Resummations in the Bloch-Nordsieck Model

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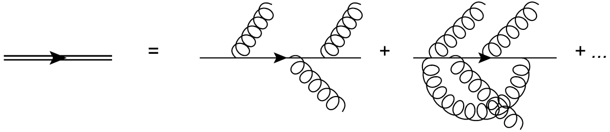
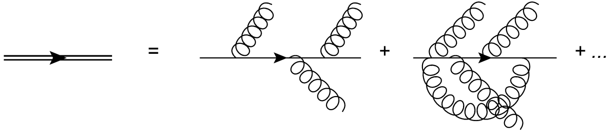
Contents

Fermion propagation in an external field $\rightarrow$ soft gauge bosons emitted, absorbed (gluons, photons, etc...)



What's the dressed propagator? In general hard to answer...

Fermion propagation in an external field $\rightarrow$ soft gauge bosons emitted, absorbed (gluons, photons, etc...)



What's the dressed propagator? In general hard to answer...

Except for the B-N model.

- Bloch-Nordsieck Model: *solution by path-integral*

EXACT

Difficult to generalize to other guage theories

$\Updownarrow$

PERTURBATION THEORY:

- tree diagram
- one-loop correction

IR problem at higher orders...

$\Updownarrow$

RESUMMATIONS:

- 2PI resummation
- Exact resummation (D-S)

Hopefully can be generalized

- A method to treat *infrared physics*

The (Toy)Model

QED Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m - eA\!\!\!/\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
- $\partial\!\!\!/ = \gamma_\mu \partial^\mu$
- m: fermion mass, e: coupling constant

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- $\partial\!\!\!/ = \gamma_\mu \partial^\mu$
- m: fermion mass, e: coupling constant

$$\Downarrow$$

Bloch-Nordsieck Lagrangian

$$\mathcal{L} = \psi^\dagger u^0 \left(i u^\mu \partial_\mu - m - e u^\mu A_\mu\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- $u^\mu \in M^4$ és $u_\mu u^\mu = 1$

The Free Fermion Green's Function

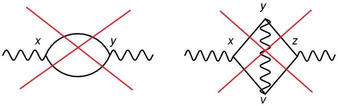
The partial differential equation:

$$(iu^\mu \partial_\mu - m) G_0(x-y) = \delta(x-y)$$

The propagator in momentum space:

$$\hat{G}_0(p) = \frac{1}{u_\mu p^\mu - m + i\epsilon}$$

- Retarded: $G_0(x-y) = 0$, if $x^0 < y^0 \rightarrow \nexists$ antifermion
- Closed fermion loops are cancelled!



Closed fermion loops

Fermion Moving In Classical Field

The partial differential equation:

$$[u^\mu (i\partial_\mu + eA_\mu(x)) - m] G(x, y|A) = \delta(x - y)$$

Solution:

$$G(x, y|A) = i \frac{1}{2\pi^4} \int_0^\infty d\nu \int dp \exp\{-ip(x-y) - i\nu(up - m + i\epsilon) + iK(\nu|A)\}$$

Where: $K = \frac{e}{(2\pi)^2} \int dk (uA(k)) e^{-ikx} \int_0^\nu d\nu' e^{i(uk)\nu'}$ és $\nu \in \mathbb{R}_+$ (linear in A)

The gauge field is separated!

The "Dressed" Fermion Propagator

Connection between the two Green's function *with functional averaging*:

$$G(x,y) = \frac{\int G(x,y|A)\langle T \exp\left\{ie\int \bar{\psi}(z)A(z)\psi(z)dz\right\}\rangle_{F_0}\mathcal{D}A}{\int \langle T \exp\left\{ie\int \bar{\psi}(z)A(z)\psi(z)dz\right\}\rangle_{F_0}\mathcal{D}A}$$

Bloch-Nordsieck

$\nexists$ antifermion $\rightarrow \langle T \exp\left\{ie\int \bar{\psi}(z)A(z)\psi(z)dz\right\}\rangle_{F_0} = 1$

