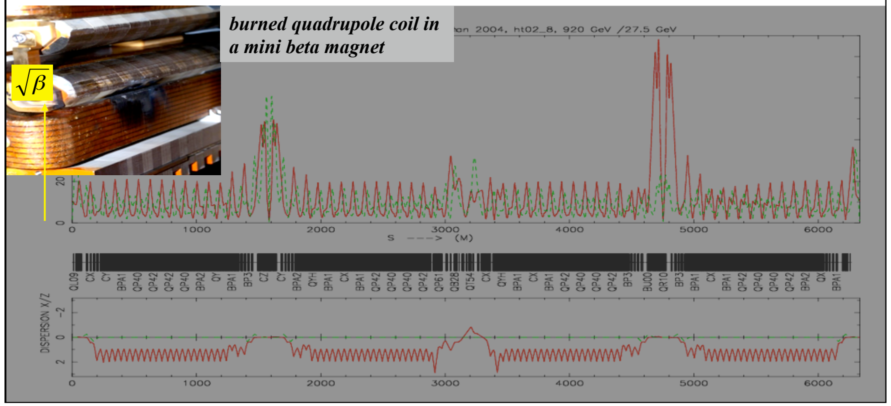
Assume: $v \approx c$

$$\rightarrow\ \frac{\delta T}{T_0} = \frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Introduction to Transverse Beam Optics

IV.) Errors in Field and Gradient



burned quadrupole coil in a mini beta magnet

20.) Errors in Field and Gradient

Reminder: Linear Beam Dynamics

The derivation of the equation of motion is based on the presumption that

… in our accerlator there are only linear magnetic fields ….

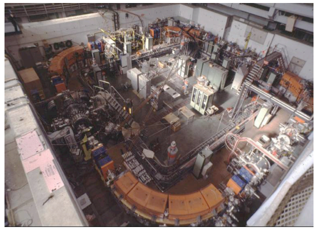
$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k * x + \frac{1}{2!} m x^2 + \frac{1}{3!} n x^3 + \dots$$

dipole ***quadrupole***

Multipole expansion of magnetic field:

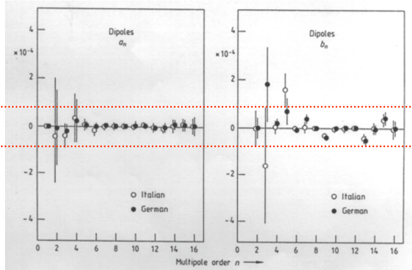
$$B_\theta(r,\theta) = B_{main} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(b_n \cos(n\theta) + a_n \sin(n\theta)\right)$$

$$B_r(r,\theta) = B_{main} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(-a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$



magnet structure of LEAR (CERN)

Example: ***HERA multipole coefficients of sc. dipole magnets***



$$B_\theta(r,\theta) = B_{main} * \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (b_n \cos n\theta + a_n \sin n\theta)$$

$$b_n\,, a_n \approx 1 \ldots 2*10^{-4}$$

systematic — *uncertainty* — *random*

```
col := 0.0000 ;
col := 0.0000 ;
col := 0.8900 ;
col := 0.6400 ;
an in collision
b6M_MQXCD_col := 0.0000 ; b6U_MQXCD_col := 1.7700 ; b6R_M
b7M_MQXCD_col := 0.0000 ; b7U_MQXCD_col := 0.2100 ; b7R_M
b8M_MQXCD_col := 0.0000 ; b8U_MQXCD_col := 0.1600 ; b8R_M
b9M_MQXCD_col := 0.0000 ; b9U_MQXCD_col := 0.0800 ; b9R_M
b10M_MQXCD_col := 0.0000 ; b10U_MQXCD_col := 0.2000 ; b10R_
b11M_MQXCD
b12M_MQXCD
b13M_MQXCD_col := 0.0000 ; b13U_MQXCD_col := 0.0200 ; b13R_
b14M_MQXCD_col := 0.0000 ; b14U_MQXCD_col := 0.0400 ; b14R_
b15M_MQXCD_col := 0.0000 ; b15U_MQXCD_col := 0.0000 ; b15R_

a1M_MQXCD_col := 0.0000 ; a1U_MQXCD_col := 0.0000 ; a1R_MQXCD_col := 0.0000 ;
a2M_MQXCD_col := 0.0000 ; a2U_MQXCD_col := 0.0000 ; a2R_MQXCD_col := 0.0000 ;
a3M_MQXCD_col := 0.0000 ; a3U_MQXCD_col := 0.8900 ; a3R_MQXCD_col := 0.8900 ;
a4M_MQXCD_col := 0.0000 ; a4U_MQXCD_col := 0.6400 ; a4R_MQXCD_col := 0.6400 ;
a5M_MQXCD_col := 0.0000 ; a5U_MQXCD_col := 0.4600 ; a5R_MQXCD_col := 0.4600 ;
QXCD_col := 0.3300 ;
QXCD_col := 0.2100 ;
a8M_MQXCD_col := 0.0000 ; a8U_MQXCD_col := 0.1600 ; a8R_MQXCD_col := 0.1600 ;
a9M_MQXCD_col := 0.0000 ; a9U_MQXCD_col := 0.0800 ; a9R_MQXCD_col := 0.0800 ;
a10M_MQXCD_col := 0.0000 ; a10U_MQXCD_col := 0.1400 ; a10R_MQXCD_col := 0.0600 ;
a11M_MQXCD_col := 0.0000 ; a11U_MQXCD_col := 0.0300 ; a11R_MQXCD_col := 0.0300 ;
a12M_MQXCD_col := 0.0000 ; a12U_MQXCD_col := 0.0200 ; a12R_MQXCD_col := 0.0200 ;
a13M_MQXCD_col := 0.0000 ; a13U_MQXCD_col := 0.0100 ; a13R_MQXCD_col := 0.0100 ;
a14M_MQXCD_col := 0.0000 ; a14U_MQXCD_col := 0.0300 ; a14R_MQXCD_col := 0.0100 ;
a15M_MQXCD_col := 0.0000 ; a15U_MQXCD_col := 0.0000 ; a15R_MQXCD_col := 0.0000 ;
```

