New Relativistic Dissipative Fluid Dynamics from Kinetic Theory

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Heavy-Ion Collisions in the LHC Era
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Fluid Dynamics / Hydrodynamics — what, where, why

- Kinetic/Transport Theory: Microscopic theory.
- Fluid Dynamics: Effective theory that describes the slow, long-wavelength motion of a fluid close to equilibrium.
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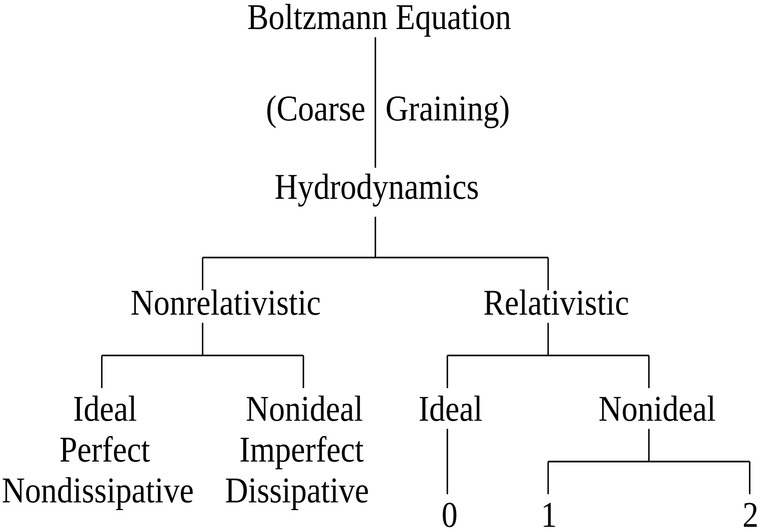

2: Currently under intense investigation

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Ideal and Dissipative Hydrodynamics

Ideal	Dissipative
$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$	$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$
$N^\mu = n u^\mu$	$N^\mu = n u^\mu + n^\mu$
Unknowns: $\underbrace{\epsilon,\ P,\ n,\ u^\mu}_{1+1+\ 1+\ 3} = 6$	$\underbrace{\epsilon,\ P,\ n,\ u^\mu,\ \Pi,\ \pi^{\mu\nu},\ n^\mu}_{1+1+\ 1\ +\ 3\ +\ 1\ +\ 5\ +\ 3} = 15$
Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad EOS}_{4 \quad + \quad 1 \quad + \quad 1} \quad = 6$	
Closed set of equations	9 more equations required

- $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to u^μ $(u_\mu \Delta^{\mu\nu} = 0\ ;\ u^\mu u_\mu = 1)$

Question:

How to derive the extra nine equations which would give us a closed set of equations?

Boltzmann equation provides a way.

Boltzmann Equation

- Boltzmann equation:

$$p^\mu \partial_\mu f(x,p) = C[f].$$

- The collision term $C[f]$ for $2 \leftrightarrow 2$ elastic collisions:

$$C[f] = \frac{1}{2}\int dp' dk\ dk'\ W_{pp'\to kk'}\left(f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} - f_p f_{p'} \tilde{f}_k \tilde{f}_{k'}\right),$$

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Properties of the Collision Term

- Should be consistent with the second law of thermodynamics
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Despite the long history of the Boltzmann equation, a non-local collision term, to our knowledge, has never been used to derive hydrodynamic equations.

From Israel-Stewart's classic paper [1979]

copic transport equations. We require only the following general properties:

(i) $\mathscr{C}$ is a purely local function or functional of N, independent of $\partial_\mu N$.

(ii) The form of $\mathscr{C}$ is consistent with conservation of 4-momentum and number of particles at collisions [see (3.6)].

(iii) $\mathscr{C}$ yields a non-negative expression for the entropy production [see (3.12)] and does not vanish unless N has the form of a local equilibrium distribution (see beginning of Sec. 4).

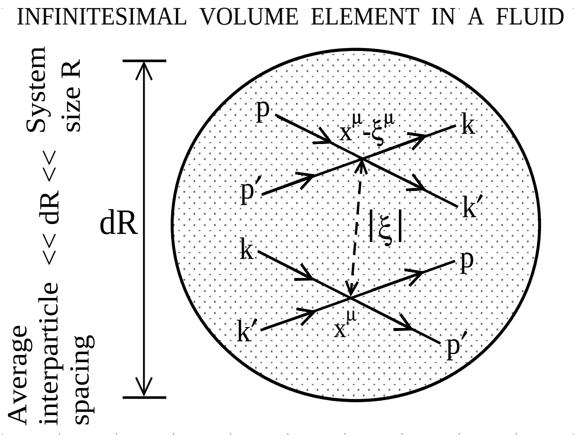
These requirements are of course met by Boltzmann's ansatz for 2-particle collisions, and, indeed, one may hope that they hold somewhat more generally, although the locality assumption (i) is a powerful restriction.

- Note the words hope and assumption.
- The assumption of the local collision term is questionable.

Infinitesimal Volume Element in a Fluid

Landau-Lifshitz, *Fluid Mechanics*, page 1, §1, para 1:

"Any small volume element in the fluid is always supposed so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are 'physically' infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules".