- $G(x,y) = \int G(x,y|A)\mathcal{D}A$
- $\hat{G}(p) = i\int\limits_0^\infty e^{-i\nu(up-m+i\epsilon)}d\nu \int e^{iK(\nu|A)}\mathcal{D}A$ (Gaussian type)

The Exact Result

$$\boxed{\hat{G}(p) = \frac{1}{(up)-m}\left(\frac{(up)}{m} - 1\right)^{-\frac{e^2(3-\xi)}{8\pi^2}}} \quad (\hat{G} \sim (up-m)^{\gamma} \text{ power law})$$

$$\left(\frac{(up)}{m} - 1\right)^{-\frac{e^2(3-\xi)}{8\pi^2}} \approx 1 - \frac{e^2}{4\pi^2}(3-\xi)\ln\left(\frac{(up)}{m} - 1\right) + \dots$$

Where ξ is the gauge fixing parameter

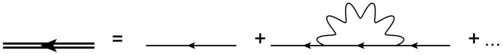
Divergencies:

- UV no
- IR ($|up| \approx m$) YES $\longrightarrow$ photon accumulation on the mass shell!

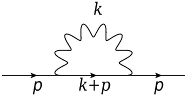
Bogoliubov N.N., Shirkov D.V., Introduction to the theory of quantized fields (Wiley, 1980)

Perturbative Theory, One-loop Integral

Expand by the *coupling constant* of the interaction:



Loop integrals$\longrightarrow$*divergencies*(IR,UV)



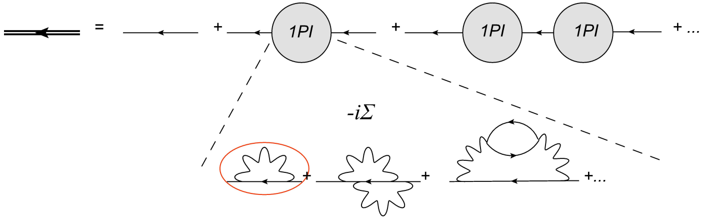
$$\equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} u^\mu \frac{i}{(p^\alpha + k^\alpha)u_\alpha - m + i\epsilon} u^\nu \left(-i \frac{g_{\mu\nu}}{k^2 + i\epsilon}\right)$$

$$\Updownarrow$$

$$\boxed{I = -ie^2 \frac{p^0 - m + i\epsilon}{4\pi^2}\left[-\frac{2}{\delta} + \gamma_E - 1 - \frac{1}{2}\ln\pi + \ln\left(\frac{m - p^0 - i\epsilon}{\lambda}\right)\right]}$$

Fermion Self-energy

1PI series:



Geometric series (Dyson): $i\hat{G}(p^0) = \frac{i}{G_0^{-1} - \Sigma}$

$$\Sigma(p^0) = iI = e^2 \frac{p^0 - m + i\epsilon}{4\pi^2}\left[-\frac{2}{\delta} + \gamma_E - 1 - \frac{1}{2}\ln\pi + \ln\left(\frac{m - p^0 - i\epsilon}{\lambda}\right)\right]$$

$$\Sigma = \Sigma_{fin} + \Sigma_{div} \quad (\delta \to 0)$$

Renormalization

Lagrangian

$$\mathcal{L} = \psi^\dagger u^0 \left(iu^\mu \partial_\mu - m - eu^\mu A_\mu\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$$

- $\mathcal{L}_r = \psi^\dagger u^0 iu^\mu \partial_\mu \psi - m_r \psi^\dagger u^0 \psi - e_r \psi^\dagger u^0 u^\mu A_\mu \psi < \infty$
- $\mathcal{L}_{ct} = \delta Z \psi^\dagger u^0 iu^\mu \partial_\mu \psi - \delta m \psi^\dagger u^0 \psi - \delta e \psi^\dagger u^0 u^\mu A_\mu \psi = \infty$

$$m = m_r + \underbrace{\delta m}_{\infty}, \quad e = e_r + \underbrace{\delta e}_{\infty}, \quad Z = 1 + \underbrace{\delta Z}_{\infty}$$