Example: ***LHC multipole coefficients of sc. triplet quadrupoles***

general rule: ***multipole errors should be in the range of „some 10^{-4}"***

Sources of field errors

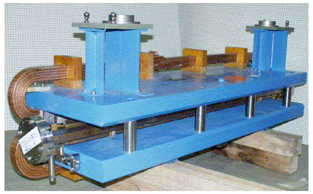
i.) power supply errors:

dipole error: ***remember from lecture N° 1:***

$$B = \frac{\mu_0\, n\, I}{h}$$

Error in dipole strength: the gap

Yoke production: laminations, made by stamping out of steel sheet.
variations of gap „h" by wear out of die or use of multiple dies



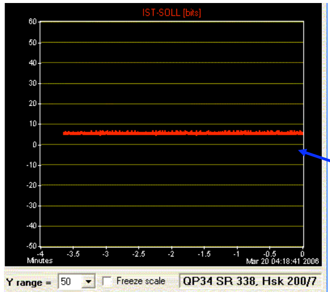
Tolerance:

$$\left.\begin{array}{l} h = 5\,cm \\ \Delta h = 25\,\mu m \end{array}\right\} \quad \frac{\Delta B}{B} = \left|\frac{\Delta h}{h}\right| = \frac{25\mu m}{5cm} = 5*10^{-4}$$

Sources of field errors

power supply stability:

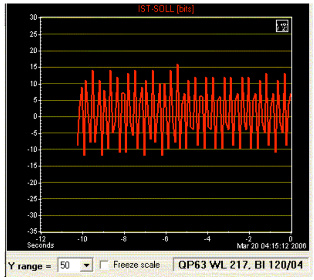
16 bit digital electronic for current control and stabilisation



survey of power supply electronics: bit stability

$$2^{16} = 65536 \qquad \frac{1bit}{2^{16}} \leftrightarrow 1.5*10^{-5}$$

require $\frac{\Delta I}{I} \leq 5*10^{-5}$



$$\Delta I \approx \pm 12 \quad bit$$

$$\frac{\Delta I}{I} \approx 1 \ldots 2*10^{-4}$$

21) Dipole Magnet errors: closed orbit distortion

The sum of all dipole magnets in a ring defines a curve that we call closed orbit.
perfect situation ↔ design orbit

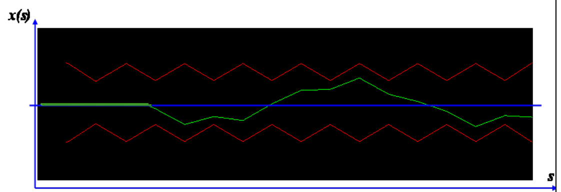
normalised effect on the beam: $\int \frac{Bdl}{B\rho} = \frac{L_0}{\rho} = \alpha = 2\pi$

effect of single dipole magnet error: $\int \frac{(Bds)}{B\rho} = \int \frac{1}{\rho} ds$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere … but – if small enough – still will lead to a closed orbit !!