Assumption that the processes $(kk' \rightarrow pp')$ and $(pp' \rightarrow kk')$ occur at the same space-time point has been relaxed to include a separation ξ.

Generalization of the Collision Term

- If gradients of $f(x,p)$ are allowed in the collision term, then

$$C[f]_{\rm gen} = C[f] + \partial_\mu (A^\mu f) + \partial_\mu \partial_\nu (B^{\mu\nu} f) + \cdots ,$$

where A^μ and $B^{\mu\nu}$ are tensor coefficients in the non-local terms.

- This form can also be derived explicitly for $2 \leftrightarrow 2$ elastic collisions: Recall

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- Collision term with non-local effects

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Tensor Decomposition and Constraints on Collision Term

- A^μ and $B^{\mu\nu}$ are tensor-decomposed into available tensor degrees of freedom p^μ and $g^{\mu\nu}$ as

$$A^\mu = ap^\mu \quad ; \quad B^{\mu\nu} = b_1 g^{\mu\nu} + b_2 p^\mu p^\nu$$

- $a, \; b_1, \; b_2$ constrained by current and energy-momentum conservation, and positive entropy divergence.

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- ξ^μ is arbitrary: Conservation equations must be satisfied order-by-order in ξ. The constraints on $a, \; b_1, \; b_2$ are

$$\partial_\mu a = 0; \quad \partial^2 \left(b_1 a_{00}\right) + \partial_\mu \partial_\nu \left(b_2 I^{\mu\nu}\right) = 0;$$

$$u_\alpha \partial_\mu \partial_\nu \left(b_2 I^{\mu\nu\alpha}\right) + u_\alpha \partial^2 \left(b_1 n u^\alpha\right) = 0; \quad |a| \leq 1$$

where $a_{00} = \int dp f_0$ and $I^{\mu_1 \mu_2 \cdots \mu_n} = \int dp \; p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} f_0$.

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Deriving Evolution Equations for Dissipative Quantities

- Evolution equations for dissipative quantities are obtained from the second moment of the improved Boltzmann equation:

$$\int dp\, p^\alpha p^\beta p^\gamma \partial_\gamma f = \int dp\, p^\alpha p^\beta \left[C[f] + p^\gamma \partial_\gamma (af) + \partial^2 (b_1 f_0) + (p\cdot\partial)^2 (b_2 f_0) \right]$$

- For system close to equilibrium: $f = f_0 + \delta f$, and to second order in gradients

$$C[f]_{\rm gen} = C[f] + \partial_\mu \left(A^\mu f \right) + \partial_\mu \partial_\nu \left(B^{\mu\nu} f_0 \right); \quad f_0 = \frac{1}{\exp(\beta u\cdot p - \alpha) + r}$$

- To proceed further, Grad's 14-moment method is used:

$$f = f_0 + f_0 \tilde{f}_0 \left(\lambda_\Pi \Pi + \lambda_n n_\alpha p^\alpha + \lambda_\pi \pi_{\alpha\beta} p^\alpha p^\beta \right)$$

- Introduce first-order shear tensor $\sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}$, vorticity $\omega_{\mu\nu} = (\nabla_\mu u_\nu - \nabla_\nu u_\mu)/2$ and expansion scalar $\theta = \partial\cdot u$.

Evolution equations (Note the New Terms)

$$\Pi = \tilde{a}\Pi_{\rm NS} - \beta_{\dot\Pi}\tau_\Pi\dot\Pi + \tau_{\Pi n} n\cdot\dot u - l_{\Pi n}\partial\cdot n - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi n} n\cdot\nabla\alpha$$
$$+ \lambda_{\Pi\pi}\pi_{\mu\nu}\sigma^{\mu\nu} + \Lambda_{\Pi\dot u}\dot u\cdot\dot u + \Lambda_{\Pi\omega}\omega_{\mu\nu}\omega^{\nu\mu} + (8\ \text{terms}),$$

$$n^\mu = \tilde{a} n^\mu_{\rm NS} - \beta_{\dot n}\tau_n\dot n^{\langle\mu\rangle} + \lambda_{nn} n_\nu\omega^{\nu\mu} - \delta_{nn} n^\mu\theta + l_{n\Pi}\nabla^\mu\Pi - l_{n\pi}\Delta^{\mu\nu}\partial_\gamma\pi^\gamma_\nu$$
$$- \tau_{n\Pi}\Pi\dot u^\mu - \tau_{n\pi}\pi^{\mu\nu}\dot u_\nu + \lambda_{n\pi} n_\nu\pi^{\mu\nu} + \lambda_{n\Pi}\Pi n^\mu + \Lambda_{n\dot u}\omega^{\mu\nu}\dot u_\nu$$
$$+ \Lambda_{n\omega}\Delta^\mu_\nu\partial_\gamma\omega^{\gamma\nu} + (9\ \text{terms}),$$