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$$\hat{G}(p^0) = \frac{1}{G_0^{-1} - \Sigma} \Rightarrow \hat{G}^r(p^0) = \frac{1}{(1+\delta Z)p^0 - (m_r + \delta m) - (\Sigma_{fin} + \Sigma_{div})}$$

$$\Sigma_{ct} = \delta Z p^0 - \delta m = \infty$$
$$\Sigma_r = \Sigma_{fin} + \Sigma_{div} - \Sigma_{ct} < \infty$$

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$$\hat{G}(p^0) = \frac{1}{G_0^{-1} - \Sigma} \Rightarrow \hat{G}^r(p^0) = \frac{1}{p^0 - m_r - (\Sigma_{fin} + \Sigma_{div} - \Sigma_{ct})} = \frac{1}{p^0 - m_r - \Sigma_r}$$

$$\Sigma_{ct} = \delta Z p^0 - \delta m = \infty$$
$$\Sigma_r = \Sigma_{fin} + \Sigma_{div} - \Sigma_{ct} < \infty$$

Renormalization

The counter temrs:

- $\delta Z = -e^2 \frac{1}{4\pi^2}\left(-\frac{2}{\delta} + \ln\left(\frac{m-p_t^0}{\lambda}\right)\right) = e^2 \frac{1}{2\pi^2}\frac{1}{\delta} + \text{Finite}(p_t^0)$
- $\delta m = -e^2 \frac{m}{4\pi^2}\left(-\frac{2}{\delta} + \ln\left(\frac{m-p_t^0}{\lambda}\right)\right) = e^2 \frac{m}{2\pi^2}\frac{1}{\delta} + \text{Finite}(p_t^0)$

$$\boxed{\Sigma_r = \Sigma_{fin} + \Sigma_{div} - \Sigma_{ct} = -e^2 \frac{p^0 - m}{4\pi^2} \ln\left(\frac{m-p^0}{\lambda}\right)}$$

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$$\boxed{\Sigma_r = \Sigma_{fin} + \Sigma_{div} - \Sigma_{ct} = -e^2 \frac{p^0 - m}{4\pi^2} \ln\left(\frac{m - p^0}{\lambda} \right)}$$

Exact ($\hat{G}^{ex}$)

$$\frac{1}{(up) - m} \left[1 - \frac{e^2}{8\pi^2} (3 - \xi) \ln\left(\frac{(up)}{m} - 1 \right) \right]$$
$$\xi = 1$$

One-loop correction ($\hat{G}^{1loop}$)

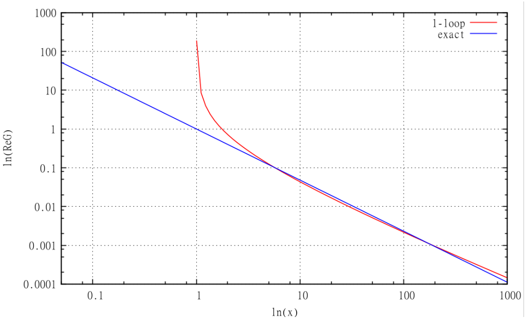
$$\frac{1}{p^0 - m - \Sigma_r} = \frac{1}{p^0 - m} \left[1 - \frac{e^2}{4\pi^2} \ln\left(\frac{p^0 - m}{\lambda} \right) \right]$$

Perturbation theory (1st order)✔

$$ReG_{ex}(x) = \frac{1}{x^{(1+\alpha/\pi)}}$$

$$ReG_{1-l}(x) \sim \frac{1}{(\alpha/\pi)x\log|x|}$$

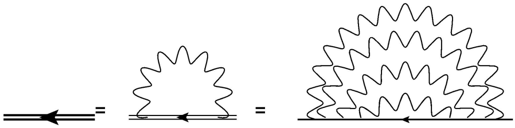
Where $x := p_0 - m$ and $\alpha = e^2/4\pi$



IR problems at the mass-shell⇒ perturbation theory breaks down

2PI (rainbow)