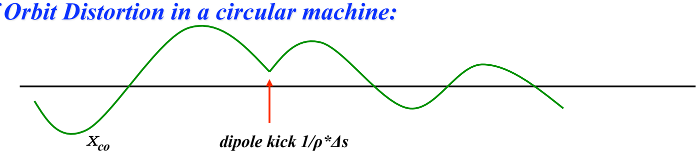
Assume one single diple error in a linac,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{lattice} * \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_{s0}$$



Overall amplitude of a single particle trajectory: $x = x_{co}(s) + x_\beta(s) + x_D(s)$

Calculation of Orbit Distortion in a circular machine:



periodicity condition still has to be fulfilled: ***we still get (!) a closed orbit***

in any case: ***distorted orbit will be a betatron oscillation.***

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$ ***a = orbit amplitude, φ = initial phase***

put starting conditions: $s = 0\ ,\quad \psi(s) = 0$

boundary condition (1): $x_d(s+L) = x_d(s)$ ***periodic closed orbit***

boundary condition (2): $x_d'(s+L) + \frac{\Delta s}{\rho} = x_d'(s)$ ***at the place of the distortion, s = 0, ψ = 0***

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$

Nota bene: * ***orbit distortion is visible at any position „s" in the ring,***
… even if the dipole error is located at one single point „s1".

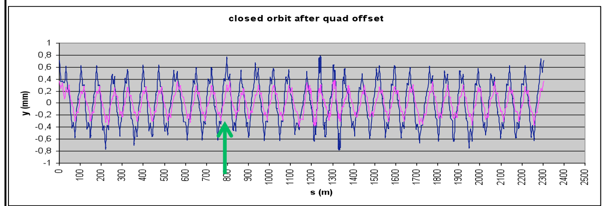
* ***the β function describes the sensitivity of the beam to external fields***

* ***the β function acts as amplification factor for the orbit amplitude at the given observation point***

* ***in any case we (clearly …) will obtain a cosine-like orbit travelling around the ring … but being closed !!! after one turn.***

* ***there is a resonance denominator***

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$



PETRA III Light Source:

closed orbit error after offset of 0.3mm in 2 quadrupole magnets

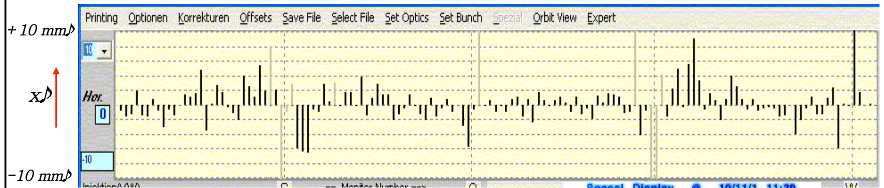
Example: „bad orbit“, i.e. closed orbit that contains large oscillation amplitudes

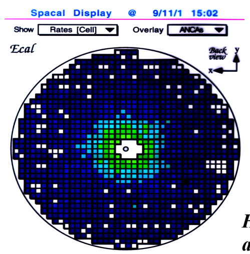
→ eats up available magnet aperture

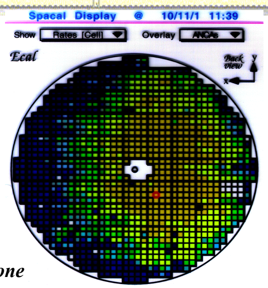
$$x(s) = x_\beta(s) + x_D(s) + x_{co}(s)$$

→ particle trajectories pass nonlinear field regions

→ detector components suffer from beam halo particles & light

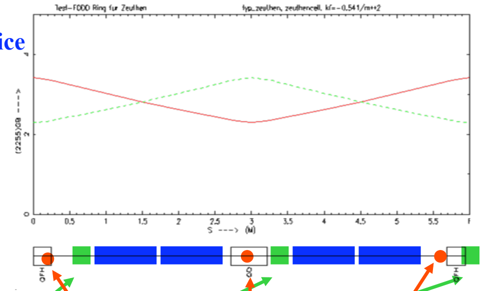






H1 drift chamber during a good run … or a „bad“ one

Orbit distortions in a periodic lattice



field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin \pi Q} \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\phi(\tilde{s}) - \phi(s)| - \pi Q) d\tilde{s}$$

*** the orbit amplitude will be large if the β function at the location of the kick is large**
$\beta(\tilde{s})$ indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice

*** the orbit amplitude will be large at places where in the lattice β(s) is large → here beam position monitors should be installed**

22.) Finally: .... Resonances

closed orbit distortion:

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$

remember from lecture 1: *μ = phase advance per revolution in general measured and expressed in units of 2π … and called „Tune“ Q*

$$Q = \frac{\mu}{2\pi}$$

… and it depends on the focusing strength of the lattice cells.