$$\pi^{\mu\nu} = \tilde{a}\pi^{\mu\nu}_{\rm NS} - \beta_{\dot\pi}\tau_\pi\dot\pi^{\langle\mu\nu\rangle} + \tau_{\pi n} n^{\langle\mu}\dot u^{\nu\rangle} + l_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi\pi}\pi^{\langle\mu}_\rho\omega^{\nu\rangle\rho}$$
$$- \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha - \tau_{\pi\pi}\pi^{\langle\mu}_\rho\sigma^{\nu\rangle\rho} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \Lambda_{\pi\dot u}\dot u^{\langle\mu}\dot u^{\nu\rangle}$$
$$+ \Lambda_{\pi\omega}\omega^{\langle\mu}_\rho\omega^{\nu\rangle\rho} + \chi_1\dot b_2\pi^{\mu\nu} + \chi_2\dot u^{\langle\mu}\nabla^{\nu\rangle}b_2 + \chi_3\nabla^{\langle\mu}\nabla^{\nu\rangle}b_2,$$

- where $\tilde a = (1-a)$, $\dot X = u^\mu\partial_\mu X$ and "8 terms" ("9 terms") involve second-order, scalar (vector) combinations of derivatives of b_1, b_2.

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Evolution Equations in Bjorken Model

- With $\pi \equiv \Phi = -\tau^2 \pi^{\eta\eta}$, the shear evolution equation is

$$\frac{\pi}{\tau_\pi} + \beta_{\dot\pi} \frac{d\pi}{d\tau} = \beta_\pi \frac{4}{3\tau} - \lambda \frac{\pi}{\tau} - \psi \pi \frac{db_2}{d\tau}$$

where

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$$\psi = \frac{11(\epsilon_0 + P_0)}{6\tilde{a}\beta\eta} \quad ; \quad \lambda = 2\tilde{a} - \frac{b_1\beta^2 - 20 b_2}{6\tilde{a}\beta\eta}(\epsilon_0 + P_0).$$

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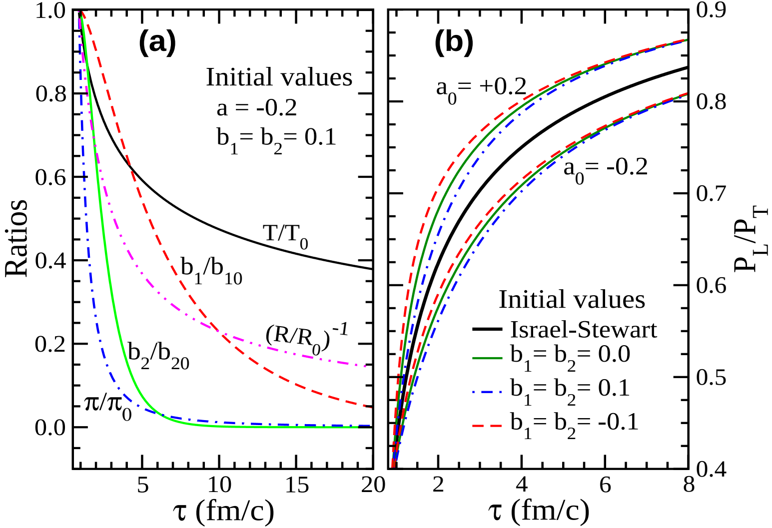
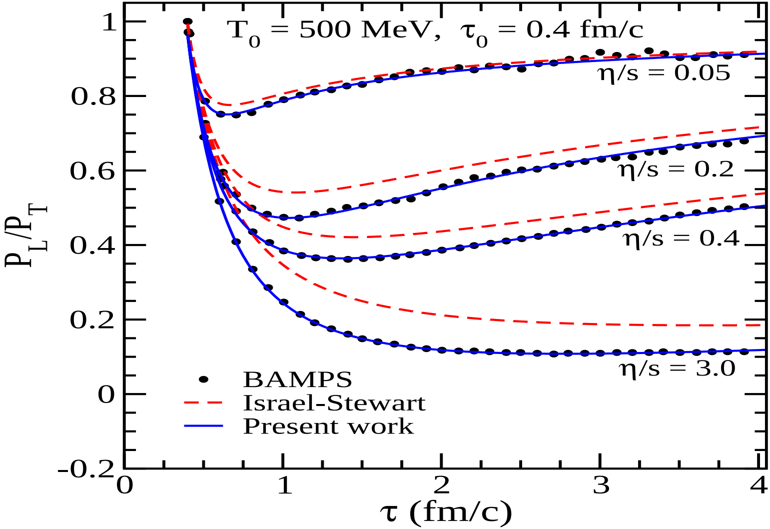
Evolution of various quantities and P_L/P_T



b_1, b_2 behave in a regular fashion   Non-local effects are important

Hydro with non-local effects can match transport



Summary

- $C[f]$ in Boltzmann equation generalized to include non-local effects.
- Non-locality parameterized and constrained from underlying physics.
- Formulated complete second-order dissipative hydrodynamics from Boltzmann equation with the generalized collision term.
- The formulation captures all the second-order terms that are allowed and the coefficients of the existing terms are also modified.
- Evolution of non-local parameters shows regular behaviour and non-local effects are important.
- Even the (first-order) Navier-Stokes equation receives a small correction.

THANK YOU