Dealing with the IR limit: must *reshuffle* the perturbation series→resummation



2PI resummation

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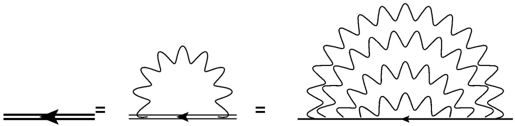


2PI resummation

$$G[\Sigma] \Leftrightarrow \Sigma[G]$$

Self-consistent equations

⇓

Numerical solution of the 2PI equations

Antal Jakovác, Phys.Rev.D76:125004,2007. hep-ph/0612268

The Analytic Solution

The self-energy:

$$\Sigma(p^0) = \frac{-ie^2}{(2\pi)^4} \int dk^4 \, \frac{G(p^0 - k^0)}{k^2 + i\epsilon}$$

With the spectral representation $G(p^0 - k^0) = \int_0^\infty d\omega \frac{\rho(\omega)}{p^0 - k^0 - \omega + i\epsilon}$

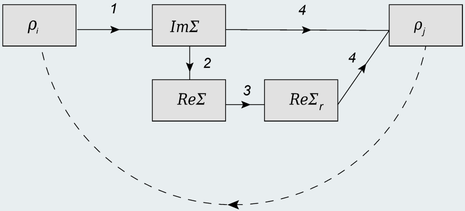
$$\Sigma(p^0) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega) \Sigma_{1-loop}(p^0, \omega)$$

$$G(p^0) \sim \frac{1}{p^0 - m}, \text{ (for small } p^0\text{)}$$

$$G(p^0) \sim \frac{1}{(p^0 - m)\sqrt{\ln(p^0 - m)}}, \text{ (for large } p^0\text{)}$$

The Numerical Solution

The algorithm



1. $\mathrm{Im}\Sigma(p^0) = \frac{1}{4\pi^2} \int_0^{p^0} dk^0 \; k^0 \rho^{FR}(p^0 - k^0)$

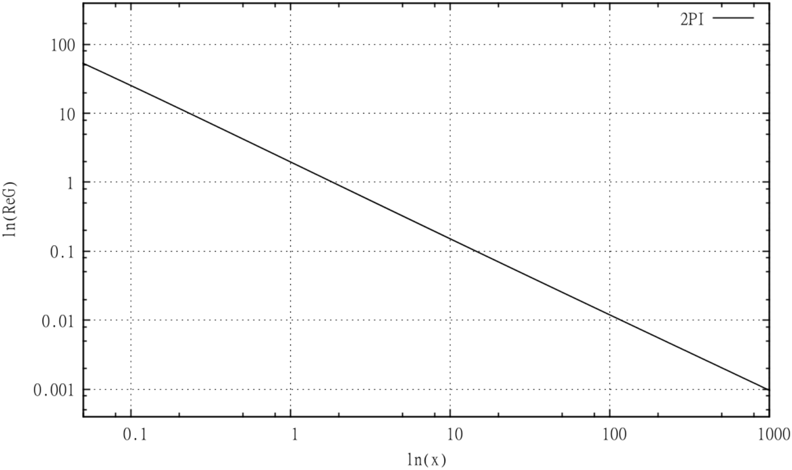
2. $\mathrm{Re}\Sigma(p^0) = \mathcal{P}\frac{1}{\pi} \int_{-\infty}^{\infty} dq^0 \; \frac{\mathrm{Im}\Sigma(q^0)}{q^0 - p^0}$ (Kramers-Kronig)

3. $\mathrm{Re}\Sigma_r(p^0) = \mathrm{Re}\Sigma(p^0) - \left(\mathrm{Re}\Sigma(p^0_t) - \left.\frac{\partial \mathrm{Re}\Sigma(p^0_t)}{\partial p^0}\right|_{p^0_t} (p^0 - p^0_t) \right)$ (renormalization: physical scheme)

4. $\rho^{FR}(p^0) = \frac{2\mathrm{Im}\Sigma_r}{\mathrm{Re}[G_0^{-1} - \Sigma_r]^2 + [\mathrm{Im}\Sigma_r]^2}$, $(\rho(x) = \langle\{\psi(x), \psi^\dagger(0)\}\rangle_0)$ (spectral function)