Tune: number of oscillations per turn

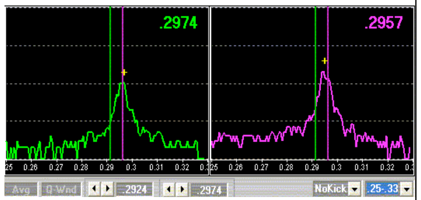
31.292
32.297

Relevant for beam stability:
non integer part

HERA revolution frequency: 47.3 kHz

$0.292 * 47.3\ kHz = 13.81\ kHz$

permanent tune measurement …and control in both planes

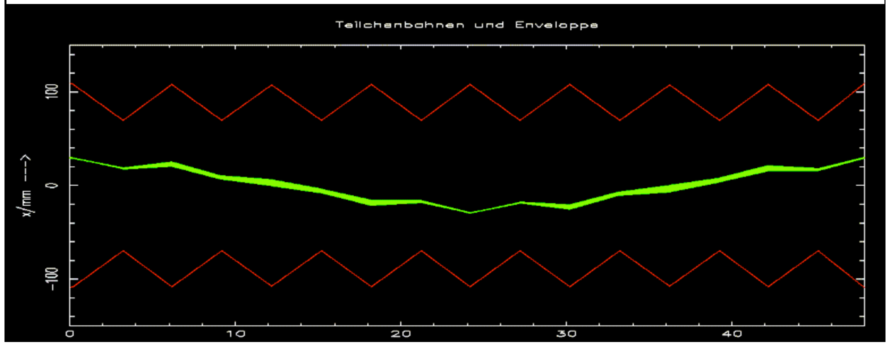


$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(\psi_{s1} - \psi_s - \pi Q)\, ds}{2 \sin \pi Q}$$

Assume: Tune = integer $Q = 1 \rightarrow 2\sin\frac{\mu}{2} = 2\sin\pi = 0$

Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.

Qualitatively spoken:

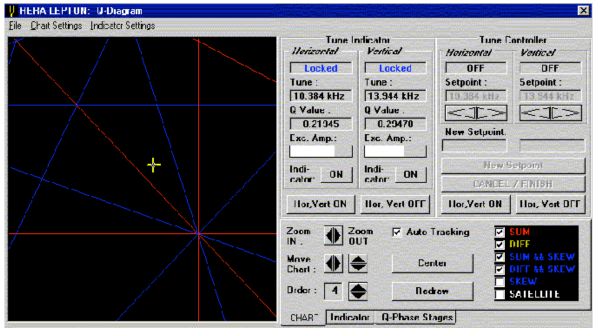


Tune & Resonances

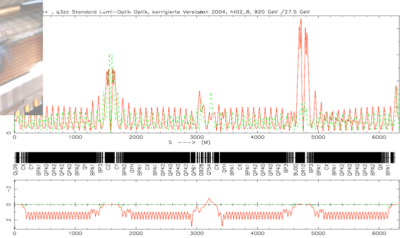
The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.
→ particle loss within a short number of turns.

→ *avoid large magnet errors*
→ *avoid forbidden tune values in both planes*

$$m * Q_x + n * Q_y = p$$ *n,m,p = integer numbers*



23.) Quadrupole Errors:



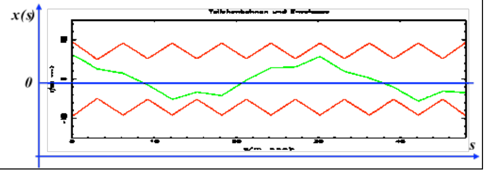
go back to Lecture I, page 1
single particle trajectory

Solution of equation of motion $x = x_0 * \cos(\sqrt{k} * l) + x_0' * \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l)$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad M_{QF} = \begin{pmatrix} \cos(\sqrt{k} * l) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l) \\ -\sqrt{k} \sin(\sqrt{k} * l) & \cos(\sqrt{k} * l) \end{pmatrix}$$

Definition: *phase advance of the particle oscillation per revolution in units of 2π is called tune*

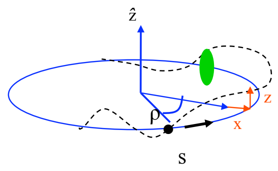
$$Q = \frac{\Delta\psi_{turn}}{2\pi} = \frac{\mu}{2\pi}$$



Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta kds(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta kds\,\beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

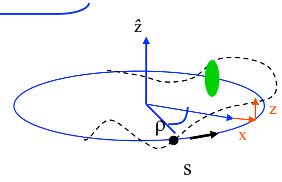
rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta kds\beta \sin\psi_0$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta kds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta kds(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta kds\,\beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta kds\beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\psi = \psi_0 + \Delta\psi \quad \longrightarrow \quad \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds\,\beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds\,\beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s) ds}{4\pi}$$

! the tune shift is proportional to the β-function at the quadrupole

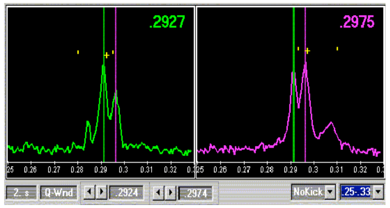
!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam

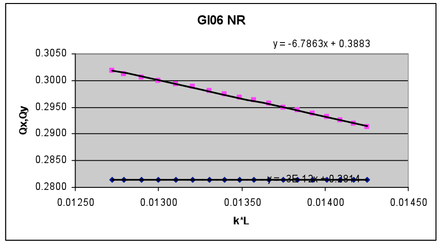
Example: deliberate change of quadrupole strength in a synchrotron:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)}{4\pi} ds \approx \frac{\Delta K(s) * l_{quad} * \bar{\beta}}{4\pi}$$



tune spectrum ...

... for heaven's sake:
why do we get three peaks ????



tune shift as a function of a gadient change

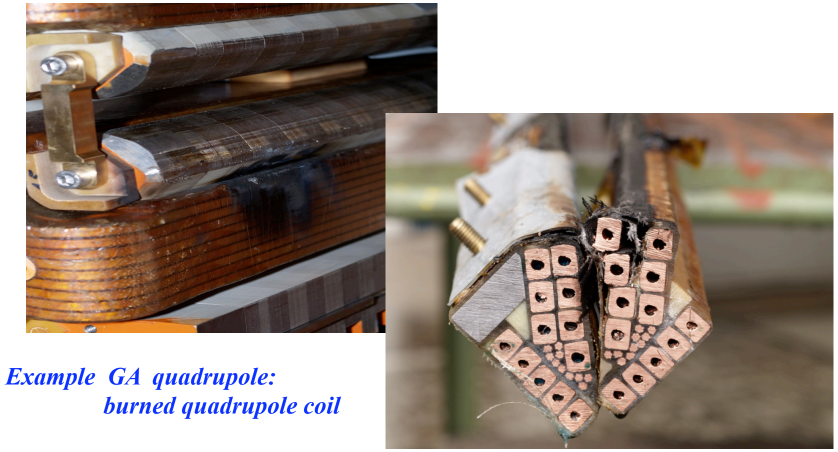
Clearly there is another problem:

a focussing error at any location in the machine

... will shift the tune

... and distort the optics

... at any place in the ring

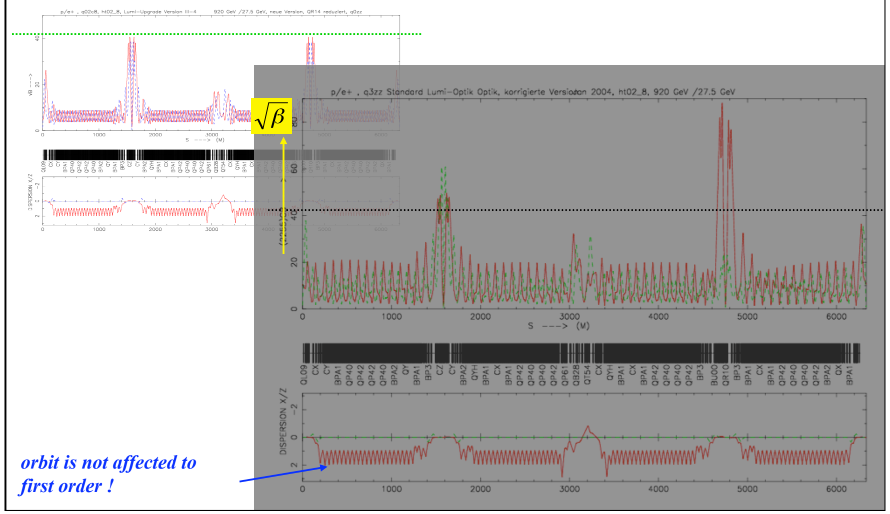


Example GA quadrupole:

burned quadrupole coil

Quadrupole error → Beta Beat

$$\Delta\beta(s_0) = -\frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1}-\psi_{s0})-2\pi Q\right)ds$$

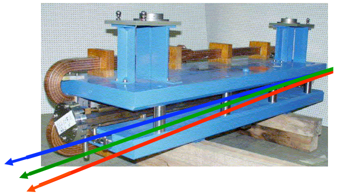


orbit is not affected to first order !

24.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

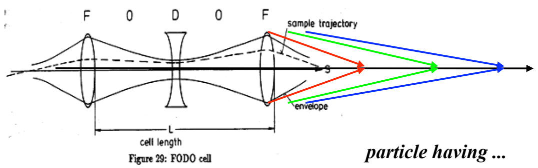
Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $\alpha = \frac{\int B\, dl}{p/e}$



$x_D(s) = D(s)\frac{\Delta p}{p}$

focusing lens $k = \frac{g}{p/e}$



particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$k = \frac{g}{p/e}$ $\qquad p = p_0 + \Delta p$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \quad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s) ds$$

Where is the Problem ?