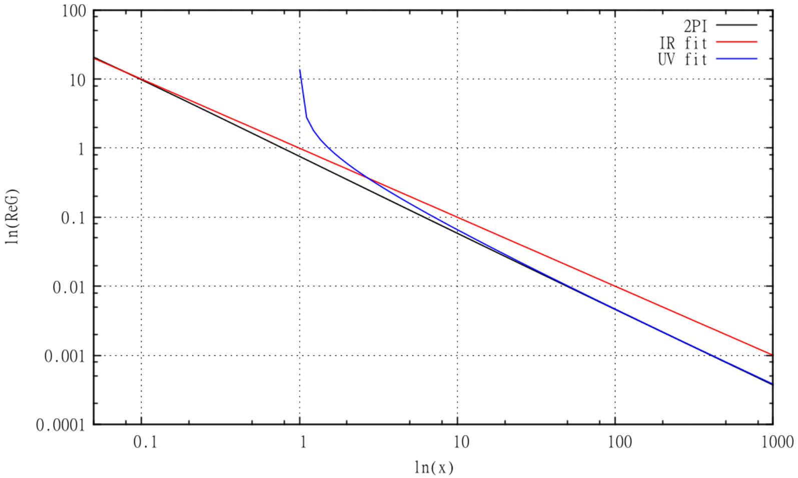
Results

From $\rho(p^0)$ we can reconstruct $\mathrm{Re}G(p^0)$



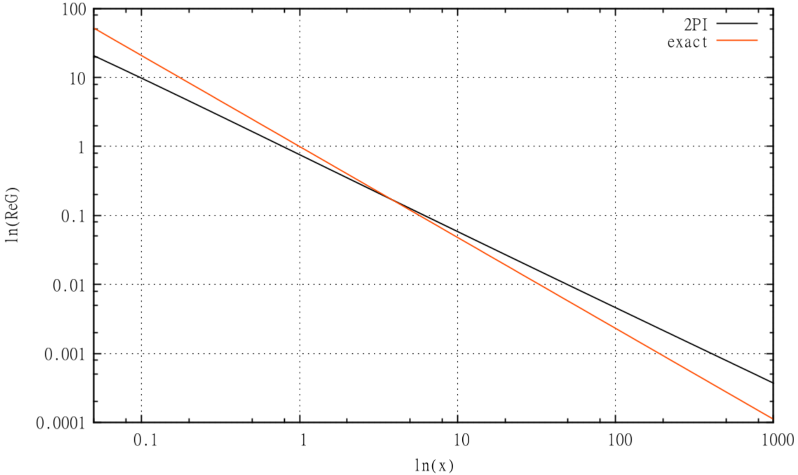
Results

Perfect match with the analytic formulas



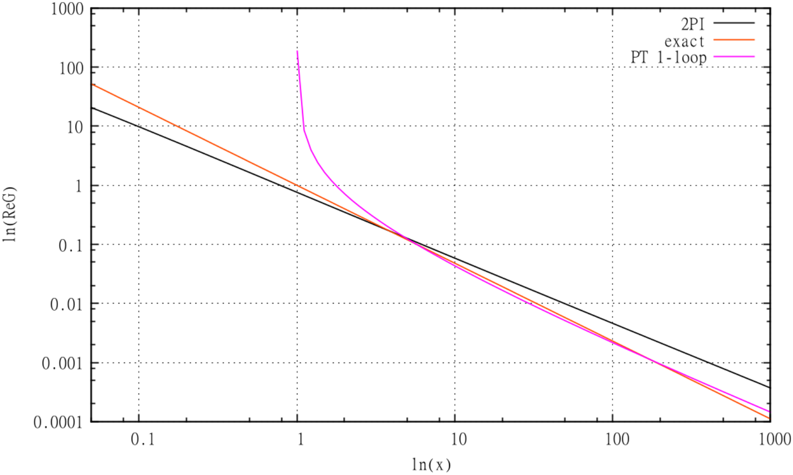
Results

Compared to the exact B-N solution... Bad news :-(



Results

At least *works in IR* unlike the perturbation theory!