… what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ *it is determined by the focusing strength k of all quadrupoles*

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

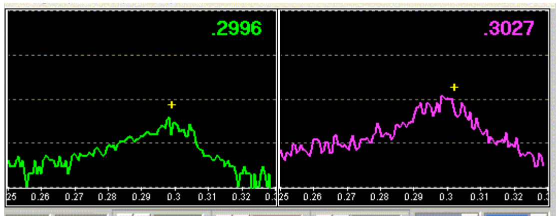
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

Q' = 250
$\Delta p/p = +/- 0.2 * 10^{-3}$
$\Delta Q = 0.256 \dots 0.36$

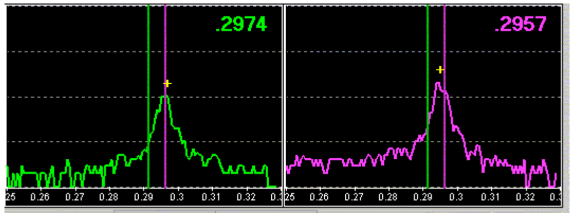
→ *Some particles get very close to resonances and are lost*

in other words: the tune is not a point it is a pancake



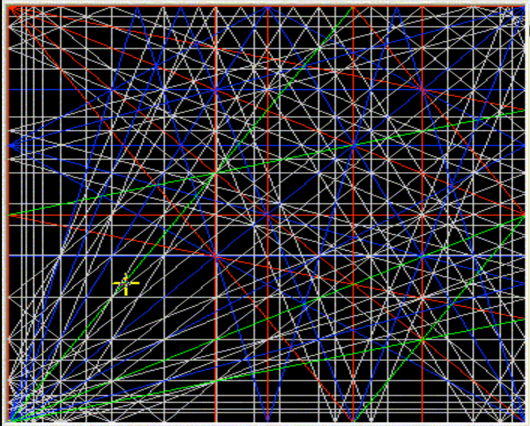
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m{*}Q_x + n{*}Q_y + l{*}Q_s = integer$$



RA e Tune diagram up to 3rd order

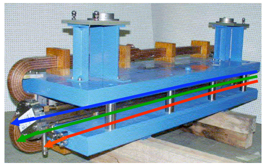
… and up to 7th order

Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

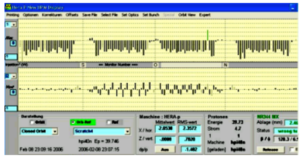
Correction of Q':

Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum $x_D(s) = D(s)\frac{\Delta p}{p}$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Correction of Q':

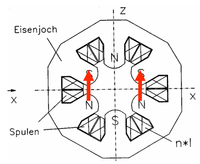
k_1 normalised quadrupole strength

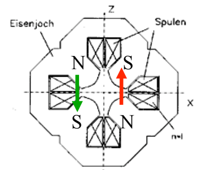
k_2 normalised sextupole strength

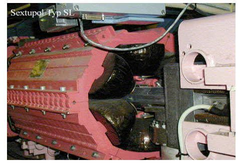
$$k_1(sext) = \frac{\tilde{g}\,x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$

Sextupole Magnets:







corrected chromaticity

considering a single cell:

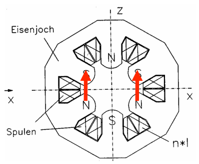
$$Q'_x = \frac{-1}{4\pi} * \oint k_1(s)\beta(s)\,ds + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

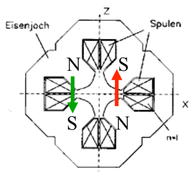
Correction of Q':

k_1 normalised quadrupole strength

k_2 normalised sextupole strength

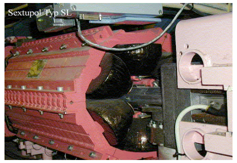
Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}\, x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$



more in detail: we have to correct the chromaticity in the two planes ...
... and in each plane the sextu[pole fields will contribute with different signs to the Q‘

$$Q'_{cell\_x} = -\frac{1}{4\pi}\left\{ k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd} \right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = -\frac{1}{4\pi}\left\{ -k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd} \right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi} ds \approx \frac{\Delta k(s) * l_{quad} * \bar{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right) ds$$

chromaticity

$$\Delta Q = Q' * \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)\,ds$$

in a FoDo

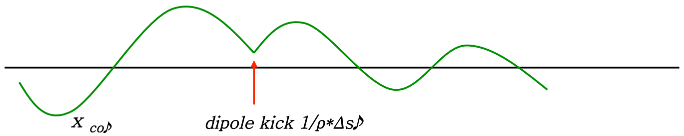
$$Q'_{cell} = -\frac{1}{\pi}\tan\frac{\mu}{2}$$

corrected chromaticity

$$Q'_x = \frac{-1}{4\pi} * \oint k_1(s)\beta(s)\,ds + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Appendix: Closed Orbit Distortion

Calculation of Orbit Distortion in a circular machine:



periodicity condition still has to be fulfilled: ***we still get (!) a closed orbit***

in any case: ***distorted orbit will be a betatron oscillation.***

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi) \qquad a = \text{orbit amplitude, } \varphi = \text{initial phase}$$

put starting conditions: $s = 0 \;, \quad \psi(s) = 0$

boundary condition (1): $x_d(s+L) = x_d(s)$ *periodic closed orbit*

$$a\sqrt{\beta(s+L)} * \cos(\psi(s) + 2\pi Q - \varphi) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

$$\varphi = \pi Q$$

Calculation of Orbit Distortion:

angle x′: $x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$

$$x_d'(s) = -a\sqrt{\beta} * \sin(\psi(s) - \varphi) * \psi'(s) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

remember: $\psi'(s) = \frac{1}{\beta}$

$$x_d'(s) = \frac{-a}{\sqrt{\beta}} \sin(\psi(s) - \varphi) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

boundary condition (2): $x_d'(s+L) + \frac{\Delta s}{\rho} = x_d'(s)$ *at the place of the distortion, s = 0, ψ = 0*

$$\frac{-a}{\sqrt{\beta(s+L)}} \sin(2\pi Q - \varphi) + \frac{\beta'(s+L)}{2\sqrt{\beta(s+L)}} a * \cos(2\pi Q - \varphi) + \frac{\Delta s}{\rho} =$$

$$= \frac{-a}{\sqrt{\beta(s)}} \sin(-\varphi) + \frac{\beta'(s)}{2\sqrt{\beta(s)}} a * \cos(-\varphi)$$

periodicity: $\beta(s) = \beta(s+L), \quad \varphi = \pi Q$

$$\frac{-a}{\sqrt{\beta}} \sin(\pi Q) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\pi Q) + \frac{\Delta s}{\rho} = \frac{-a}{\sqrt{\beta}} \sin(-\pi Q) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(-\pi Q)$$

remember: $\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x)$

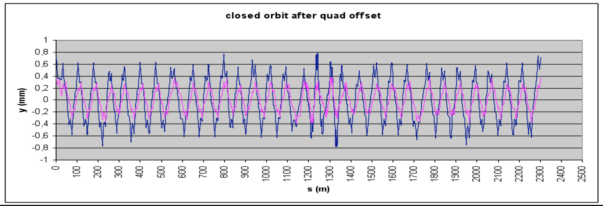
$$\frac{-a}{\sqrt{\beta}} \sin(\pi Q) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\pi Q) + \frac{\Delta s}{\rho} = \frac{a}{\sqrt{\beta}} \sin(\pi Q) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\pi Q)$$

$$\frac{\Delta s}{\rho} = \frac{2a}{\sqrt{\beta}} \sin(\pi Q) \quad \longrightarrow \quad a = \frac{\Delta s/\rho \sqrt{\beta}}{2\sin(\pi Q)}$$

put into orbit equation:

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \pi Q) = \frac{\delta_1 * \sqrt{\beta(s)\beta_1}}{2\sin(\pi Q)} * \cos(\psi(s) - \pi Q)$$

where $\delta = \Delta s/\rho$ *denotes the orbit kick*



PETRA III Light Source:

closed orbit error after offset of 0.3mm in 2 quadrupole magnets *calculation in full detail – i.e. for arbitrary initial phase ψ(s1) – yields♪*

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$

Nota bene:

* ***orbit distortion os visible at any position "s" in the ring***
  ... even if the dipole error is located at one single point "s_1"

* ***the beta function describes the sensitivity of the beam to external fields***

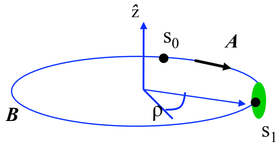
* ***the beta function acts as amplifcation factor for the orbit amplitude at the given observation point***

* ***in any case ... we clearly will obtain a cosine-like orbit travelling around the ring ... but being closed !!! after one turn.***

* ***there is a resonance denominator***

Appendix: Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"