CAN WE DO BETTER??

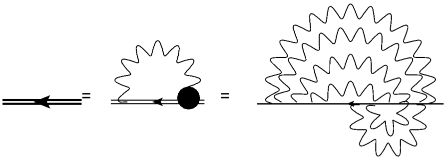
CAN WE DO BETTER??

YES, WE CAN!

(Keywords: Dyson-Schwinger + Ward-identities)

Exact Resummation

"Modified 2PI" = 2PI + vertex corrections (exact Ward-identities)



- $G[\Sigma] \Leftrightarrow \Sigma[G]$
- $k_0\Gamma^0(p, p+k, k) = G^{-1}(p) - G^{-1}(p-k)$

Where $\Gamma^0(p, p+k, k)$ is the *vertexfunction*
It has one-to-one correspondence to $G(p)$!

Self-consistent equations

The self-energy:

$$\Sigma(p) = \frac{-ie^2}{(2\pi)^4} \int dk^4 \frac{1}{k^2 + i\epsilon} G(p-k) u_\mu \Gamma^\mu(k; p-k, p)$$

Since $\Gamma^0(p, p-k, k) = \frac{G^{-1}(p) - G^{-1}(p-k)}{k_0}$

And $u = (1, 0, 0, 0)$

$$\Sigma(p^0) = \frac{-ie^2}{(2\pi)^4} G(p^0) \int dk^4 \frac{1}{k^2 + i\epsilon} \frac{G(p^0 - k^0)}{k_0}$$

From this we can get

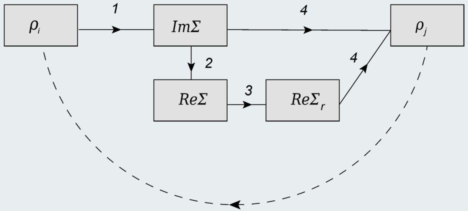
$$(p^0 - m) G(p^0) = \frac{\alpha}{\pi} \int_{p_0}^{M} d\omega G(\omega)$$

Its solution:

$$G(p^0) = \frac{const.}{(p^0 - m)^{(1 + \frac{\alpha}{\pi})}} \equiv \text{B-N}$$

The Numerical Solution (possible!)

The algorithm



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3. $\mathrm{Re}\Sigma_r(p^0) = \mathrm{Re}\Sigma(p^0) - \left(\mathrm{Re}\Sigma(p_t^0) - \frac{\partial \mathrm{Re}\Sigma(p_t^0)}{\partial p^0}\Big|_{p_t^0}(p^0-p_t^0)\right)$ (renormalization)

4. $\rho^{FR}(p^0) = \frac{2\mathrm{Im}\Sigma_r}{\mathrm{Re}[G_0^{-1}-\Sigma_r]^2+[\mathrm{Im}\Sigma_r]^2}$ (retarded spectralfunction)

Conclusion:

- Exact B-N solution:
  - power law: $\rho_{eg} \sim (p-m)^\gamma$
  - hard to generalize to other gauge theories
- One-loop correction:
  - not power law
  - breaks down at IR (because of $\ln(x)$)
- 2PI resummation:
  - can deal with the IR div.
  - but poor approximation of the exact
- Exact resummation:
  - can deal with the IR div.
  - power law (exactly the B-N)
  - a new solution of the model
  - hopefully can be generalized

Outlook, Literature

Outlook:

- adapting the method to QED (a possibly good approx.!)
- applications at ELI (strong fields vs. nonperurbative method)
- finite temperature calculation (automatic regularization of IR)
- examine bounded states (IR physics questions)
- adapting the method to QCD (???)

Literature:

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R. J. Rivers *Path Integral Methods in Quantum Field Theory, Cambridge University Press (1987)*

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Antal Jakovác *Phys.Rev.D76:125004,2007. hep-ph/0612268*

THANK YOU FOR YOUR ATTENTION!