$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

distorted matrix $$M_{dist} = \begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{21} & m^*_{22} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta k ds a_{11} + a_{12} & -\Delta k ds a_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11} a_{12} + b_{12}(-\Delta k ds a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12", which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m^*_{12} = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta k ds$$

$$(1) \quad m^*_{12} = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M* is still a matrix for one complete turn we still can express the element m_{12} in twiss form:

$$(2) \quad m^*_{12} = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\beta_0 \cancel{\sin 2\pi Q} - a_{12}b_{12}\Delta k ds = \beta_0 \cancel{\sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ} \cos 2\pi Q$$

ignoring second order terms

$$-a_{12}b_{12}\Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \frac{\Delta k \beta_1 ds}{4\pi}$

(index „1" refers to location of the error)

$$-a_{12}b_{12}\Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \{ 2a_{12}b_{12} + \beta_0 \beta_1 \cos 2\pi Q \} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}} (\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1\cos 2\pi Q\right\}\Delta k ds$$

$$a_{12} = \sqrt{\beta_0\beta_1}\,\sin\Delta\psi_{0\to 1}$$
$$b_{12} = \sqrt{\beta_0\beta_1}\,\sin(2\pi Q - \Delta\psi_{0\to 1})$$

$$\boldsymbol{d\beta_0} = \frac{-\beta_0\beta_1}{2\sin 2\pi \boldsymbol{Q}}\left\{2\sin\Delta\psi_{01}\sin(2\pi \boldsymbol{Q} - \Delta\psi_{01}) + \cos 2\pi \boldsymbol{Q}\right\}\boldsymbol{\Delta k ds}$$

... after some TLC transformations ... $= \cos(2\Delta\psi_{01} - 2\pi Q)$

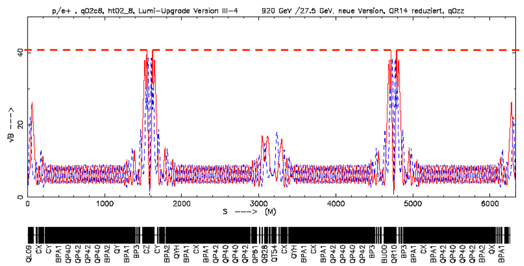
$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right)ds$$

Nota bene: **! the beta beat is *proportional to the strength of the error Δk***

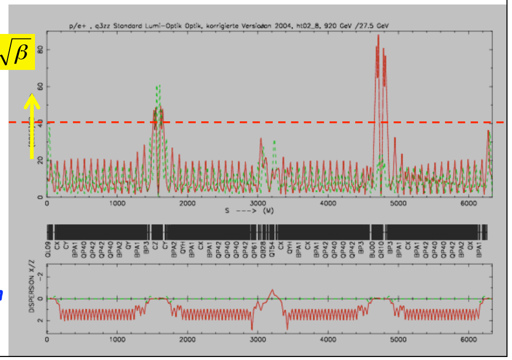
!! and to the *β function* at the place of the error ,

!!! and to the *β function* at the observation point,
(… remember orbit distortion !!!)

!!!! there is a *resonance denominator*



ideal i.e. unperturbed beam optics for luminosity operation



perturbed bam optics due to broken quadrupole winding

Appendix: Dispersion

Solution of the inhomogenious equation of motion

Ansatz: $D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$

$$D'(s) = S'*\int\frac{1}{\rho}C\,dt + S\frac{1}{\rho}C - C'*\int\frac{1}{\rho}S\,dt - C\frac{1}{\rho}S$$

$$D'(s) = S'*\int\frac{C}{\rho}dt \; - \; C'*\int\frac{S}{\rho}dt$$

$$D''(s) = S''*\int\frac{C}{\rho}d\tilde{s} + S'\frac{C}{\rho} \; - \; C''*\int\frac{S}{\rho}d\tilde{s} \; - \; C'\frac{S}{\rho}$$

$$= S''*\int\frac{C}{\rho}d\tilde{s} - C''*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}\underbrace{(CS' - S\,C')}_{=\det M = 1}$$

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remember: for C(s) and S(s) to be independent solutions the Wronski determinant has to meet the condition $W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$

and as it is independent of the variable „s“ $\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$

we get for the initital conditions that we had chosen ... $C_0 = 1, \; C_0' = 0$; $S_0 = 0, \; S_0' = 1$ $\Rightarrow$ $W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$

$$D'' = S''*\int\frac{C}{\rho}d\tilde{s} - C''*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion: $S'' + K*S = 0$, $C'' + K*C = 0$

$$D'' = -K*S*\int\frac{C}{\rho}d\tilde{s} + K*C*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K*\underbrace{\left\{S\int\frac{C}{\rho}d\tilde{s} + C\int\frac{S}{\rho}d\tilde{s}\right\}}_{=D(s)} + \frac{1}{\rho}$$

$D'' = -K*D + \frac{1}{\rho}$ **... or** $\underline{\underline{D'' + K*D = \frac{1}{\rho}}}$ ***qed***